EQUIVALENT STATIC LOADS FOR RANDOM VIBRATION
Revision N

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October 4, 2012

The following approach in the main text is intended primarily for single-degree-of-freedom systems. Some consideration is also given for multi-degree-of-freedom systems.

Introduction

A particular engineering design problem is to determine the equivalent static load for equipment subjected to base excitation random vibration. The goal is to determine peak response values.

The resulting peak values may be used in a quasi-static analysis, or perhaps in a fatigue calculation.   The response levels could be used to analyze the stress in brackets and mounting hardware, for example.

Limitations

Limitations of this approach are discussed in Appendices F through K.

A particular concern for either a multi-degree-of-freedom system or a continuous system is that the static deflection shape may not properly simulate the predominant dynamic mode shape. In this case, the equivalent static load may be as much as one order of magnitude more conservative than the true dynamic load in terms of the resulting stress levels.

Load Specification

Ideally, the dynamics engineer and the static stress engineer would mutually understand, agree upon, and document the following parameters for the given component.

1. Mass, center-of-gravity, and inertia properties
2. Effective modal mass and participation factors
3. Stiffness
4. Damping
5. Natural frequencies
6. Dynamic mode shapes
7. Static deflection shape
8. Response acceleration
9. Modal velocity
10. Relative displacement

11. Transmitted force from the base to the component in each of three axes
12. Bending moment at the base interface about each of three axes
13. The manner in which the equivalent static loads and moments will be applied to the component, such as point load, body load, distributed load, etc.
14. Dynamic stress and strain at critical locations if the component is best represented as a continuous system
15. Response limit criteria, such as yield stress, ultimate stress, fatigue, or loss of clearance

Each of the response parameters should be given in terms of frequency response function, power spectral density, and an overall response level.

Furthermore, assumptions must be documented, including a discussion of conservatism.

Again, this list is very idealistic.

Importance of Modal Velocity

Bateman wrote in Reference 24:

> Of the three motion parameters (displacement, velocity, and acceleration) describing a shock spectrum, velocity is the parameter of greatest interest from the viewpoint of damage potential. This is because the maximum stresses in a structure subjected to a dynamic load typically are due to the responses of the normal modes of the structure, that is, the responses at natural frequencies. At any given natural frequency, stress is proportional to the modal (relative) response velocity. Specifically,
>
> $$\sigma_{max} = C\,V_{max}\sqrt{E\,\rho} \tag{1}$$
>
> where
>
> σ_{max} = Maximum modal stress in the structure
>
> V_{max} = Maximum modal velocity of the structural response
>
> E = Elastic modulus
>
> ρ = Mass density of the structural material
>
> C = Constant of proportionality dependent upon the geometry of the structure (often assumed for complex equipment to be $4 < C < 8$)

Some additional research is needed to further develop equation 1 so that it can be used for equivalent quasi-static loads for random vibration. Its fundamental principle is valid, however.

Further information on the relationship between stress and velocity is given in Reference 25.

Importance of Relative Displacement

Relative displacement is needed for the spring force calculation. Note that the transmitted force for an SDOF system is simply the mass times the response acceleration.

Specifying the relative displacement for an SDOF system may seem redundant because the relative displacement can be calculated from the response acceleration and the natural frequency per equation (7) given later in this paper.

But specifying the relative displacement for an SDOF system is a good habit.

The reason is that the relationship between the relative displacement and the response acceleration for a multi-degree-of-freedom (MDOF) or continuous system is complex. Any offset of the component's center-of-gravity (CG) further complicates the calculation due to coupling between translational and rotational motion in the modal responses.

The relative displacement calculation for an MDOF system is beyond the scope of a hand calculation, but the calculation can be made via a suitable Matlab script. A dynamic model is required as shown in Appendices H and I.

Furthermore, examples of continuous structures are shown in Appendices J & K. The structures are beams. The bending stress for the equivalent static analysis of each beam correlates better with relative displacement than with response acceleration.

Model

The first step is to determine the acceleration response of the component.

Model the component as an SDOF system, if appropriate, as shown in Figure 1.

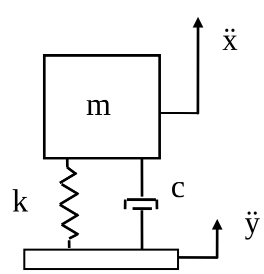

Figure 1.

where

M	is the mass
C	is the viscous damping coefficient
K	is the stiffness
X	is the absolute displacement of the mass
Y	is the base input displacement

Furthermore, the relative displacement z is

$$z = x - y \tag{2}$$

The natural frequency of the system fn is

$$fn = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3}$$

<u>Acceleration Response</u>

The Miles' equation is a simplified method of calculating the response of a single-degree-of-freedom system to a random vibration base input, where the input is in the form of a power spectral density.

The overall acceleration response $\ddot{x}_{GRMS}$ is

$$\ddot{x}_{GRMS}\left(f_n, \xi\right) = \sqrt{\left(\frac{\pi}{2}\right)\left(\frac{fn}{2\xi}\right)P} \tag{4}$$

where

Fn	is the natural frequency
P	is the base input acceleration power spectral density at the natural frequency
ξ	is the damping ratio

Note that the damping is often represented in terms of the quality factor Q.

$$Q = \frac{1}{2\xi} \tag{5}$$

Equation (4), or an equivalent form, is given in numerous references, including those listed in Table 1.

Table 1. Miles' equation References			
Reference	Author	Equation	Page
1	Himelblau	(10.3)	246
2	Fackler	(4-7)	76
3	Steinberg	(8-36)	225
4	Luhrs	-	59
5	Mil-Std-810G	-	516.6-12
6	Caruso	(1)	28

Furthermore, the Miles' equation is an approximate formula that assumes a flat power spectral density from zero to infinity Hz. As a rule-of-thumb, it may be used if the power spectral density is flat over at least two octaves centered at the natural frequency.

An alternate response equation that allows for a shaped power spectral density input is given in Appendix A.

Relative Displacement & Spring Force

Consider a single-degree-of-freedom (SDOF) system subject to a white noise base input and with constant damping. The Miles' equation set shows the following with respect to the natural frequency fn:

$$\text{Response Acceleration} \quad \propto \sqrt{f_n} \tag{6}$$

$$\text{Relative Displacement} \quad \propto 1 / f_n^{1.5} \tag{7}$$

$$\text{Relative Displacement} \quad = \text{Response Acceleration} \;/\; \omega_n^2 \tag{8}$$

where $\omega_n = 2\pi f_n$

Equation (8) is derived in Reference 18.

Consider that the stress is proportional to the force transmitted through the mounting spring. The spring force F is equal to the stiffness k times the relative displacement z.

$$F = k\,z \tag{9}$$

RMS and Standard Deviation

The RMS value is related the mean and standard deviation σ values as follows:

$$\text{RMS}^2 = \text{mean}^2 + \sigma^2 \tag{10}$$

Note that the RMS value is equal to the 1σ value assuming a zero mean.

A 3σ value is thus three times the RMS value for a zero mean.

Peak Acceleration

There is no method to predict the exact peak acceleration value for a random time history.

An instantaneous peak value of 3σ is often taken as the peak equivalent static acceleration. A higher or lower value may be appropriate for given situation.

Some sample guidelines for peak acceleration are given in Table 2. Some of the authors have intended their respective equations for design purposes. Others have intended their equations for "Test Damage Potential."

Table 2.
Sample Design Guidelines for Peak Response Acceleration or Transmitted Force

Refer.	Author	Design or Test Equation	Page	Qualifying Statements
1	Himelblau, et al	3σ	190	However, the response may be non-linear and non-Gaussian
2	Fackler	3σ	76	3σ is the usual assumption for the equivalent peak sinusoidal level.
4	Luhrs	3σ	59	Theoretically, any large acceleration may occur.

Table 2. Sample Design Guidelines for Peak Response Acceleration or Transmitted Force (continued)				
Refer.	Author	Design or Test Equation	Page	Qualifying Statements
7	NASA	3σ for STS Payloads 2σ for ELV Payloads	2.4-3	Minimum Probability Level Requirements
8	McDonnell Douglas	4σ	4-16	Equivalent Static Load
10	Scharton & Pankow	5σ	-	See Appendix C.
11	DiMaggio, Sako, Rubin	nσ	Eq (22)	See Appendices B and D for the equation to calculate n via the Rayleigh distribution.
12	Ahlin	Cn	-	See Appendix E for equation to calculate Cn.

Furthermore, some references are concerned with fatigue rather than peak acceleration, as shown in Table 3.

Table 3. Design Guidelines for Fatigue based on Miner's Cumulative Damage Index		
Reference	Author	Page
3	Steinberg	229
6	Caruso	29

Note that the Miner's Index considers the number of stress cycles at the 1σ, 2σ, and 3σ levels.

Modal Transient Analysis

The input acceleration may be available as a measured time history. If so, a modal transient analysis can be performed. The numerical engine may be the same as that used in the shock response spectrum calculation. The advantage of this approach is that it accounts for the response peaks that are potentially above 3σ. It is also useful when the base input is non-stationary or when its histogram deviates from the normal ideal.

The modal transient approach can still be used if a power spectral density function is given without a corresponding time history. In this case a time history can be synthesized to meet the power spectral density, as shown in Appendix B. This approach effectively requires the time history to be stationary with a normal distribution.

Furthermore, a time domain analysis would be useful if fatigue is a concern. In this case, the rainflow cycle counting method could be used.

Special Case

Consider a system that has a natural frequency that is much higher than the maximum base input frequency. An example would be a very stiff bar that was subjected to a low frequency base excitation in the bar's longitudinal axis.

This case is beyond the scope of Miles' equation, since the Miles' equation takes the input power spectral density at the natural frequency. The formula in Appendix A can handle this case, however.

As the natural frequency becomes increasingly higher than the maximum frequency of the input acceleration, the following responses occur:

1. The response acceleration converges to the input acceleration.
2. The relative displacement approaches zero.

Furthermore, the following rule-of-thumb is given in Reference 24:

> Quasi-static acceleration includes pure static acceleration as well as low-frequency excitations. The range of frequencies that can be considered quasi-static is a function of the first normal mode of vibration of the equipment. Any dynamic excitation at a frequency less than about 20 percent of the lowest normal mode (natural) frequency of the equipment can be considered quasi-static. For example, an earthquake excitation that could cause severe damage to a building could be considered quasi-static to an automobile radio.

Case History

A case history for random load factor derivation for a NASA programs is given in Reference 22.

Error Source Summary

Here is a list of error sources discussed in this paper, including the appendices.

1. An SDOF system may be an inadequate model for a component or structure.
2. An SDOF model cannot account for spatial variation in either the input or the response.
3. A CG offset leads to coupling between translational and rotational modes, thus causing the transmitted forces to vary between the mounting springs.

4. Instantaneous peak values can occur in the time domain as high as 5σ, depending on the duration and natural frequency.

5. The static deflection shape is not the same as the dynamic mode shape, thus affecting the strain calculations.

Base Input & Component Response Concerns

The derivation of the base input level is beyond the scope of this paper, but a few points are mentioned here as an aside.

1. The base input time history may have a histogram which departs from the Gaussian ideal, with a kurtosis value > 3. A solution for this problem is given in Reference 19.

2. Consider a component in its field or flight environment. The base excitation at the component's respective input points may vary by location in terms of amplitude and phase. As a first approximation, the field response of the component would be less than if the loads were uniform and in phase at the input points, which would be the case during a shaker table test. On the other hand, consider a beam simply-supported at each end. A uniform base input would not excite the beam's second bending mode. However, this mode could be excited in a field environment where the inputs were non-uniform.

3. The base input level might not account for any force-limiting or mass-loading effects from the component.

4. A structure or component may have a nonlinear response. Consider a component mounted to a plate or shell, where the mounting structure is excited by acoustical energy on the opposite side. At higher acoustic levels, the structure will undergo membrane effects which limit its vibration response, thus limiting the base input to the attached component.

5. Component damping tends to be non-linear. The damping tends to increase as the input level increases. This increase can be due to joint slipping for example. This should be considered in the context of adding margin to the input levels.

6. Conservative enveloping may have been used to derive the component base input level. In some case, the input level may be the maximum of all three axes.

Conclusion

The task of deriving an accurate equivalent static load for a component or secondary structure is very challenging.

There are numerous error sources. Some of the sources in this paper could lead to an underprediction of the load, such as omitting potential peaks > 3σ. Other sources could result in an over-prediction, as shown for the cantilever beam example in Appendix J.

Ideally, these issues could be resolved by thorough testing and analysis.

Components could be instrumented with both accelerometers and strain gages and then exposed to shaker table testing. This would allow a correlation between strain and acceleration response. The input level should be varied to evaluate potential non-linearity. The resulting stress can then be calculated from the strain.

Component modal testing would also useful to identify natural frequencies, mode shapes, and modal damping values. This can be achieved to some extent by taking transmissibility measurements during a shaker table test.

The test results could then be used to calibrate a finite element model. The calibration could be as simple as a uniform scaling of the stiffness so that the model fundamental frequency matches the measured natural frequency.

The test results would also provide the needed modal damping. Note that damping cannot be calculated from theory. It can only be measured.

Cost and schedule often limit the amount of analysis which can be performed. But ideally, the calibrated finite element model could be used for the dynamic stress calculation via a modal transient or frequency response function approach. Note that the analyst may choose to perform the post-processing via Matlab scripts using the frequency response functions from the finite element analysis.

Otherwise, the calibrated finite element model could be used for a static analysis.

The proper approach for a given component must be considered on a case-by-case basis. Engineering judgment is required.

Future Research

Further research is needed in terms of base input derivation, response analysis, and testing.

Another concern is material response. There are some references that report that steel and other materials are able to withstand higher stresses than their respective ultimate limits if the time history peak duration is of the order of 1 millisecond or less. See Appendix K.

Appendices

Table 4. Appendix Organization	
Appendix	Title
A	SDOF Acceleration Response
B	Normal Probability Values & Rayleigh Distribution
C	Excerpt from Reference 10
D	Excerpt from Reference 11
E	Excerpt from Reference 12
F	Excerpt from Reference 14
G	Excerpts from References 15 & 16
H	Two-degree-of-freedom System, Example 1
I	Two-degree-of-freedom System, Example 2
J	Cantilever Beam Example
K	Beam Simply-supported at each End Example
L	Material Stress Limits

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26. R. Huston and H. Josephs, Practical Stress Analysis in Engineering Design, Dekker, CRC Press, 2008. See Table 13.1.

APPENDIX A

SDOF Acceleration Response

The acceleration response $\ddot{x}_{GRMS}$ of a single-degree-of-freedom system to a base input power spectral density is

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\sum_{i=1}^{N} \frac{\left\{1+(2\xi\rho_i)^2\right\}}{\left\{\left[1-\rho_i^{\,2}\right]^2+\left[2\xi\rho_i\right]^2\right\}} \hat{Y}_{A\,PSD}(f_i)\Delta f_i}\ , \quad \rho_i = f_i / f_n \tag{A-1}$$

where

f_n	is the natural frequency
$\hat{Y}_{A\,PSD}(f_n)$	is the base input acceleration power spectral density

The corresponding relative displacement is

$$Z_{RMS} = \left(\frac{1}{2\pi fn}\right)^2 \ddot{X}_{GRMS} \tag{A-2}$$

Equation (A-1) allows for a shaped base input power spectral density, defined over a finite frequency domain. It is thus less restrictive than the Miles' equation. Equation (A-1) is derived in Reference 13.

Note that equation (A-1) is the usual method for the vibration response spectrum calculation, where the natural frequency is an independent variable.

APPENDIX B

Normal Probability Values

Note that the RMS value is equal to the 1σ value assuming a zero mean. The 1σ value is the standard deviation.

Consider a broadband random vibration time history x(t), which has a normal distribution.

The precise amplitude x(t) cannot be calculated for a given time. Nevertheless, the probability that x(t) is inside or outside of certain limits can be expressed in terms of statistical theory.

The probability values for the instantaneous amplitude are given in Tables B-1 and B-2 for selected levels in terms of the standard deviation or σ value.

Table B-1.
Probability for a Random Signal with Normal Distribution and Zero Mean

Statement	Probability Ratio	Percent Probability
$-\sigma < x < +\sigma$	0.6827	68.27%
$-2\sigma < x < +2\sigma$	0.9545	95.45%
$-3\sigma < x < +3\sigma$	0.9973	99.73%
$-4\sigma < x < +4\sigma$	0.99994	99.994%

Table B-2.
Probability for a Random Signal with Normal Distribution and Zero Mean

Statement	Probability Ratio	Percent Probability
$\lvert x \rvert > \sigma$	0.3173	31.73%
$\lvert x \rvert > 2\sigma$	0.0455	4.55%
$\lvert x \rvert > 3\sigma$	0.0027	0.27%
$\lvert x \rvert > 4\sigma$	6e-005	0.006%

Furthermore, the probability that an instantaneous amplitude is less than +3σ is P = 0.99865.

This is equivalent to saying that 1 out of every 741 points will exceed +3σ.

Rayleigh Distribution

The following section is based on Reference 9.

Consider the response of single-degree-of-freedom distribution to a broadband time history. The response is approximately a constant frequency oscillation with a slowly varying amplitude and phase.

The probability distribution of the instantaneous acceleration is the same as that for the broadband random function.

The absolute values of the response peaks, however, will have a Rayleigh distribution, as shown in Table B-3.

Table B-3. Rayleigh Distribution Probability

λ	Prob [$A > \lambda\sigma$]
0.5	88.25 %
1.0	60.65 %
1.5	32.47 %
2.0	13.53 %
2.5	4.39 %
3.0	1.11 %
3.5	0.22 %
4.0	0.034 %
4.5	4.0e-03 %
5.0	3.7e-04 %
5.5	2.7e-05 %
6.0	1.5e-06 %

The values in Table B-3 are calculated from

$$P(\lambda\sigma < A < \infty) = \exp\left\{-\frac{1}{2}\lambda^2\right\} \tag{B-1}$$

Determine the λ value for which exactly one peak is expected to occur for a natural frequency fn and a duration T.

$$\exp\left\{-\frac{1}{2}\lambda^2\right\} \text{fn T} = 1 \tag{B-2}$$

$$\exp\left\{-\frac{1}{2}\lambda^2\right\} = \frac{1}{\text{fn T}} \tag{B-3}$$

$$-\frac{1}{2}\lambda^2 = \ln\left[\frac{1}{\text{fn T}}\right] \tag{B-4}$$

$$\lambda^2 = -2\ln\left[\frac{1}{\text{fn T}}\right] \tag{B-5}$$

$$\lambda^2 = 2\ln[\text{fn T}] \tag{B-6}$$

$$\lambda = \sqrt{2\ln(\text{fn T})} \tag{B-7a}$$

Thus a 4.33σ peak response is expected for a system with a natural frequency of 200 Hz exposed a random vibration with a normal distribution over a 60 second duration.

$$\lambda = \sqrt{2\ln(\text{fn T})} = \sqrt{2\ln(200\,\text{Hz})(60\,\text{sec})} = 4.33 \tag{B-7b}$$

<u>Example</u>

An experimental verification of equation (B-7a) for a particular case is given in the following example.

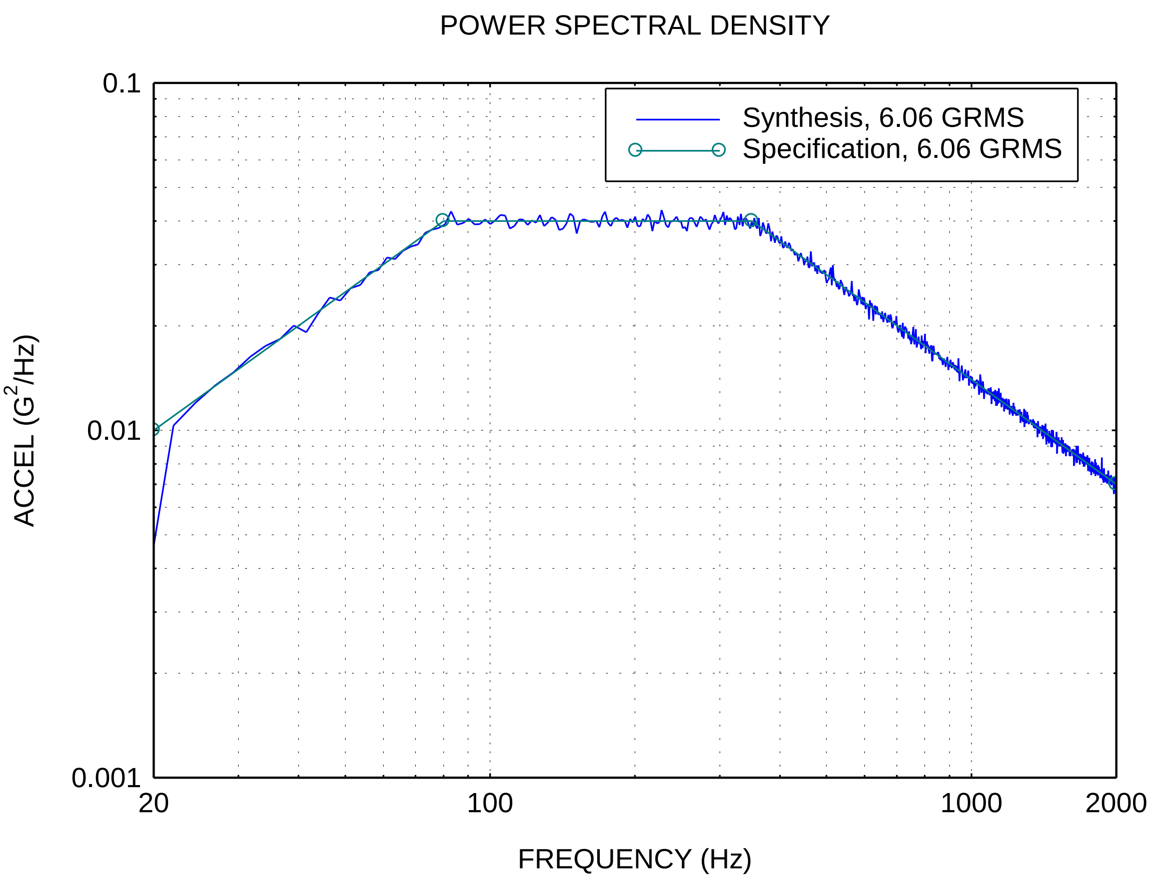


Figure B-1.

The synthesis curve agrees well with the specification, although it has a drop-out near 20 Hz.

The actual synthesized time history is shown in Figure B-2.

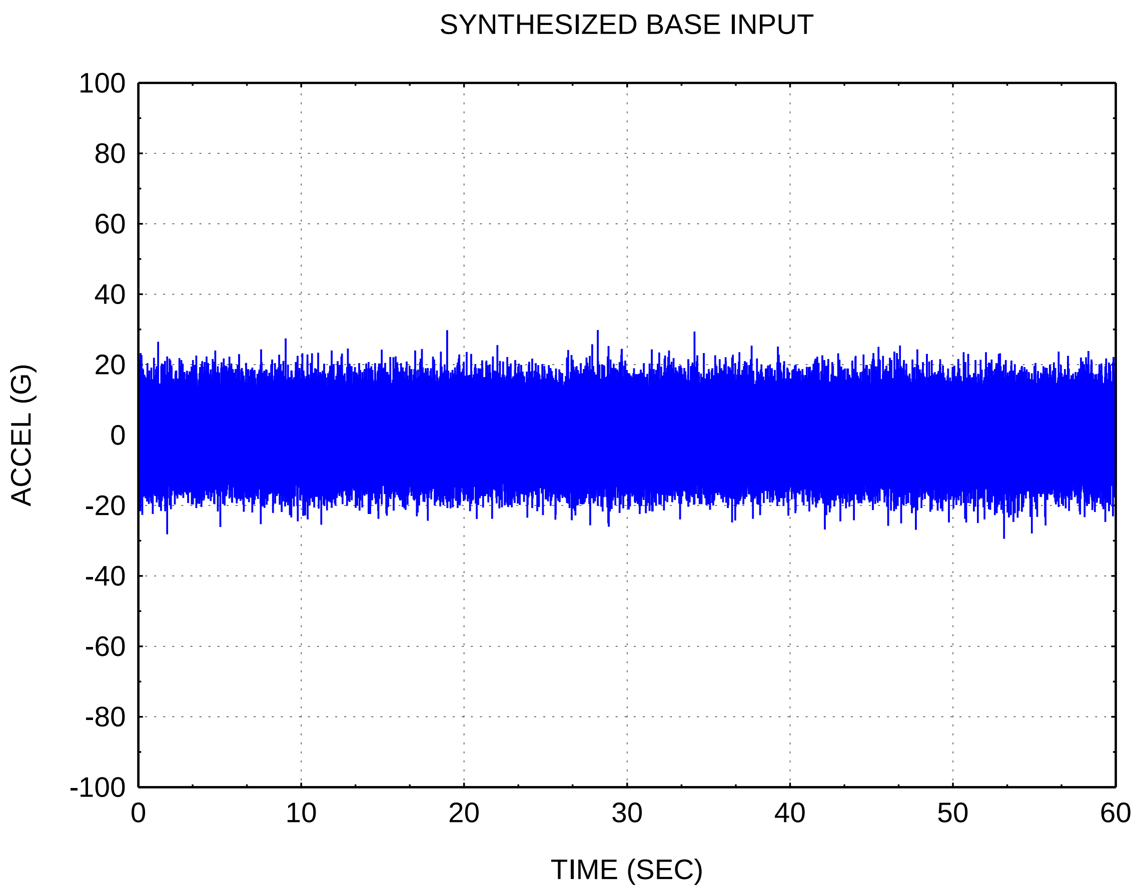


Figure B-2.

The sample rate is 20,000 samples per second.

```
Amplitude Stats

 mean = -0.0001157    std =     6.058     rms =     6.058
 max  =      29.82  at  =      28.2 sec
 min  =     -29.45  at  =     53.14 sec

 crest factor =     4.923    kurtosis  =    3.034
```

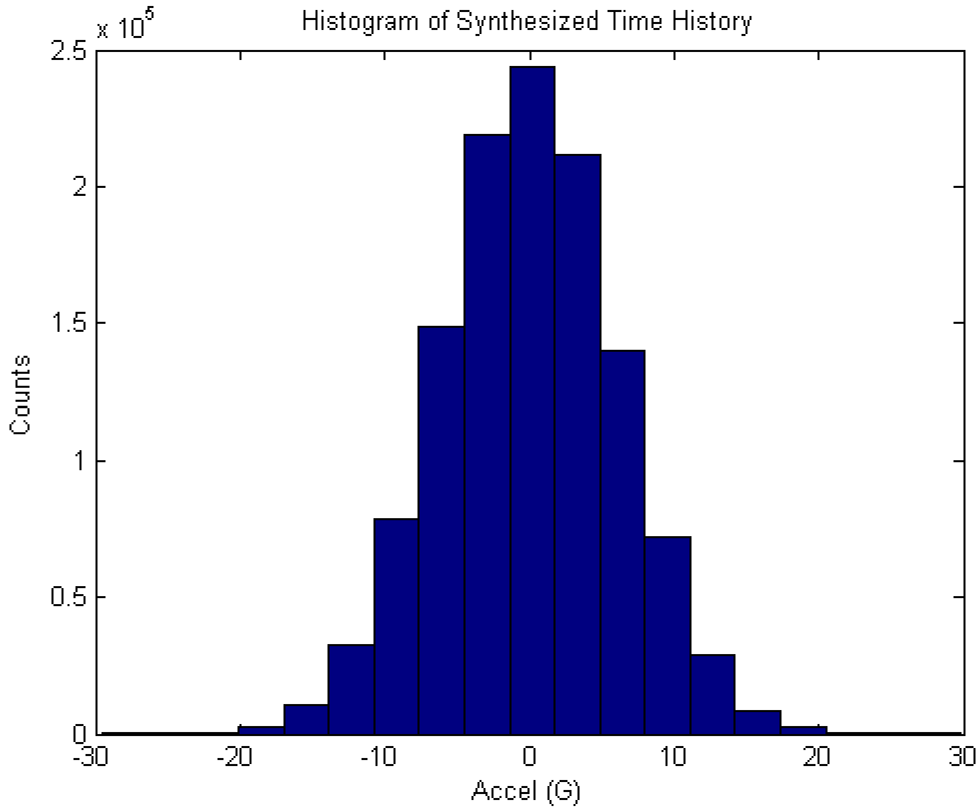


Figure B-3.

The histogram of the base input has a normal distribution.

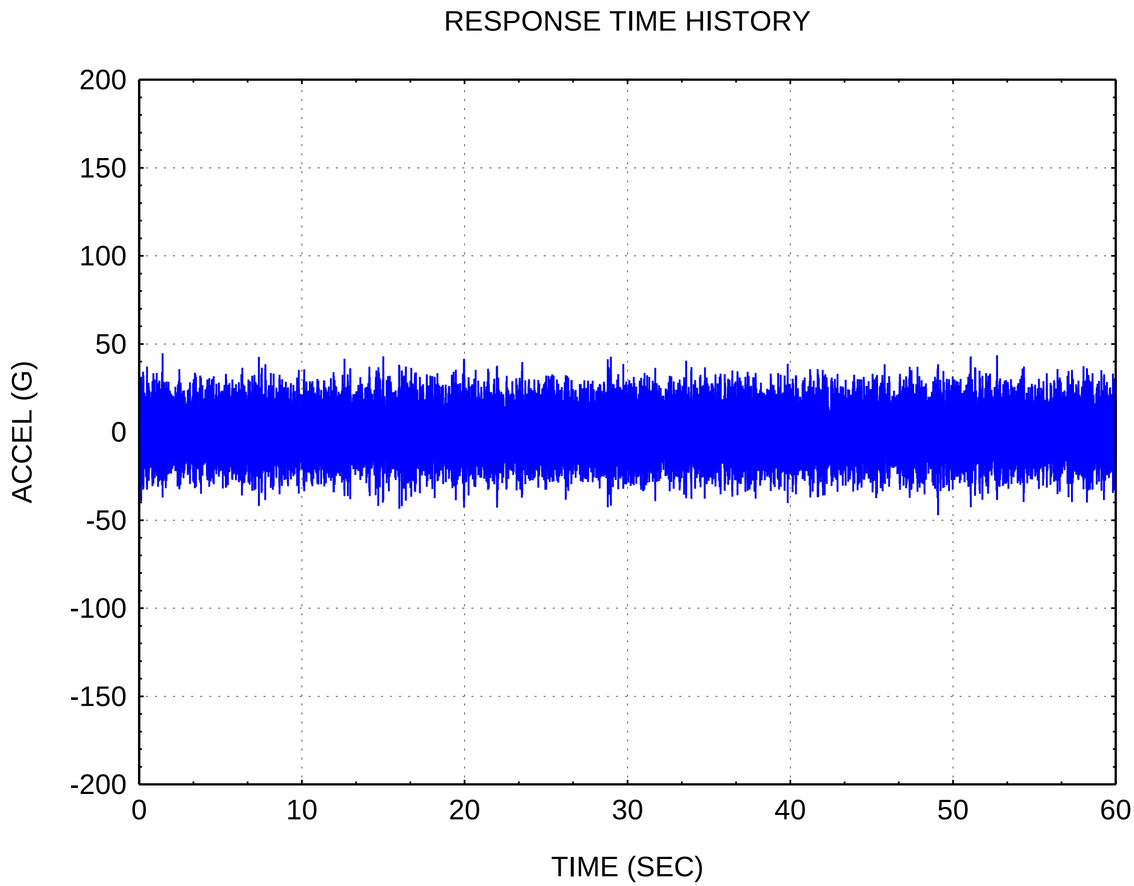


Figure B-4.

The response of single-degree-of-freedom is shown in Figure B-4. The system has a natural frequency of 200 Hz and an amplification factor Q=10. The base input is the time history in Figure B-2.

```
Amplitude Stats

mean = -2.133e-005     std =     11.19     rms =     11.19
max  =      44.57   at  =      1.44 sec
min  =     -47.18   at  =     49.07 sec

           crest factor =     4.218    kurtosis  =      2.93
```

The theoretical crest factor is

$$\lambda = \sqrt{2\ln(\mathrm{fn}\,T)} = \sqrt{2\ln(200\,\mathrm{Hz})(60\,\mathrm{sec})} = 4.33 \tag{B-8}$$

The theoretical value is 2.7% higher than the experiment value.

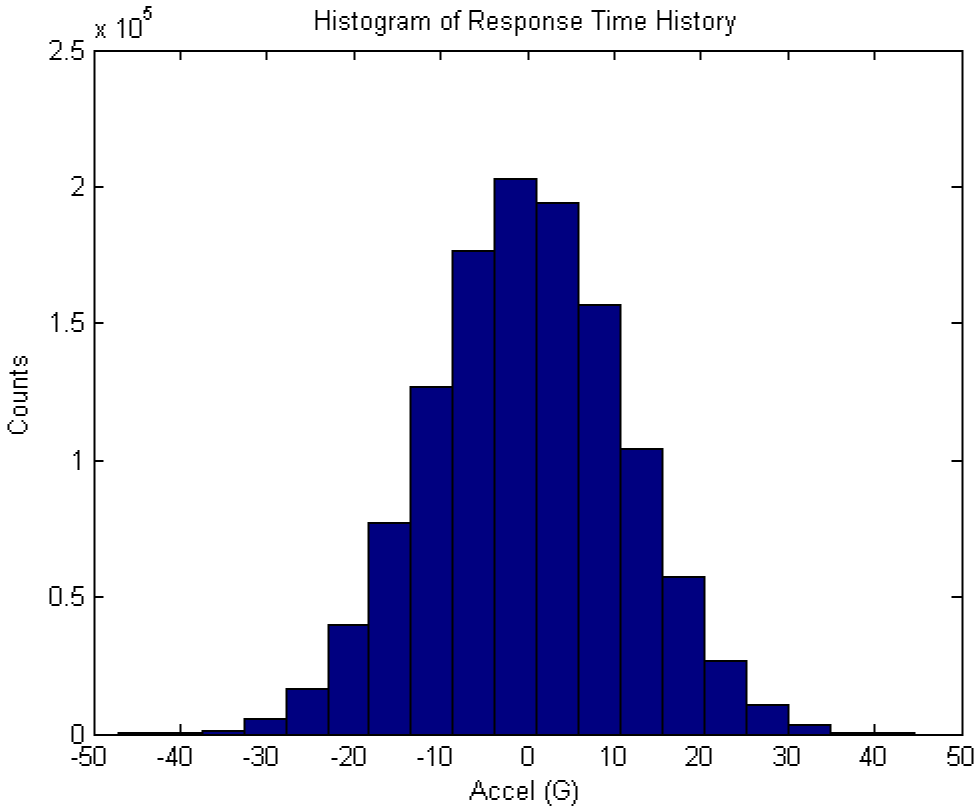


Figure B-5.

The histogram follows a normal distribution.

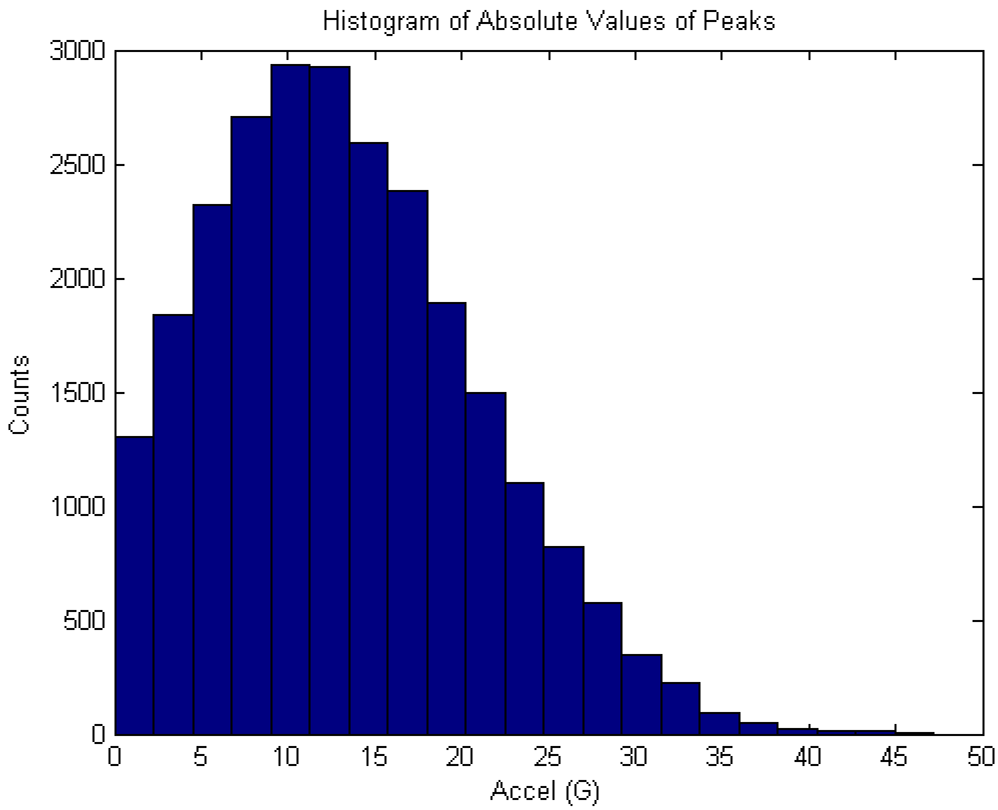


Figure B-6.

The histogram of the absolute peak from the response time history follows a Rayleigh distribution.

APPENDIX C

<u>Excerpt from Reference 10</u>

Test data shows that occurrence of extreme peaks exceeds that predicted by Rayleigh distribution.

For a Gaussian probability distribution, the probability of $|x| > 5\sigma$ is 6E-7.

If one has 60 seconds of white noise digitized at 20,000 points per seconds, the probability of a peak exceeding 5σ is 60 *20,000*6E-7=0.73

This may explain why one sees more extreme peaks than predicted by a Rayleigh distribution.

Extreme single peaks in the input acceleration are less of a concern because they do not appear to produce near-resonant amplification of the response.

Extreme peaks in the base force and acceleration responses are a very real threat.

A simple experiment indicted that the strength of hard steel and carbon rods, and presumably of other brittle materials, does not increase with frequencies up to 500 Hz.

Data in the literature indicate that the increase in strength of aluminum and many other materials is small even at considerably higher frequencies.

Given the frequent observance of five sigma test peaks in time histories of responses in random vibration tests, three sigma design strength requirements, such as those in NASA-STD-5002, appear inconsistent.

The options are to increase mission limit loads, or to decrease test margins.

APPENDIX D

Excerpt from Reference 11

Reference 11 uses the Rayleigh distribution.

Equation (22) from this reference is

$$\frac{A_{max}^2}{\sigma^2} = 2 \ln (fnT)$$

This equation is equivalent to equations (B-6) and (B-7) in Appendix B of this paper.

APPENDIX E

Excerpt from Reference 12

The following formula is intended for comparing random vibration to shock. It is not a design level per se, but may be used as such.

Consider a single-degree-of-freedom system with the index n. The maximum response $\max_n$ can be estimated by the following equations.

$$c_n = \sqrt{2 \ln (fn\ T)} \tag{E-1}$$

$$C_n = c_n + \frac{0.5772}{c_n} \tag{E-2}$$

$$\max_n = C_n\ \sigma_n \tag{E-3}$$

where

fn	is the natural frequency
T	is the duration
ln	is the natural logarithm function
σ_n	is the standard deviation of the oscillator response

APPENDIX F

Excerpt from Reference 14

The authors of Reference 14 give the following warning:

> MILES' EQUATION DOES NOT GIVE AN EQUIVALENT STATIC LOAD –
>
> Calculating the GRMS value at a resonant peak after a random vibration test and multiplying it by the test article mass does not mean that the test article was subjected to that same, equivalent static load. It simply provides a statistical calculation of the peak load for a SDOF system. The actual loading on a multiple DOF system due to random input depends on the response of multiple modes, the mode shapes and the amount of effective mass participating in each mode. Static testing must still be done.

APPENDIX G

Excerpt from References 15 & 16

NASA engineers performed experimental static and vibration testing an “AEPI fiberglass pedestal” structure.

The following conclusions were made:

1. Strain, in general, is lower during random testing than during an equivalent static loading as predicted by Miles’ equation.
2. The Miles’ equation equivalent static loading clearly develops stresses an order of magnitude above those created by the random environments.

The reference also noted:

> A study completed in 1993 by the Marshall Space Flight Center (MSFC) Random Loads/Criteria Issues Team concluded, after an extensive literature search, that almost no analytical or empirical documentation exists on the subject of the relationship between random limit load (stress) and static limit load (stress). The consensus of the team was that it is a complex subject and requires a carefully planned effort to produce and effective, yet practical solution.

APPENDIX H

Two-degree-of-Freedom-System, Example 1

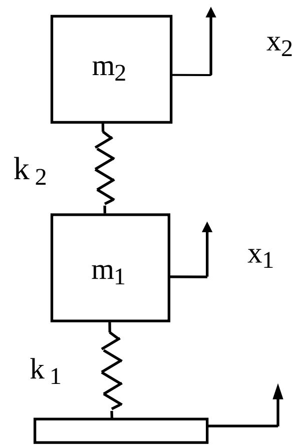


Figure H-1.

The variables are

y	is the base displacement
x_i	is the absolute displacement of mass i
z_i	is the relative displacement of mass i
ξ_j	is the modal damping for mode j
ω_j	is the natural frequency (rad/sec) for mode j
ω	is the base excitation frequency
A	is the base acceleration amplitude
k_i	is the stiffness for spring i
m_i	is mass i

The mass-normalized eigenvectors in column format are

$$\begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix}$$

The equation of motion from Reference 17 is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{H-1}$$

Define participation factors Γ_1 and Γ_2.

$$\Gamma_1 = \hat{q}_{11}m_1 + \hat{q}_{21}m_2 \tag{H-2}$$

$$\Gamma_2 = \hat{q}_{12}m_1 + \hat{q}_{22}m_2 \tag{H-3}$$

The relative displacement transfer functions are

$$\hat{Z}_1(f)/A = \frac{-\hat{q}_{11}\Gamma_1}{[\omega_1^2 - \omega^2] + j\,2\xi_1\,\omega_1\,\omega} + \frac{-\hat{q}_{12}\Gamma_2}{[\omega_2^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega} \tag{H-4}$$

$$\hat{Z}_2(f)/A = \frac{-\hat{q}_{21}\Gamma_1}{[\omega_1^2 - \omega^2] + j\,2\xi_1\,\omega_1\,\omega} + \frac{-\hat{q}_{22}\Gamma_2}{[\omega_2^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega} \tag{H-5}$$

<u>Example</u>

Consider the system in Figure H-1. Assign the following values.

Table H-1. Parameters	
Variable	Value
m_1	20.0 lbm
m_2	10.0 lbm
k_1	60,000 lbf/in
k_2	40,000 lbf/in

Furthermore, assume that each mode has a damping value of 5%.

The following parameters were calculated for the sample system via a Matlab script.

```
Natural Frequencies =

     126.2 Hz
     268.5 Hz

  Modes Shapes (column format) =

       -2.823          -3.366
        -4.76           3.993

Participation Factors =

   -0.2696
  -0.07097

 Effective Modal Mass =

     28.06   lbm
     1.944   lbm

 Total Modal Mass =      30.0000 lbm
```

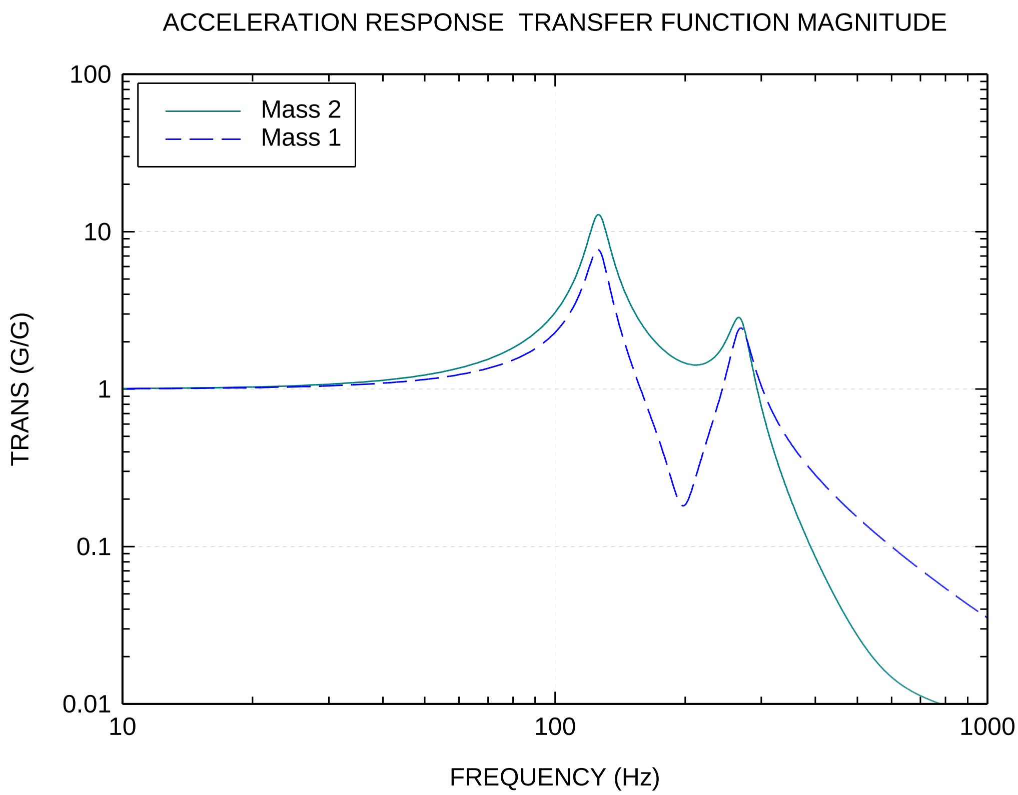


Figure H-1.

The resulting transfer functions were also calculated via a Matlab script.

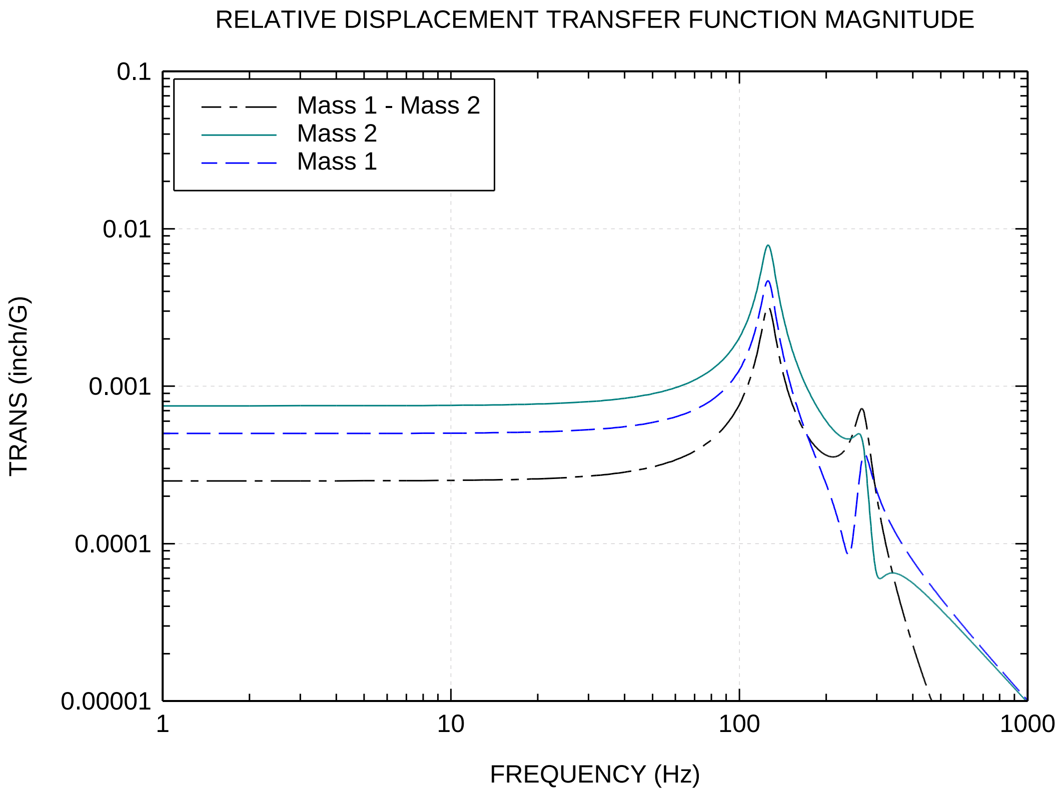


Figure H-2.

Table H-2. Magnitude Values at Fundamental Mode, Frequency = 126.2 Hz		
Magnitude Parameter	Mass 1	Mass 2
Acceleration Response (G/G)	7.7	12.8
Relative Displacement (inch/G)	0.0047	0.0079

Note that:

$$\text{Relative Displacement} = \text{Acceleration Response} / \omega_1^{\,2}$$

This equation is true for the transfer function values, at least for a two-degree-of-freedom system with well-separated modal frequencies. It will have some error, however, for the response acceleration and relative displacement overall RMS values, as shown later in this appendix.

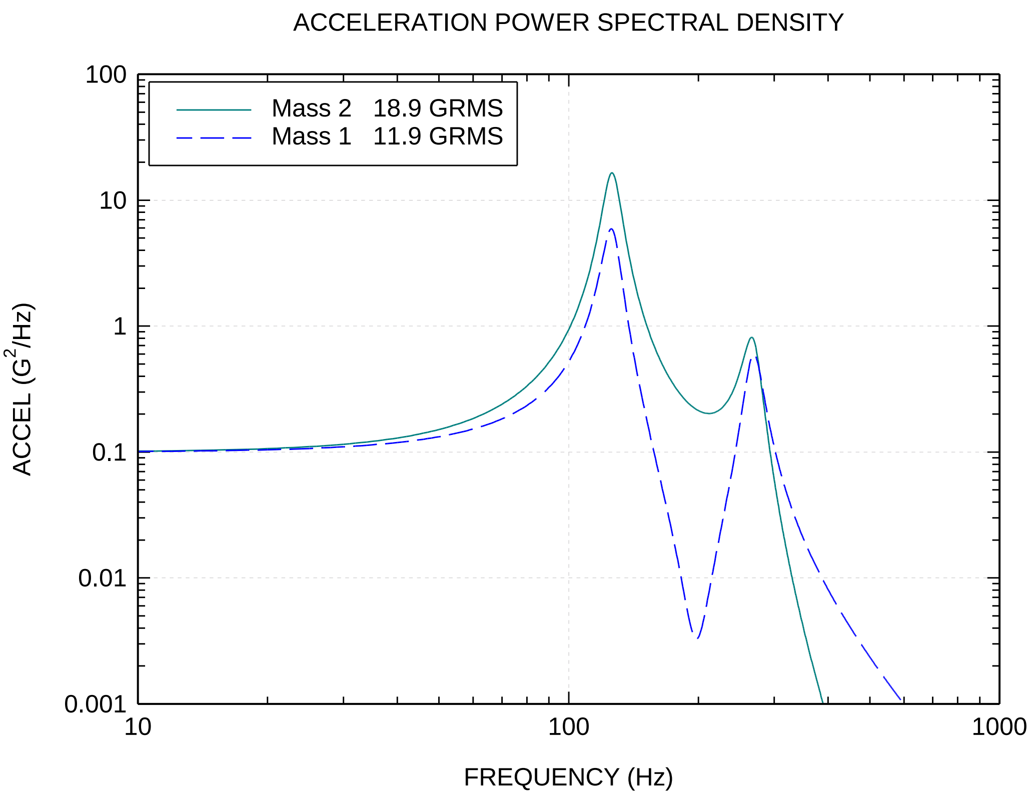


Figure H-3.

Table H-3. PSD, Base Input, 14.1 GRMS	
Freq (Hz)	Accel (G^2/Hz)
10	0.1
2000	0.1

The PSD in Table H-3 is applied to the system in Figure H-1. The acceleration response is shown in Figure H-3.

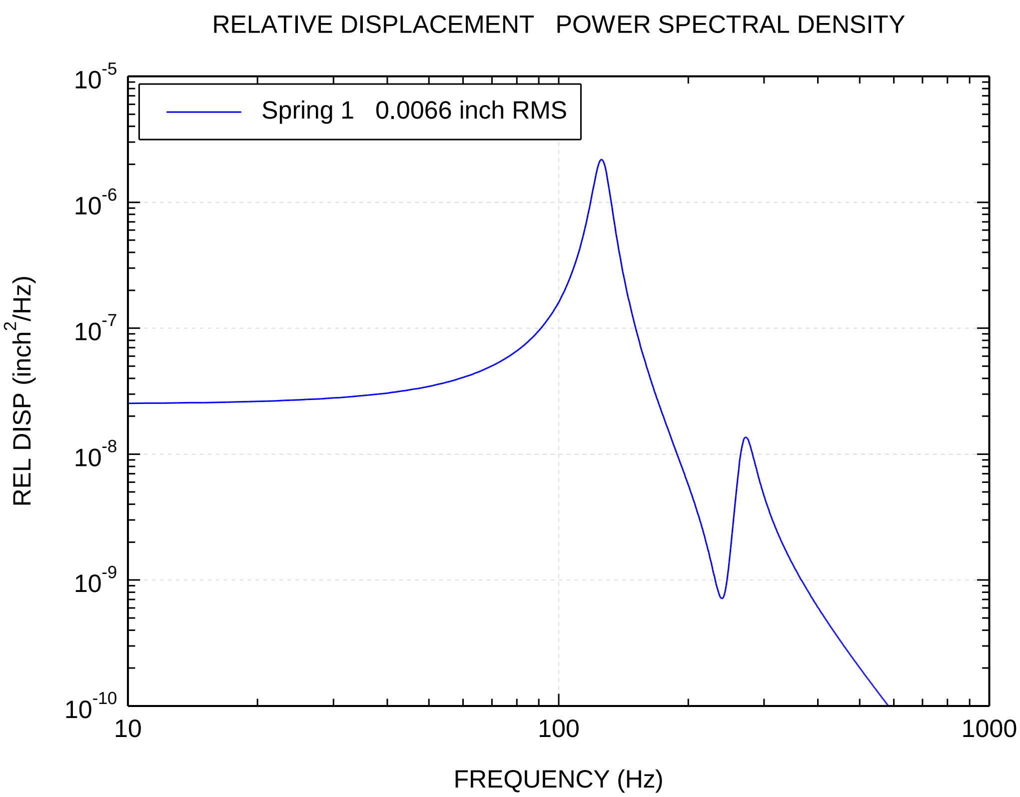


Figure H-4.

Table H-4. Overall Levels			
Element	Parameter	RMS Value	3σ Value
Mass 1	Acceleration Response	11.9 G	35.7 G
Spring 1	Relative Displacement	0.0066 inch	0.0198 inch

Note that Spring 1 is the base spring. The force transmitted through Spring 1 is 1188 lbf 3σ.

Also note that for the RMS values

$$\frac{11.9\text{ G}\left[\left(386\,\text{in}/\text{sec}^2\right)/\text{G}\right]}{\left[2\pi\left(126.2\text{ Hz}\right)\right]^2} = 0.0073\text{ inch} > 0.0066\text{ inch}$$

The reason for this difference is that the second modal peak has a higher contribution in the acceleration transfer function than in the relative displacement transfer function. This difference is amplified when each of the transfer functions is squared for the PSD response calculations.

Equivalent SDOF System

How would the two-degree-of-freedom system be modeled as an SDOF system?

A conservative approach would be to taken the highest acceleration response from Figure H-3 and apply it to the total mass.

The maximum response acceleration was 18.9 GRMS, or 56.7 G 3σ.

The combined mass is 30 lbm.

The force transmitted via the base spring to the boundary interface is thus 1701 lbf 3σ.

The base spring has a stiffness of 60,000 lbf/in

Thus the base spring relative displacement would be 0.0284 inch 3σ.

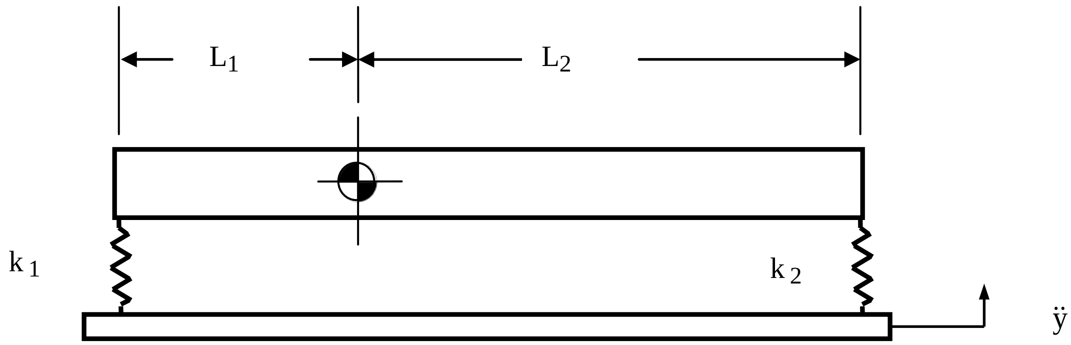
Table H-5. Spring 1 Results, Model Comparison

Model	Transmitted Force (lbf 3σ)	Relative Displacement (inch 3σ)
SDOF	1701	0.0284
2-DOF	1188	0.0198

The SDOF result is 3.1 dB higher for each parameter.

APPENDIX I

<u>Two-degree-of-Freedom-System, Example 2</u>



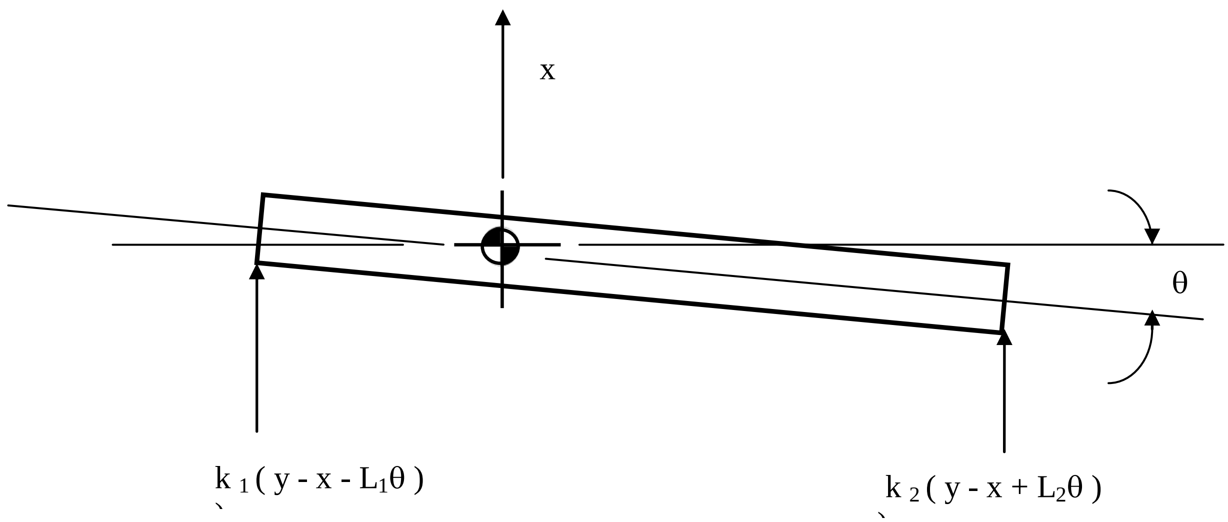


Figure I-1.

The system has a CG offset if $L_1 \neq L_2$.

The system is statically coupled if $k_1 L_1 \neq k_2 L_2$.

The rotation is positive in the clockwise direction.

The variables are

y	is the base displacement
x	is the translation of the CG
θ	is the rotation about the CG
m	is the mass
J	is the polar mass moment of inertia
k_i	is the stiffness for spring i
z_i	is the relative displacement for spring i
ξ_j	is the modal damping for mode j
ω_j	is the natural frequency (rad/sec) for mode j
ω	is the base excitation frequency
A	is the base acceleration amplitude

The mass-normalized eigenvectors in column format are

$$\begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix}$$

The equation of motion are taken from Reference 17.

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} k_1+k_2 & k_1L_1-k_2L_2 \\ k_1L_1-k_2L_2 & k_1L_1^2+k_2L_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix}=\begin{bmatrix} k_1+k_2 \\ k_1L_1-k_2L_2 \end{bmatrix}\left[\,y\,\right] \tag{I-1}$$

The relative displacement for spring 1 is

$$z_1 = z + L_1\theta \tag{I-2}$$

The Fourier transform for the relative displacement in spring 1 is

$$Z_1 = m\ A\left\{\hat{q}_{11}\frac{\hat{q}_{11}+L_1\hat{q}_{21}}{[\omega_1{}^2-\omega^2]+j\,2\xi_1\,\omega_1\,\omega}+\hat{q}_{21}\frac{\hat{q}_{12}+L_1\hat{q}_{22}}{[\omega_2{}^2-\omega^2]+j\,2\xi_2\,\omega_2\,\omega}\right\} \qquad \text{(I-3)}$$

The relative displacement for spring 2 is

$$z_2 = z - L_2\theta \qquad \text{(I-4)}$$

The Fourier transform for the relative displacement in spring 2 is

$$Z_2 = m\ A\left\{\hat{q}_{11}\frac{\hat{q}_{11}-L_2\,\hat{q}_{21}}{[\omega_1{}^2-\omega^2]+j\,2\xi_1\,\omega_1\,\omega}+\hat{q}_{21}\frac{\hat{q}_{12}-L_2\hat{q}_{22}}{[\omega_2{}^2-\omega^2]+j\,2\xi_2\,\omega_2\,\omega}\right\} \qquad \text{(I-5)}$$

The Fourier transform for the translational acceleration at the CG is

$$X_a = A\left\{1+m\,\omega^2\left\{\frac{\hat{q}_{11}{}^2}{[\omega_1{}^2-\omega^2]+j\,2\xi_1\,\omega_1\,\omega}+\frac{\hat{q}_{12}\,\hat{q}_{21}}{[\omega_2{}^2-\omega^2]+j\,2\xi_2\,\omega_2\,\omega}\right\}\right\} \qquad \text{(I-6)}$$

The Fourier transform for the rotational acceleration is

$$\Theta_1 = m\ A\left\{\frac{\hat{q}_{11}\,\hat{q}_{21}}{[\omega_1{}^2-\omega^2]+j\,2\xi_1\,\omega_1\,\omega}+\frac{\hat{q}_{22}\,\hat{q}_{21}}{[\omega_2{}^2-\omega^2]+j\,2\xi_2\,\omega_2\,\omega}\right\} \qquad \text{(I-7)}$$

Example

Consider the system in Figure I-1. Assign the following values. The values are based on a slender rod, aluminum, diameter =1 inch, total length=24 inch.

Table H-1. Parameters	
Variable	Value
m	18.9 lbm
J	907 lbm in^2
k_1	20,000 lbf/in
k_2	20,000 lbf/in
L_1	8 in
L_2	16 in

Furthermore, assume that each mode has a damping value of 5%.

The following parameters were calculated for the sample system via a Matlab script.

```
The mass matrix is

m =

    0.0490         0
         0    2.3497


 The stiffness matrix is

k =

       40000      160000
      160000     6400000


  Natural Frequencies =

     133.8 Hz
     267.9 Hz

  Modes Shapes (column format) =

         -4.4           1.029
       0.1486          0.6352

 Participation Factors =

    0.1336
     1.543
```

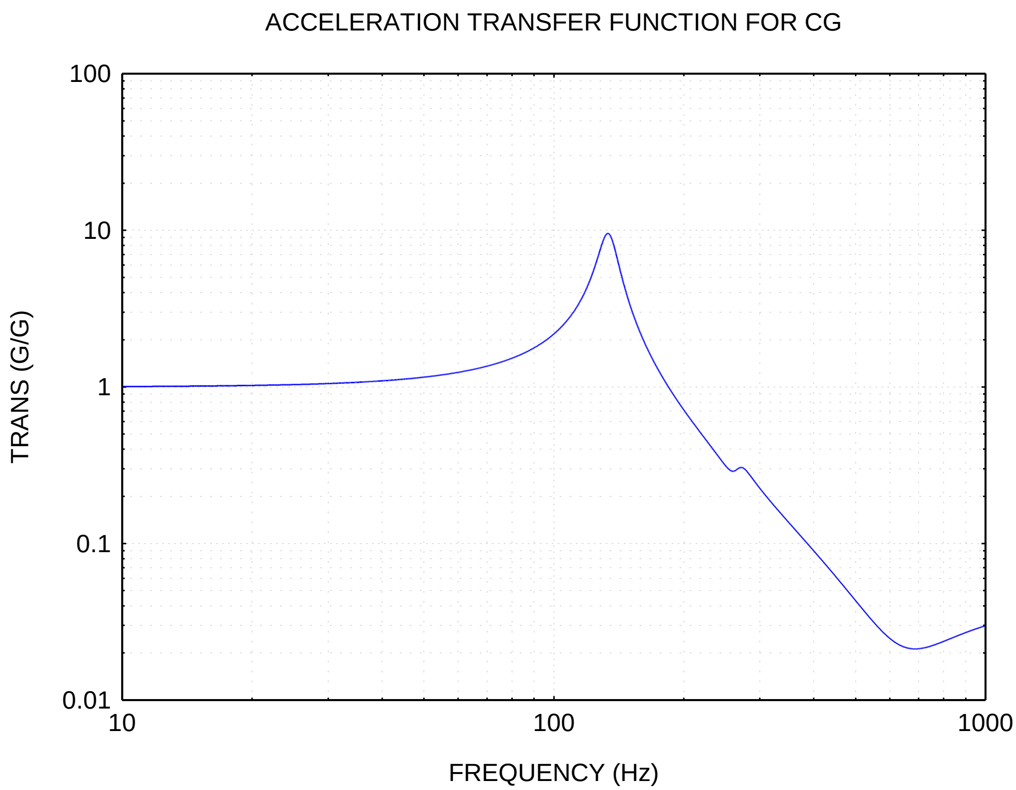

ACCELERATION TRANSFER FUNCTION FOR CG
TRANS (G/G)
100
10
1
0.1
0.01
10
100
1000
FREQUENCY (Hz)

Figure I-2.

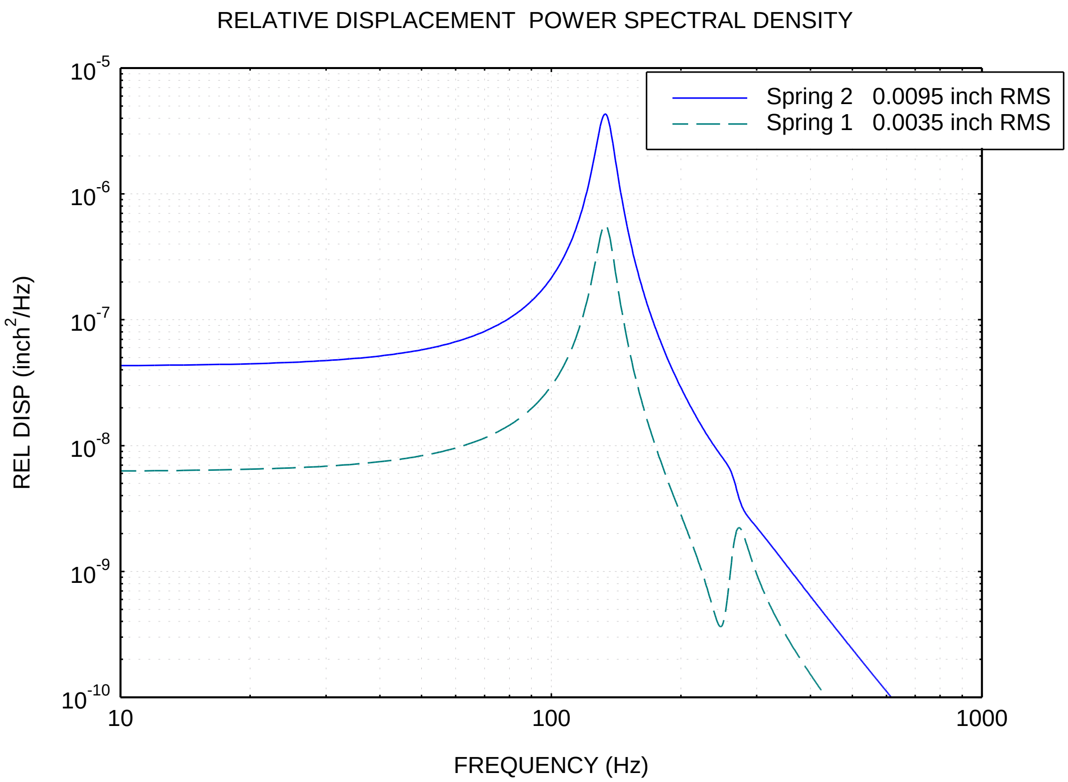

RELATIVE DISPLACEMENT POWER SPECTRAL DENSITY
Spring 2 0.0095 inch RMS
Spring 1 0.0035 inch RMS
REL DISP (inch^2/Hz)
FREQUENCY (Hz)
10^-5
10^-6
10^-7
10^-8
10^-9
10^-10
10
100
1000

Figure I-3.

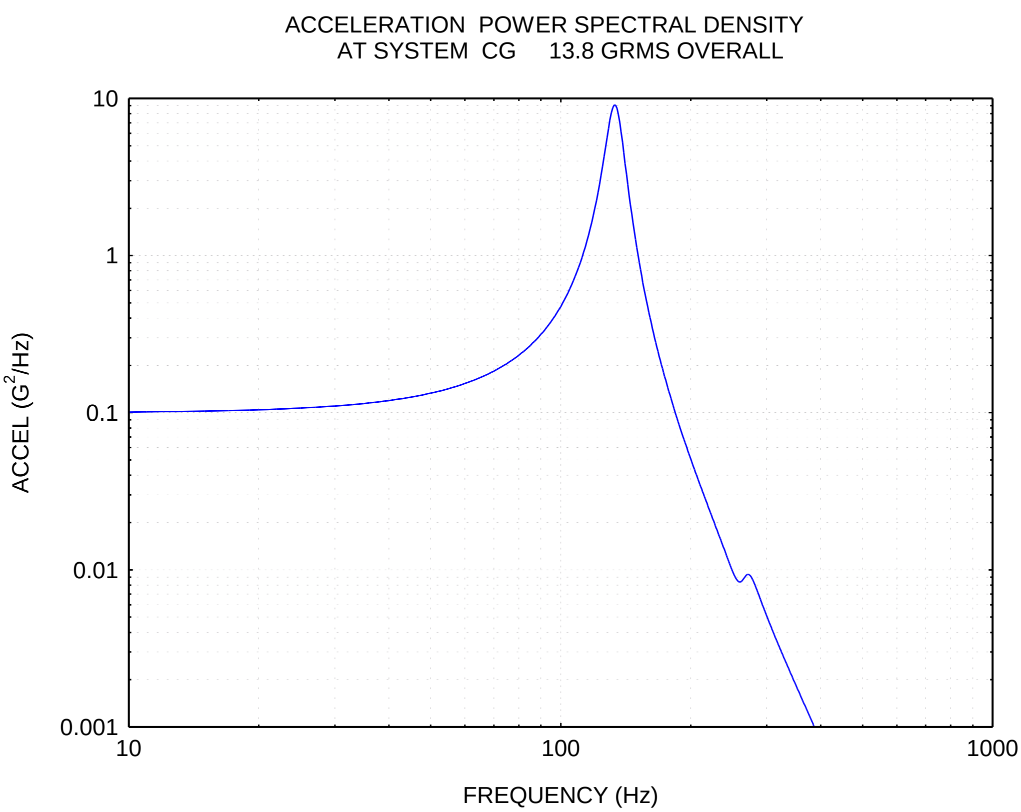


Figure I-4.

Table I-2. PSD, Base Input, 14.1 GRMS	
Freq (Hz)	Accel (G^2/Hz)
10	0.1
2000	0.1

The PSD in Table I-2 is applied to the system in Figure I-1. The acceleration response is shown in Figure I-4.

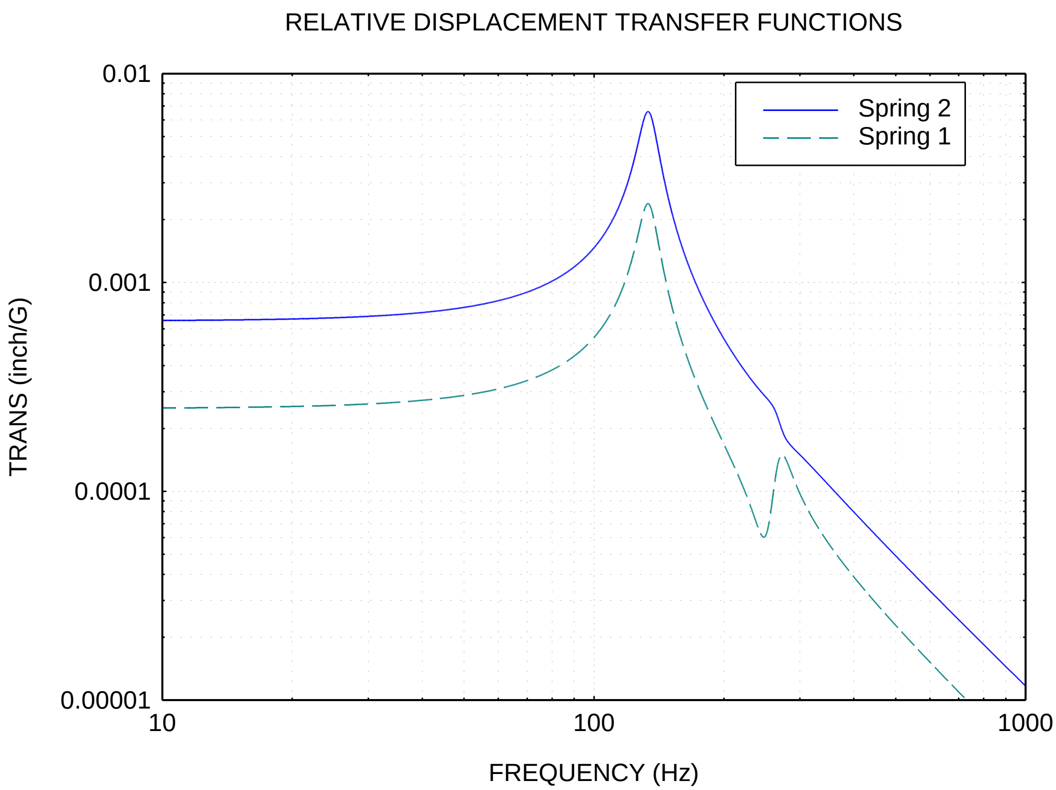


Figure I-5.

Table I-3. Overall Levels, Two-degree-of-freedom Model			
Element	Parameter	RMS Value	3σ Value
Mass	Acceleration Response	13.8 G	41.4 G
Spring 1	Relative Displacement	0.0035 inch	0.0105 inch
Spring 2	Relative Displacement	0.0095 inch	0.0285 inch

The results in Table I-3 show an error that would occur if the system had been modeled as an SDOF system with translation only. The spring displacement in this case would have been 0.0065 inches. This would be 3.3 dB less that the Spring 1 displacement in Table I-3.

APPENDIX J

<u>Cantilever Beam</u>

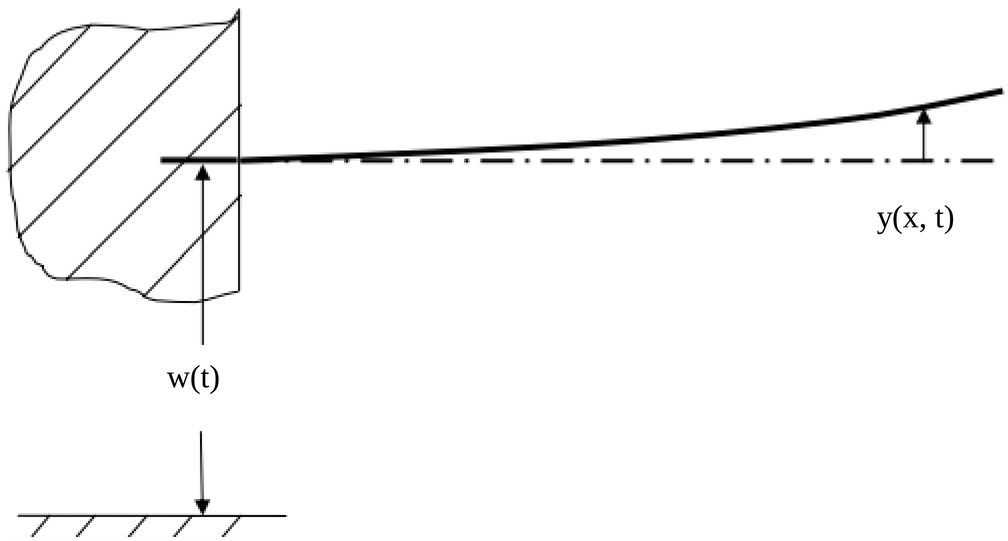


Figure J-1.

The following equations are taken from Reference 20.

The relative displacement response $Y(x,\omega)$ to base acceleration is

$$Y(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega \omega_n} \right\} \tag{J-1}$$

$$\frac{\partial}{\partial x} Y(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n \frac{d}{dx} Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega \omega_n} \right\} \tag{J-2}$$

$$\frac{\partial^2}{\partial x^2} Y(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n \frac{d^2}{dx^2} Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega\omega_n} \right\} \tag{J-3}$$

The bending moment is

$$M(x,\omega) = EI \frac{\partial^2}{\partial x^2} Y(x,\omega) \tag{J-4}$$

The bending stress is

$$\sigma = \frac{Mc}{I} \tag{J-5}$$

$$\sigma(x,\omega) = E\,c\,\frac{\partial^2}{\partial x^2} Y(x,\omega) \tag{J-6}$$

$$\sigma(x,\omega) = E\,c\,\ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n \frac{d^2}{dx^2} Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega\omega_n} \right\} \tag{J-7}$$

$$\sigma(x,f) = E\,c\,\left(2\pi\ \ddot{W}(f)\right)\left(\frac{1}{4\pi^2}\right) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n \frac{d^2}{dx^2} Y_n(x)}{\left(f_n^2 - f^2\right) + j2\xi_n\, f\, f_n} \right\} \tag{J-8}$$

$$\sigma(x,f) = \left(\frac{1}{2\pi}\right) E\,c\,\ddot{W}(f) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n \frac{d^2}{dx^2} Y_n(x)}{\left(f_n^2 - f^2\right) + j2\xi_n\, f\, f_n} \right\} \tag{J-9}$$

The bending stress transfer function is

$$H_s(x,f) = \frac{\sigma(x,f)}{\ddot{W}(f)} = \left(\frac{1}{2\pi}\right) E\,c \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n \frac{d^2}{dx^2} Y_n(x)}{\left(f_n^2 - f^2\right) + j2\xi_n\, f\, f_n} \right\} \tag{J-10}$$

The eigenvalue roots for the cantilever beam are

Table J-1. Roots	
Index n	$\beta_n L$
1	1.875104
2	4.694091
3	7.854575
4	10.99554

The first mode shape and its derivatives are

$$Y_1(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_1 x) - \cos(\beta_1 x)\right] - 0.73410\left[\sinh(\beta_1 x) - \sin(\beta_1 x)\right]\right\} \tag{J-11}$$

$$\frac{d}{dx} Y_1(x) = \left\{\frac{\beta_1}{\sqrt{\rho L}}\right\}\left\{\left[\sinh(\beta_1 x) + \sin(\beta_1 x)\right] - 0.73410\left[\cosh(\beta_1 x) - \cos(\beta_1 x)\right]\right\} \tag{J-12}$$

$$\frac{d^2}{dx^2} Y_1(x) = \left\{\frac{\beta_1^{\,2}}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_1 x) + \cos(\beta_1 x)\right] - 0.73410\left[\sinh(\beta_1 x) + \sin(\beta_1 x)\right]\right\} \tag{J-13}$$

The second mode shape and its derivatives are

$$Y_2(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\{[\cosh(\beta_2 x) - \cos(\beta_2 x)] - 1.01847\,[\sinh(\beta_2 x) - \sin(\beta_2 x)]\} \quad \text{(J-14)}$$

$$\frac{d}{dx}Y_2(x) = \left\{\frac{\beta_2}{\sqrt{\rho L}}\right\}\{[\sinh(\beta_2 x) + \sin(\beta_2 x)] - 1.01847\,[\cosh(\beta_2 x) - \cos(\beta_2 x)]\} \quad \text{(J-15)}$$

$$\frac{d^2}{dx^2}Y_2(x) = \left\{\frac{\beta_2{}^2}{\sqrt{\rho L}}\right\}\{[\cosh(\beta_2 x) + \cos(\beta_2 x)] - 1.01847\,[\sinh(\beta_2 x) + \sin(\beta_2 x)]\} \quad \text{(J-16)}$$

The third mode shape and its derivatives are

$$Y_3(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\{[\cosh(\beta_3 x) - \cos(\beta_3 x)] - 0.99922\,[\sinh(\beta_3 x) - \sin(\beta_3 x)]\} \quad \text{(J-17)}$$

$$\frac{d}{dx}Y_3(x) = \left\{\frac{\beta_3}{\sqrt{\rho L}}\right\}\{[\sinh(\beta_3 x) + \sin(\beta_3 x)] - 0.99922\,[\cosh(\beta_3 x) - \cos(\beta_3 x)]\} \quad \text{(J-18)}$$

$$\frac{d^2}{dx^2}Y_3(x) = \left\{\frac{\beta_3{}^2}{\sqrt{\rho L}}\right\}\{[\cosh(\beta_3 x) + \cos(\beta_3 x)] - 0.99922\,[\sinh(\beta_3 x) + \sin(\beta_3 x)]\} \quad \text{(J-19)}$$

The fourth mode shape and its derivatives are

$$Y_4(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\{[\cosh(\beta_4 x) - \cos(\beta_4 x)] - 1.00003\,[\sinh(\beta_4 x) - \sin(\beta_4 x)]\} \quad \text{(J-20)}$$

$$\frac{d}{dx}Y_4(x) = \left\{\frac{\beta_4}{\sqrt{\rho L}}\right\}\{[\sinh(\beta_4 x) + \sin(\beta_4 x)] - 1.00003\,[\cosh(\beta_4 x) - \cos(\beta_4 x)]\} \quad \text{(J-21)}$$

$$\frac{d^2}{dx^2}Y_4(x) = \left\{\frac{\beta_4{}^2}{\sqrt{\rho L}}\right\}\{[\cosh(\beta_4 x) + \cos(\beta_4 x)] - 1.00003\,[\sinh(\beta_4 x) + \sin(\beta_4 x)]\} \quad \text{(J-22)}$$

Example

Consider a beam with the following properties:

Cross-Section	Circular
Boundary Conditions	Fixed-Free
Material	Aluminum

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	24 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.471 lbm (1.22E-03 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via Matlab script: continuous_base_base_accel.m. The normal modes results are:

Table J-2. Natural Frequency Results, Cantilever Beam

Mode	fn (Hz)	Participation Factor	Effective Modal Mass (lbf sec^2/in)	Effective Modal Mass (lbm)
1	23.86	0.02736	0.000748	0.289
2	149.53	0.01516	0.00023	0.089
3	418.69	0.00889	7.90E-05	0.031
4	820.47	0.00635	4.04E-05	0.016

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

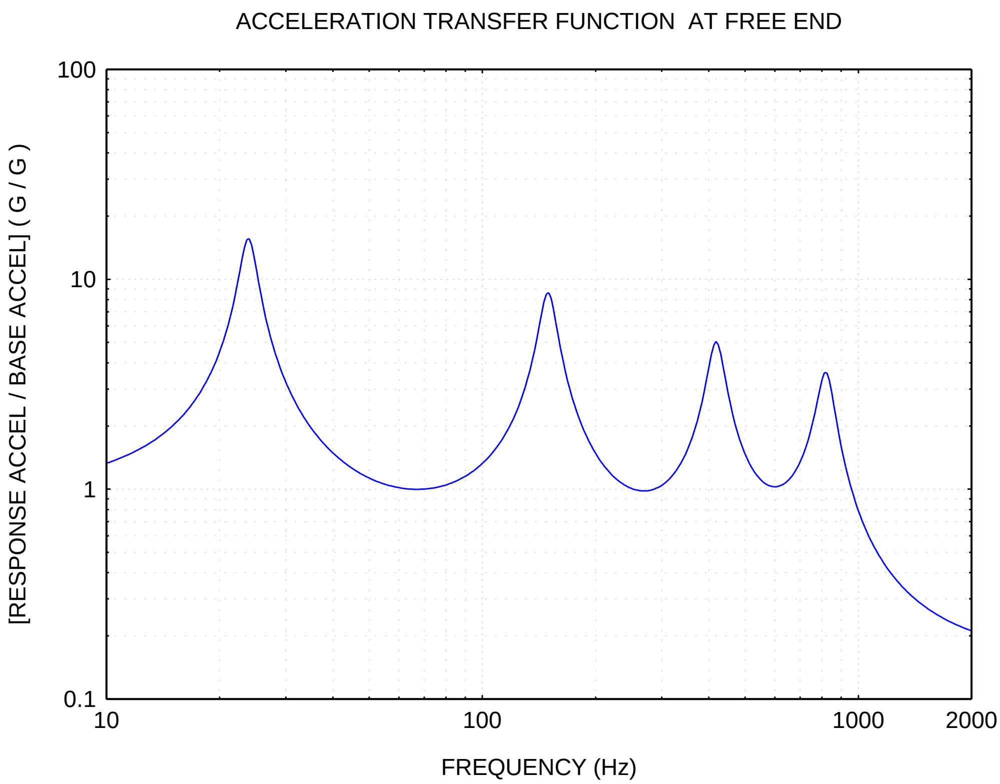

ACCELERATION TRANSFER FUNCTION AT FREE END
100
10
1
0.1
[RESPONSE ACCEL / BASE ACCEL] (G / G)
10
100
1000
2000
FREQUENCY (Hz)

Figure J-2.

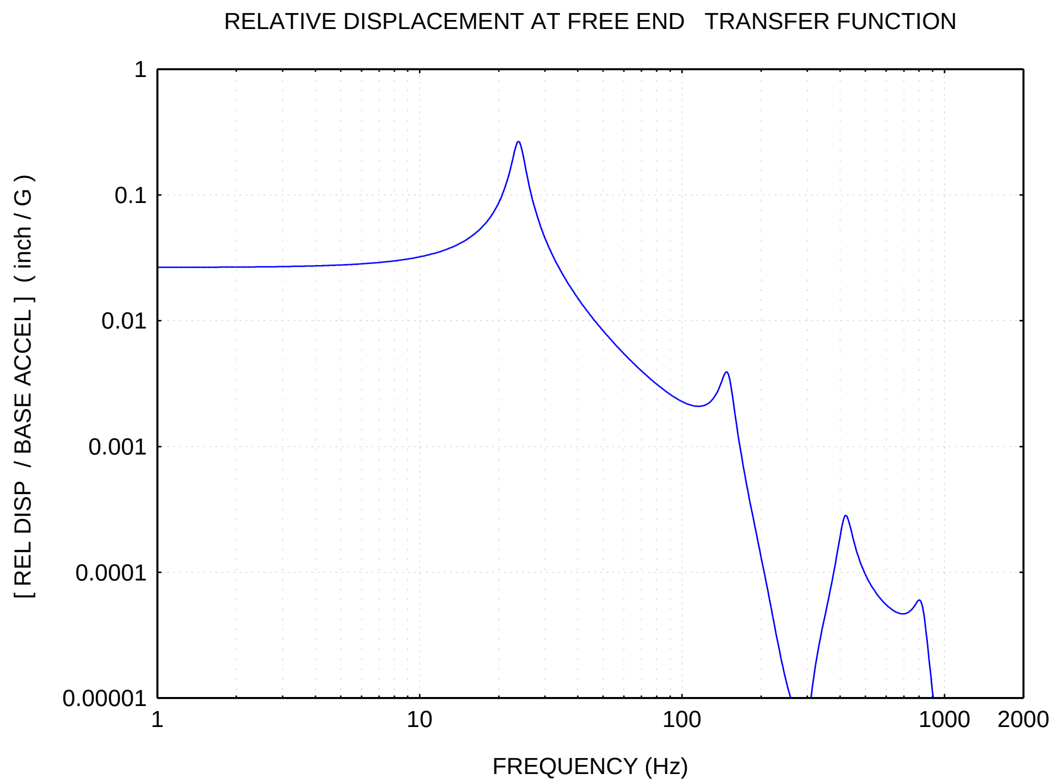

RELATIVE DISPLACEMENT AT FREE END   TRANSFER FUNCTION
[REL DISP / BASE ACCEL] (inch / G)
1
0.1
0.01
0.001
0.0001
0.00001
1
10
100
1000
2000
FREQUENCY (Hz)

Figure J-3.

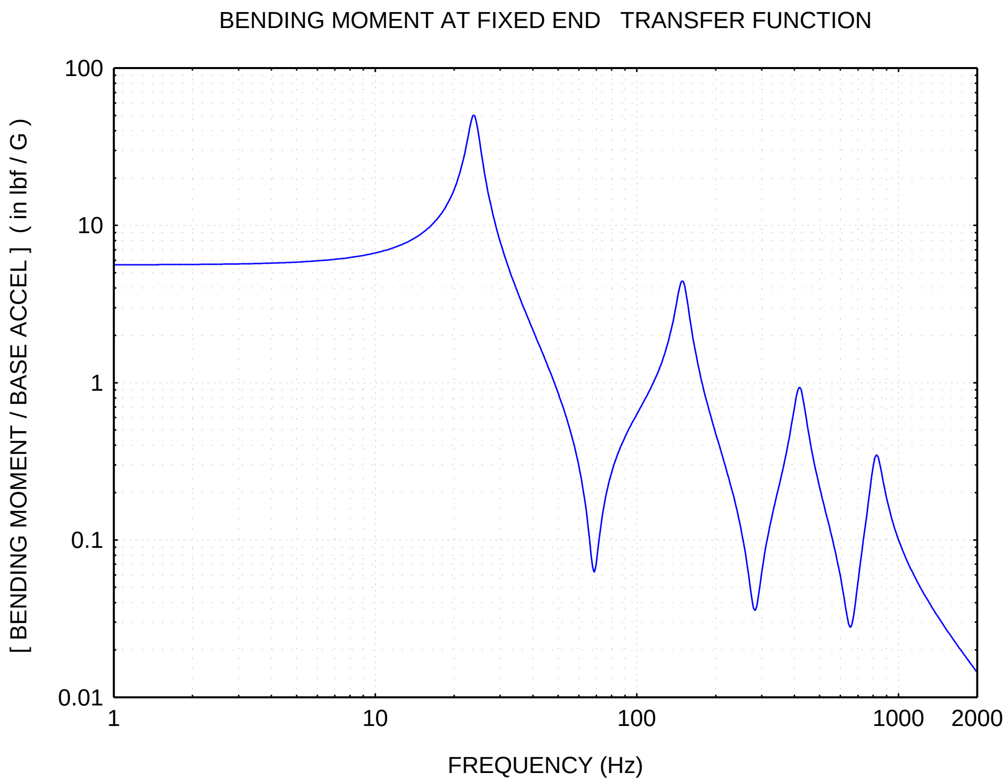

BENDING MOMENT AT FIXED END   TRANSFER FUNCTION
[BENDING MOMENT / BASE ACCEL] (in lbf / G)
100
10
1
0.1
0.01
1
10
100
1000
2000
FREQUENCY (Hz)

Figure J-4.

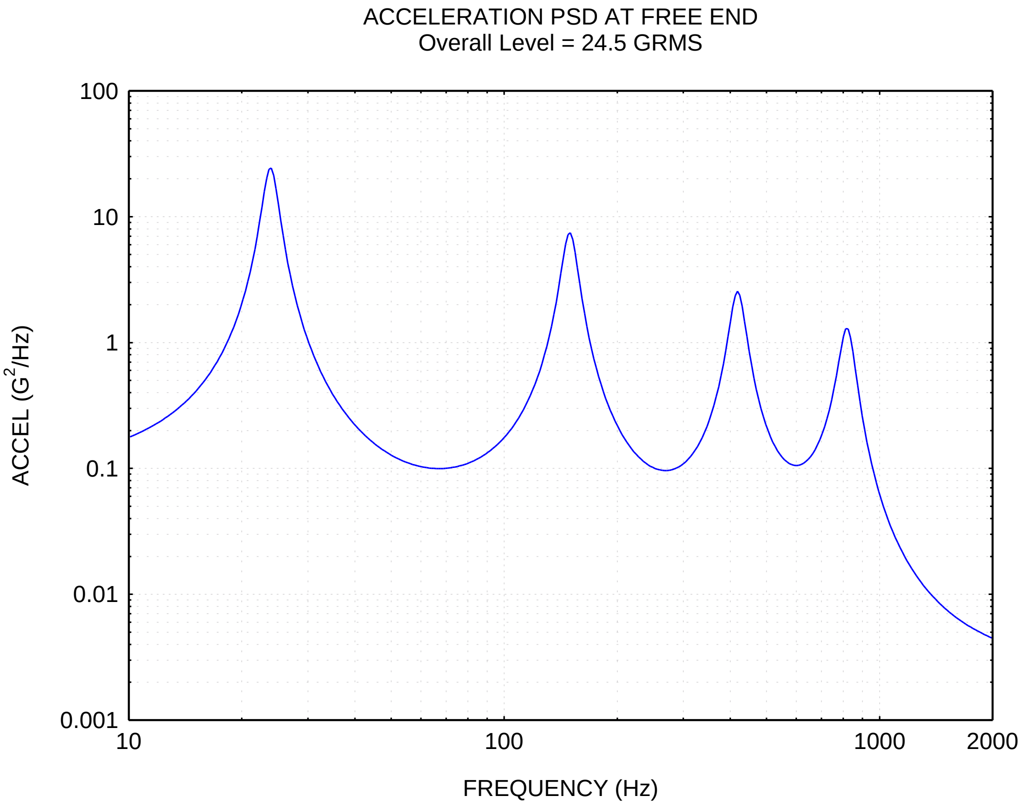


Figure J-5.

Table J-3. PSD, Base Input, 14.1 GRMS	
Freq (Hz)	Accel (G^2/Hz)
10	0.1
2000	0.1

The cantilever beam is subjected to the base input PSD in Table J-3. The resulting response PSD curves are shown in Figures J-5 through J-7 for selected parameters and locations.

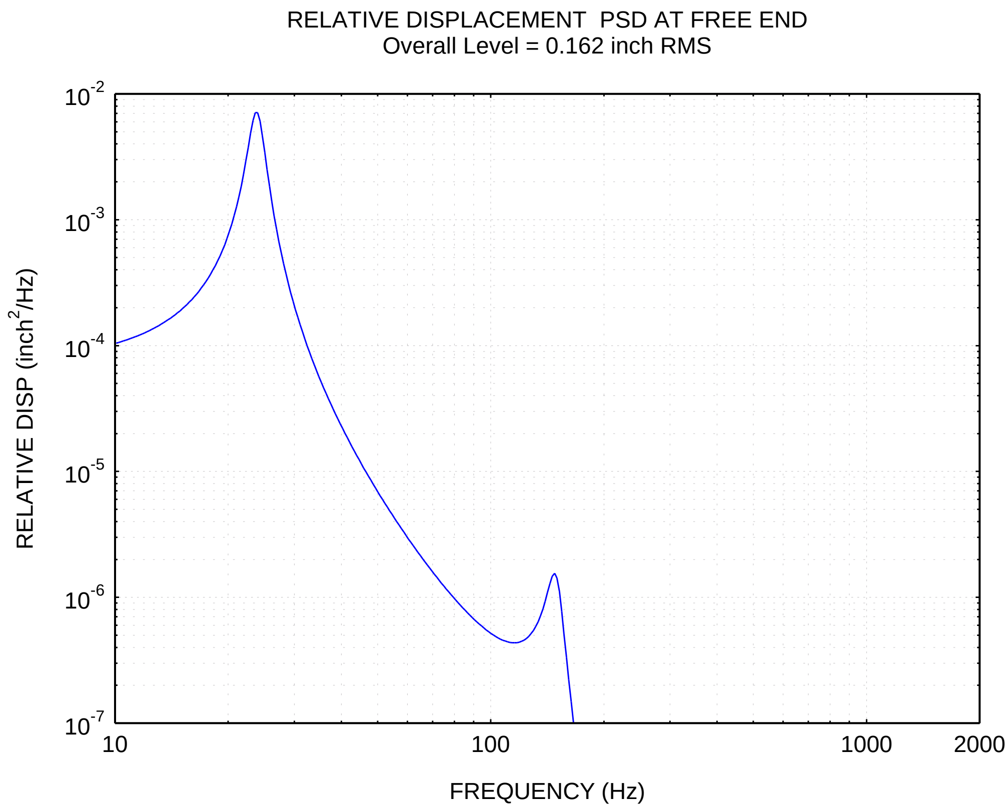

RELATIVE DISPLACEMENT PSD AT FREE END
Overall Level = 0.162 inch RMS
RELATIVE DISP (inch^2/Hz)
FREQUENCY (Hz)
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10
100
1000
2000

Figure J-6.

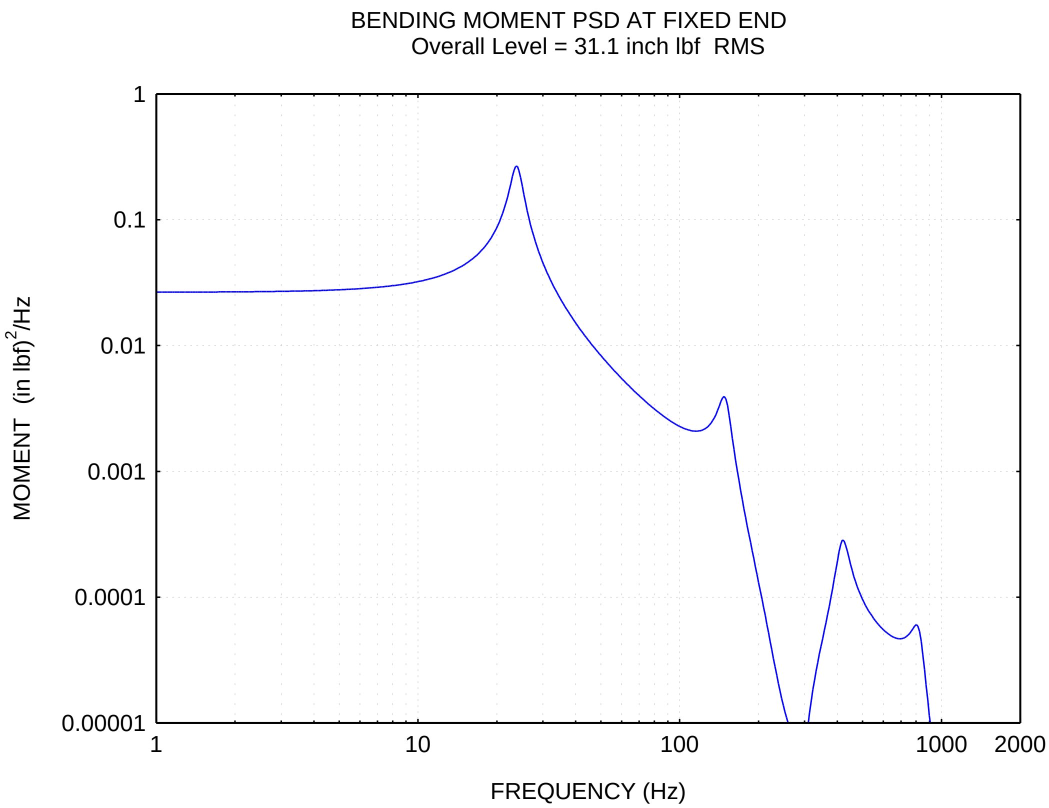


Figure J-7.

<u>Stress Calculation</u>

Ignore stress concentration factors. Neglect shear stress.

The stress results are given in Table J-4.

Table J-4. Stress and Strain Results at Fixed End		
Parameter	RMS	3σ
Bending Moment (in lbf)	31.1	93.3
Bending stress (psi)	2551	7653
micro Strain	255	765

The bending stress is the peak fiber stress for the cross-section at the fixed interface.

Equivalent Static Load

There are three candidate methods. The load will be applied as a uniformly distributed load along the beam in each case.

Method 1

This method is very conservative. Take the beam mass multiplied the acceleration response in Figure J-5.

$$(0.471 \text{ lbm })(24.5 \text{ GRMS }) = 11.5 \text{ lbf RMS} \tag{J-23}$$

The bending moment M_a at the fixed boundary for a uniform load is calculated using a formula from Reference 21.

$$M = \frac{\hat{w} L^2}{2} \tag{J-24a}$$

$$W = \hat{w} L \tag{J-24b}$$

$$M = \frac{W L}{2} \tag{J-24c}$$

$$M_a = (11.5 \text{ lbf RMS })(24 \text{ inch} / 2) = 138.5 \text{ inch lbf RMS} \tag{J-24d}$$

Method 2

The second method is similar to the first except that only the fundamental mode is considered to cause stress.

The overall response for the curve in Figure J-5 is 9.6 GRMS for the domain from 10 to 60 Hz. This response level can be approximately considered as that of the fundamental mode only.

Furthermore, the load will be calculated using the effective modal mass for the first mode.

$$(0.289 \text{ lbm })(9.6 \text{ GRMS }) = 2.77 \text{ lbf RMS} \tag{J-25}$$

The bending moment at the fixed boundary is

$$M_a = (2.77 \text{ lbf RMS })(24 \text{ inch} / 2) = 33.3 \text{ inch lbf RMS} \tag{J-26}$$

Method 3

The third method finds an equivalent load so that the static and dynamic relative displacements match at the free end. The dynamic relative displacement is taken from Figure J-6.

Let Y be the relative displacement. The distributed load W per Reference 21 is

$$W = \frac{8\,EI\,Y}{L^4} = \frac{8(30680\ \text{lbf in^2})(0.162\ \text{in RMS})}{(24\,\text{in})^4} = 0.120\ (\text{lbf / in})\ \text{RMS} \qquad \text{(J-27)}$$

The corresponding bending moment at the fixed boundary is

$$M_a = [\ 0.120\ (\text{lbf / in})\ \text{RMS}]\ [\ 24\ \text{inch}]^2 / 2 = 34.5\ \text{inch lbf RMS} \qquad \text{(J-28)}$$

Summary

Table J-5. Results Comparison RMS Values, Fixed End

Parameter	Modes Included	Bending Moment (in lbf)	Bending Stress (lbf/in^2)	micro Strain
Static Method 1, Accel	4	138.5	11,284	1128
Static Method 2, Accel	1	33.3	2713	271
Static Method 3, Relative Disp.	4	34.5	2811	281
Dynamic Analysis	4	31.1	2551	255

Methods 2 and 3 agree reasonably well with the dynamic results for each of the respective parameters.

APPENDIX K

Beam Simply-Supported at Each End

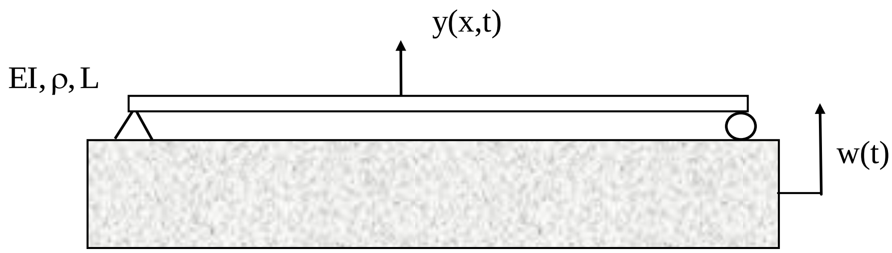


Figure K-1.

The mode shapes are

$$Y_n(x) = \sqrt{\frac{2}{\rho L}}\,\sin(\beta_n x) \tag{K-1}$$

$$\frac{d^2}{dx^2} Y_n(x) = -\beta_n{}^2 \sqrt{\frac{2}{\rho L}}\,\sin(\beta_n x) \tag{K-2}$$

The eigenvalues are

$$\beta_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \tag{K-3}$$

The natural frequencies are

$$\omega_n = \beta_n{}^2 \sqrt{EI/\rho} \tag{K-4}$$

The relative displacement response Y(x, ω) to base acceleration is

$$Y(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega\omega_n} \right\} \tag{K-5}$$

Example

Use the same beam from Appendix J except change the boundary conditions to simply-supported at each end.

The normal modes and frequency response function analysis are performed via Matlab script: continuous_base_base_accel.m. The normal modes results are:

Table K-1.
Natural Frequency Results, Beam Simply-Supported at Each End

Mode	fn (Hz)	Participation Factor	Effective Modal Mass (lbf sec^2/in)	Effective Modal Mass (lbm)
1	66.97	0.0315	0.9896e-03	0.3820
2	267.9	0	0	0
3	603.8	0.0105	0.1100e-03	0.0424
4	1072	0	0	0

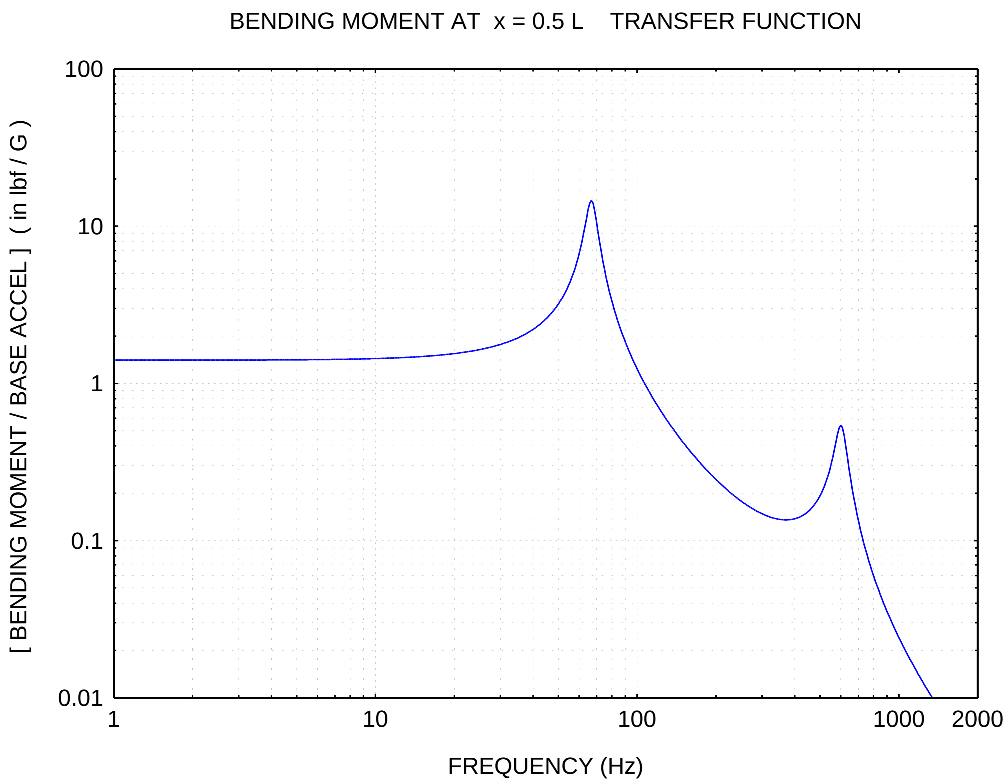

BENDING MOMENT AT x = 0.5 L TRANSFER FUNCTION
[BENDING MOMENT / BASE ACCEL] (in lbf / G)
100
10
1
0.1
0.01
1
10
100
1000
2000
FREQUENCY (Hz)

Figure K-2.

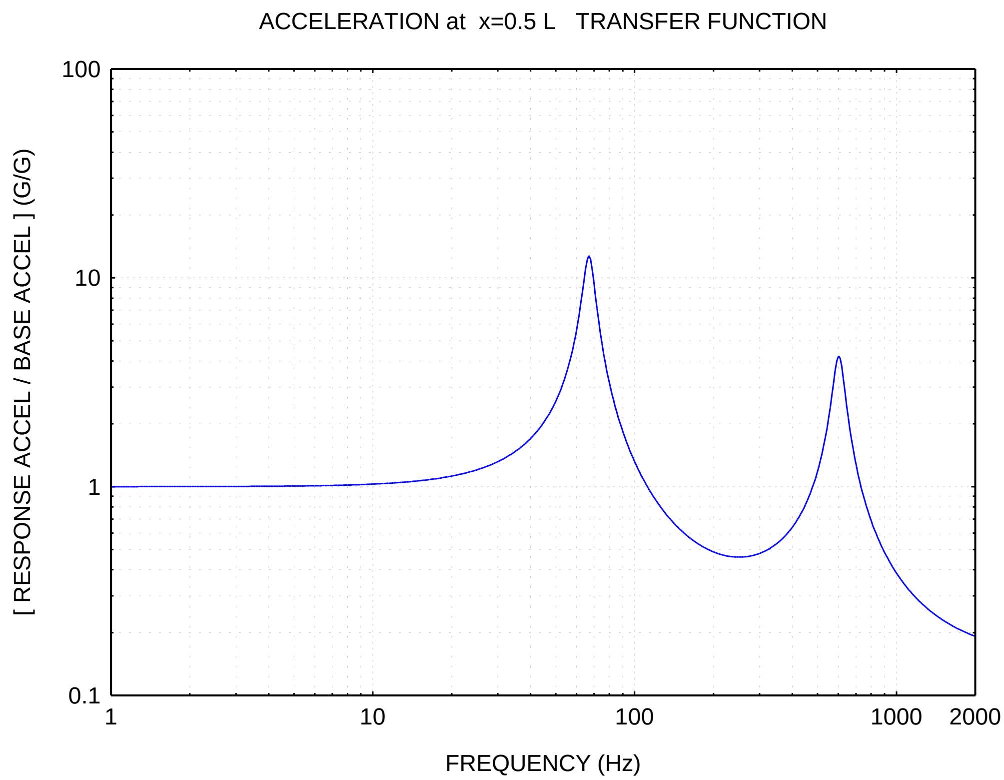

ACCELERATION at x=0.5 L TRANSFER FUNCTION
[RESPONSE ACCEL / BASE ACCEL] (G/G)
100
10
1
0.1
1
10
100
1000
2000
FREQUENCY (Hz)

Figure K-3.

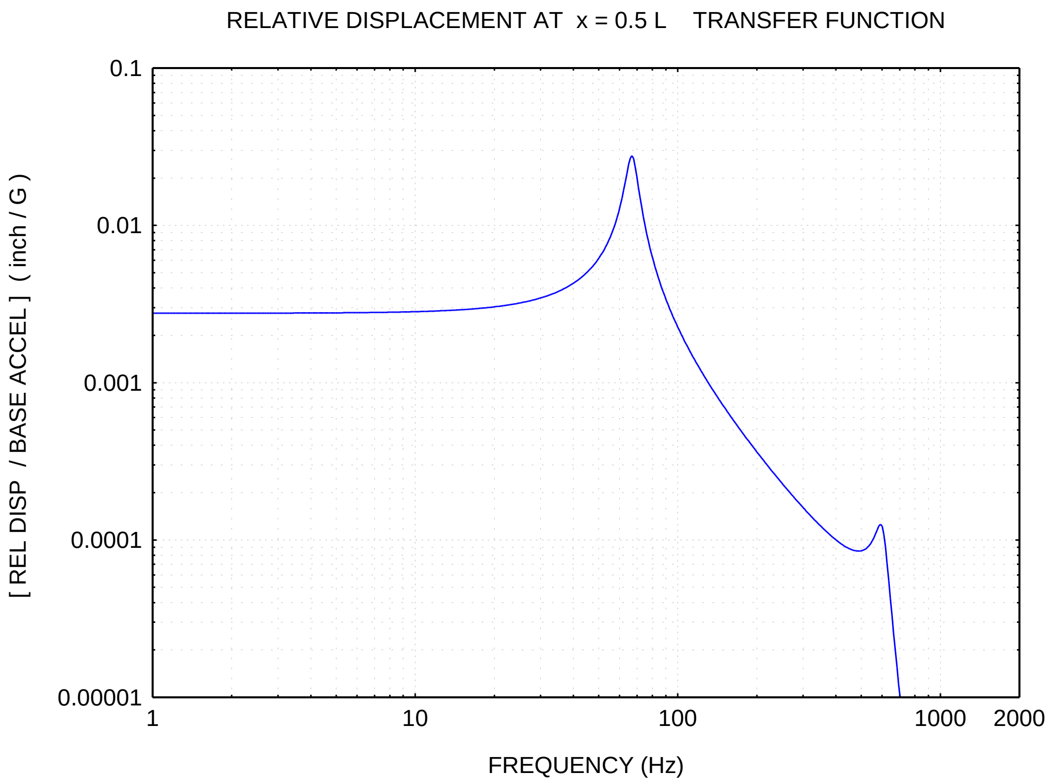

RELATIVE DISPLACEMENT AT x = 0.5 L TRANSFER FUNCTION
[REL DISP / BASE ACCEL] (inch / G)
0.1
0.01
0.001
0.0001
0.00001
1
10
100
1000
2000
FREQUENCY (Hz)

Figure K-4.

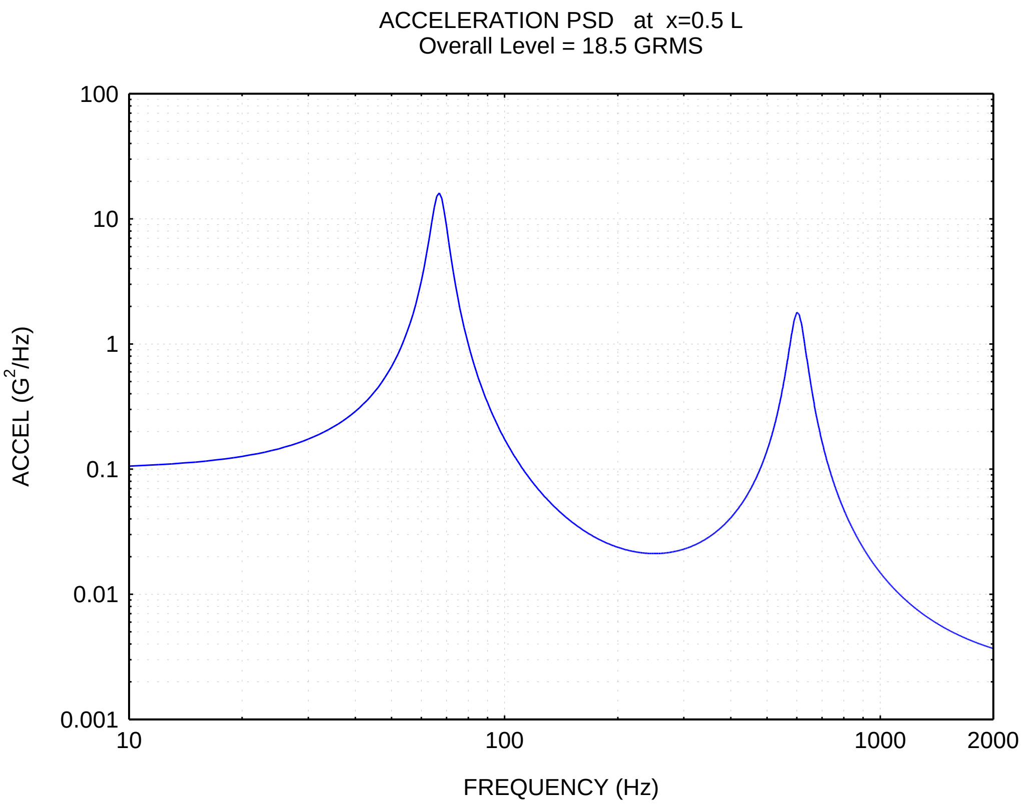


Figure K-5.

Table K-2. PSD, Base Input, 14.1 GRMS	
Freq (Hz)	Accel (G^2/Hz)
10	0.1
2000	0.1

The simply-supported beam is subjected to the base input PSD in Table K-3. The resulting response PSD curves are shown in Figures K-5 through K-7 for selected parameters and locations.

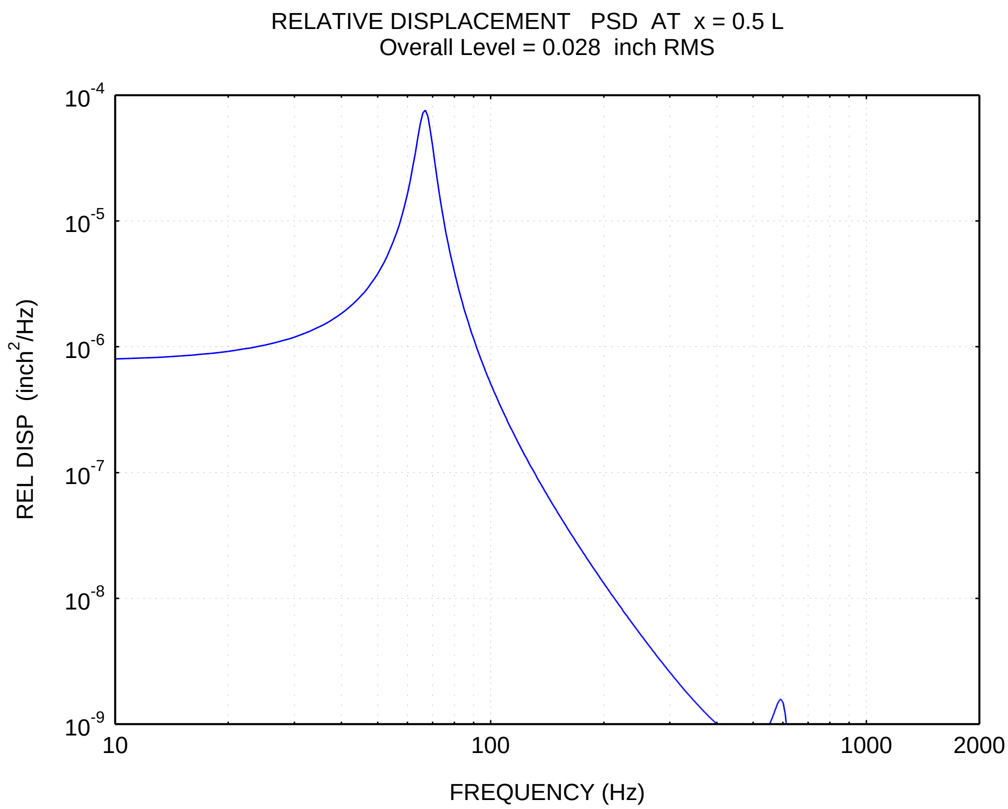

RELATIVE DISPLACEMENT PSD AT x = 0.5 L
Overall Level = 0.028 inch RMS
REL DISP (inch^2/Hz)
10^-4
10^-5
10^-6
10^-7
10^-8
10^-9
10
100
1000
2000
FREQUENCY (Hz)

Figure K-6.

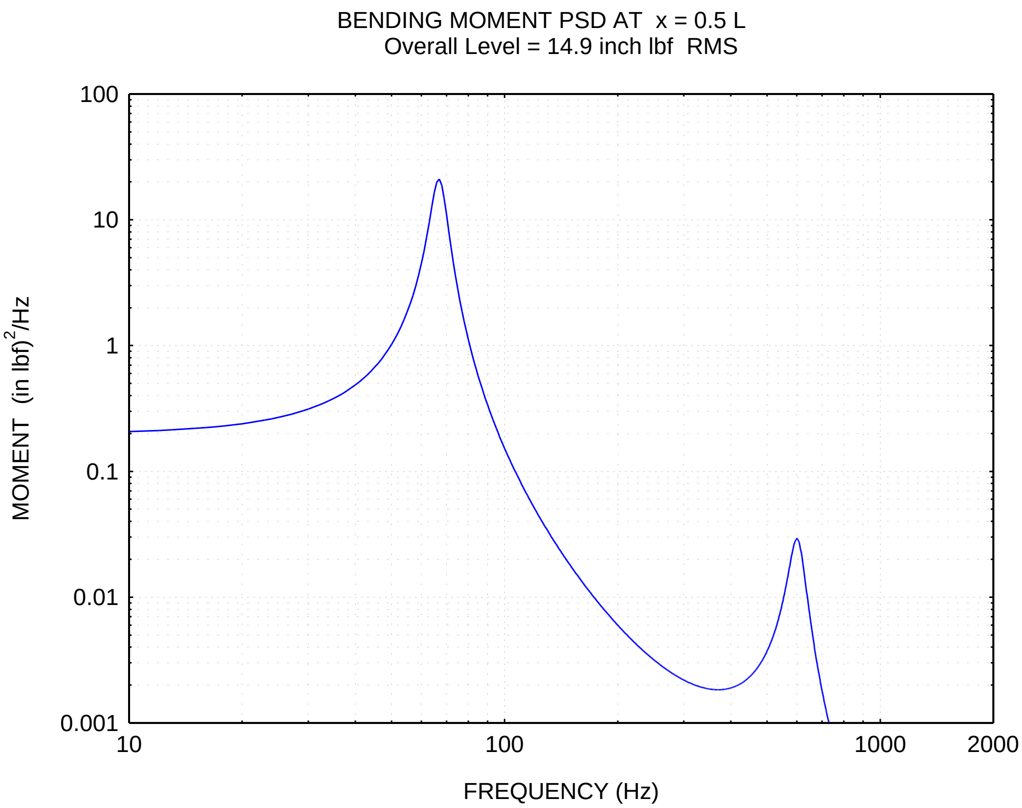

BENDING MOMENT PSD AT x = 0.5 L
Overall Level = 14.9 inch lbf RMS
100
10
1
0.1
0.01
0.001
MOMENT (in lbf)^2/Hz
10
100
1000
2000
FREQUENCY (Hz)

Figure K-7.

Stress Calculation

Ignore stress concentration factors. Neglect shear stress.

The stress results are given in Table K-3.

Table K-3. Stress and Strain Results at x = 0.5 L		
Parameter	RMS	3σ
Bending Moment (in lbf)	14.9	44.7
Bending stress (psi)	1214	3642
micro Strain	121.4	364

Equivalent Static Load

There are three candidate methods. The load will be applied as a uniformly distributed load along the beam in each case.

Method 1

This method is very conservative. Take the beam mass multiplied the acceleration response in Figure K-5.

$$(\ 0.471 \text{ lbm })(\ 18.5 \text{ GRMS }) = 8.7 \text{ lbf RMS} \tag{K-6}$$

The bending moment M at the midpoint for a uniform load is calculated using a formula from Reference 21.

$$M = \frac{\hat{w} L^2}{8} \tag{K-7a}$$

$$W = \hat{w} L \tag{K-7b}$$

$$M = \frac{W L}{8} \tag{K-7c}$$

$$M = \left(\frac{1}{8}\right)\left[8.7 \text{ lbf RMS}\right]\left(24\,\text{inch}\right) \quad = 26.1 \text{ inch lbf RMS} \tag{K-7d}$$

Method 2

The second method is similar to the first except that only the fundamental mode is considered to cause stress.

The overall response for the curve in Figure K-5 is 13.0 GRMS for the domain from 10 to 200 Hz. This response level can be approximately considered as that of the fundamental mode only.

Furthermore, the load will be calculated using the effective modal mass for the first mode.

$$(0.382 \text{ lbm})(13.0 \text{ GRMS}) = 5.0 \text{ lbf RMS} \tag{K-8}$$

The bending moment for a uniform load at the midpoint is

$$M = \frac{W\,L}{8} = \left(\frac{1}{8}\right)\left[\ 5.0 \text{ RMS}\right]\left(24\,\text{inch}\right) \quad = \ 15 \ \text{ inch lbf RMS} \tag{K-9}$$

Method 3

The third method finds an equivalent load so that the static and dynamic relative displacements match at the free end. The dynamic relative displacement is taken from Figure K-6.

Let Y be the relative displacement. The distributed load $\hat{w}$ per Reference 21 is

$$\hat{w} = \frac{384\,EI\,Y}{5L^4} = \frac{384\left(30680 \text{ lbf in\^{}2}\right)\left(0.028 \ \text{ in RMS}\right)}{5\,(24\,\text{in})^4} = 0.199 \ \text{(lbf / in) RMS} \tag{K-10}$$

The corresponding bending moment at the midpoint is

$$M = \frac{\hat{w}\,L^2}{8} = \left(\frac{1}{8}\right)\left[0.199 \ \text{(lbf / in) RMS}\right]\left(24\,\text{inch}\right)^2 \tag{K-11a}$$

$$M = 14.3 \quad \text{in lbf} \tag{K-11b}$$

Summary

Table K-4. Results Comparison RMS Values, at x = 0.5 L				
Parameter	Modes Included	Bending Moment (in lbf)	Bending Stress (lbf/in^2)	micro Strain
Static Method 1, Accel	4	26.1	2130	213.0
Static Method 2, Accel	1	15.0	1222	122.2
Static Method 3, Relative Disp.	4	14.3	1165	116.5
Dynamic Analysis	4	14.9	1214	121.4

Again, Methods 2 and 3 agree reasonably well with the dynamic results for each of the respective parameters.

APPENDIX L

Material Stress Limits

The following is an excerpt from Reference (23) with some minor editing.

> A material can sometimes sustain an important dynamic load without damage, whereas the same load, statically, would lead to plastic deformation or to failure. Many materials subjected to short duration loads have ultimate strengths higher than those observed when they are static.
>
> Hopkinson noted that copper and steel wire can withstand stresses that are higher than their static elastic limit and are well beyond the static ultimate limit without separating proportionality between the stresses and the strains. This is provided that the length of time during which the stress exceeds the yield stress is of the order of 1 millisecond or less.
>
> From tests carried out on steel (annealed steel with a low percentage of carbon) it was noted that the initiation of plastic deformation requires a definite time when stresses greater than the yield stress are applied. It was observed that this time can vary between 5 milliseconds (under a stress of approximately 352 MPa) and 6 seconds with approximately 255 MPa; with the static yield stress being equal to 214 MPa). Other tests carried out on five other materials showed that this delay exists only for materials for which the curve of static stress deformation presents a definite yield stress, and the plastic deformation then occurs for the load period.

(End of Excerpt)

The equivalent units are as follows:

Table L-1. Annealed Steel Test Results

Parameter	Stress (MPa)	Stress (ksi)
5 msec for plastic deformation onset	352	51.1
6 sec for plastic deformation onset	255	37.0
Static Yield Stress	214	31.1

Dynamic Strength

Reference 26 notes:

> As far as steels and other metals are concerned, those with lower yield strength are usually more ductile than higher strength materials. That is, high yield strength materials tend to be brittle. Ductile (lower yield strength) materials are better able to withstand rapid dynamic loading than brittle (high yield strength) materials. Interestingly, during repeated dynamic loadings low yield strength ductile materials tend to increase their yield strength, whereas high yield strength brittle materials tend to fracture and shatter under rapid loading.

Reference 26 includes the following table where the data was obtained for uniaxial testing using an impact method.

Dynamic Strengthening of Materials

Material	Static Strength (psi)	Dynamic Strength (psi)	Impact Speed (ft/sec)
2024 Al (annealed)	65,200	68,600	>200
Magnesium Alloy	43,800	51,400	>200
Annealed Copper	29,900	36,700	>200
302 Stainless Steel	93,300	110,800	>200
SAE 4140 Steel	134,800	151,000	175
SAE 4130 Steel	80,000	440,000	235
Brass	39,000	310,000	216