NOTES ON RANDOM VIBRATION FATIGUE

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The following method is intended so that one random vibration test level can be compared with another random level, or even with a shock or sine vibration specification. This approaches uses response acceleration rather than stress.

Let R be random vibration fatigue damage. Miner's accumulative fatigue for random vibration acceleration can be expressed as

$$R = \sum_{i=1}^{N} n_i \, a_i^{b} \tag{1}$$

where

N	=	Total number of cycles
n_i	=	Number of cycles at magnitude a_i
a_i	=	Peak acceleration of each cycle
b	=	Fatigue exponent from S-N curve

Note that R has a dimension of $[\text{length}/\text{time}^2]^b$.

Consider a single-degree-of-freedom system subjected to a random vibration base input. The system will tend to behave as a bandpass filter centered at its natural frequency. Thus assume that the response occurs at the natural frequency.

The cumulative fatigue formula can thus be expressed as an integral:

$$R \;=\; f_n T \int_0^{\infty} P(A)\, A^b \, dA \tag{2}$$

where

f_n	=	natural frequency
T	=	test duration
A	=	acceleration peak, absolute value
P(A)	=	Probability of the peak occurrence

The system's response peaks will tend to follow a Rayleigh distribution.

The probability density function P(A) of a Rayleigh distribution for a peak amplitude A is given by

$$P(A) = \frac{A}{\sigma^2} \exp\left\{ -\frac{1}{2} \left[\frac{A}{\sigma} \right]^2 \right\}, \quad A > 0 \tag{3}$$

where σ is the standard deviation.

Equation (3) is taken from Reference 1.

The cumulative fatigue formula becomes

$$R \;=\; f_n T \int_{0^+}^{\infty} \frac{A}{\sigma^2} \exp\left\{ -\frac{1}{2} \left[\frac{A}{\sigma} \right]^2 \right\} A^b \, dA \tag{4}$$

$$R \;=\; \frac{f_n T}{\sigma^2} \int_{0^+}^{\infty} \exp\left\{ -\frac{1}{2} \left[\frac{A}{\sigma} \right]^2 \right\} A^{b+1} \, dA \tag{5}$$

$$R \;=\; \frac{f_n T}{\sigma^2}\int_{0^+}^{\infty} \exp\left\{-\frac{1}{2}\left[\frac{A}{\sigma}\right]^2\right\} A^n \, dA \tag{6}$$

where n = b+1

The following relation is taken from Reference 2.

$$\int_0^{\infty} x^n \exp(-ax^2)dx \;=\; \frac{(n-1)!!}{2(2a)^{n/2}}\sqrt{\frac{\pi}{a}} \qquad \text{for } a > 0 \text{ and } n > 0 \tag{7}$$

Apply this relationship to equation (6).

$$a = \frac{1}{2\sigma^2} \tag{8}$$

The cumulative fatigue is thus

$$R \;=\; \frac{f_n T}{\sigma^2}\left[\frac{(n-1)!!}{2[2/(2\sigma^2)]^{n/2}} \sqrt{\frac{\pi}{1/(2\sigma^2)}}\right] \tag{9}$$

$$R \;=\; \frac{f_n T}{\sigma^2}\left[\frac{b!!}{2[2/(2\sigma^2)]^{(b+1)/2}} \sqrt{\frac{\pi}{1/(2\sigma^2)}}\right] \tag{10}$$

$$R \;=\; \frac{f_n T}{\sigma^2}\left[\frac{b!!}{2[1/\sigma^2]^{(b+1)/2}} \sqrt{2\sigma^2\pi}\right] \tag{11}$$

$$R = \frac{f_n T}{\sigma^2}\left[\frac{b!!}{2[1/\sigma]^{(b+1)}}\ \sigma\sqrt{2\pi}\right] \tag{12}$$

$$R = \frac{\sqrt{2\pi}\ f_n T}{\sigma}\left[\frac{b!!}{2[1/\sigma]^{(b+1)}}\right] \tag{13}$$

$$R = \frac{\sqrt{2\pi}\ f_n T}{2\sigma}\left[b!!\,\sigma^{(b+1)}\right] \tag{14}$$

The cumulative fatigue simplifies to

$$R = \sqrt{\frac{\pi}{2}}\ f_n T\left[b!!\,\sigma^{b}\right] \tag{15}$$

The b!! term is a double factorial. Further information regarding the double factorial is given in Appendix A.

The double factorials for two common fatigue exponents are

b	b!!
4	8
6.4	65.93

Note that MIL-STD-1540C requires the use of b = 4 in all fatigue equivalence calculations, but Steinberg recommends b = 6.4 in many cases as given in Reference 3.

References

1. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.
2. Zwillinger, Daniel, editor, CRC Standard Mathematical Tables and Formulae, 30th Edition, CRC Press, Inc., 1996.
3. Dave S. Steinberg, Vibration Analysis for Electronic Equipment, Second Edition, Wiley-Interscience, New York, 1988.
4. T. Irvine, Time-Scaling Equivalence Methods for Random Vibration Testing, Rev F, Vibrationdata, 2002.
5. T. Irvine, An Introduction to the Vibration Response Spectrum, Rev C, Vibrationdata, 2000.

APPENDIX A

The double factorial is defined in Reference 1 for a positive, integer n is as

$$n!! = n(n-2)(n-4)\ .\ .\ . \tag{A-1}$$

The double factorial can be expressed as

$$n!! = \frac{n!(n-2)!(n-4)!\ldots}{(n-1)!(n-3)!(n-5)!\ldots} \tag{A-2}$$

A challenge is that equation (A-1) requires an integer value for n. The fatigue exponent, however, may be a non-integer, real number.

Note the relationship between the Gamma function and the factorial.

$$\Gamma(n+1) = n! \tag{A-3}$$

The double factorial formula becomes

$$n!! = \frac{\Gamma(n+1)\Gamma(n-1)!\Gamma(n-3)\ldots}{\Gamma(n)\Gamma(n-2)\Gamma(n-4)\ldots} \tag{A-4}$$

Equation (A-4) allows n to be a non-integer.

APPENDIX B

Example 1

The following example shows how the fatigue formula can be used for time-scaling.

Again the cumulative fatigue formula is

$$R = \sqrt{\frac{\pi}{2}}\, f_n T \left[b!!\, \sigma^b \right] \tag{B-1}$$

The natural frequency is not required for this example.

A time-scaling equation can be derived as follows

$$T_1\, \sigma_1{}^b = T_2\, \sigma_2{}^b \tag{B-2}$$

$$\frac{\sigma_2}{\sigma_1} = \left[\frac{T_1}{T_2} \right]^{1/b} \tag{B-3}$$

Now assume that the exponent is b = 4.

Assume that a component is to be tested for 16 hours at a given level. The goal is to reduce the test duration to 1 hour. By how much must the level be increased?

The new level at 1 hour must be twice the level at 16 hours per equation (B-3). This is equivalent to 6 dB.

Further information regarding time scaling is given in Reference 4.

APPENDIX C

Example 2

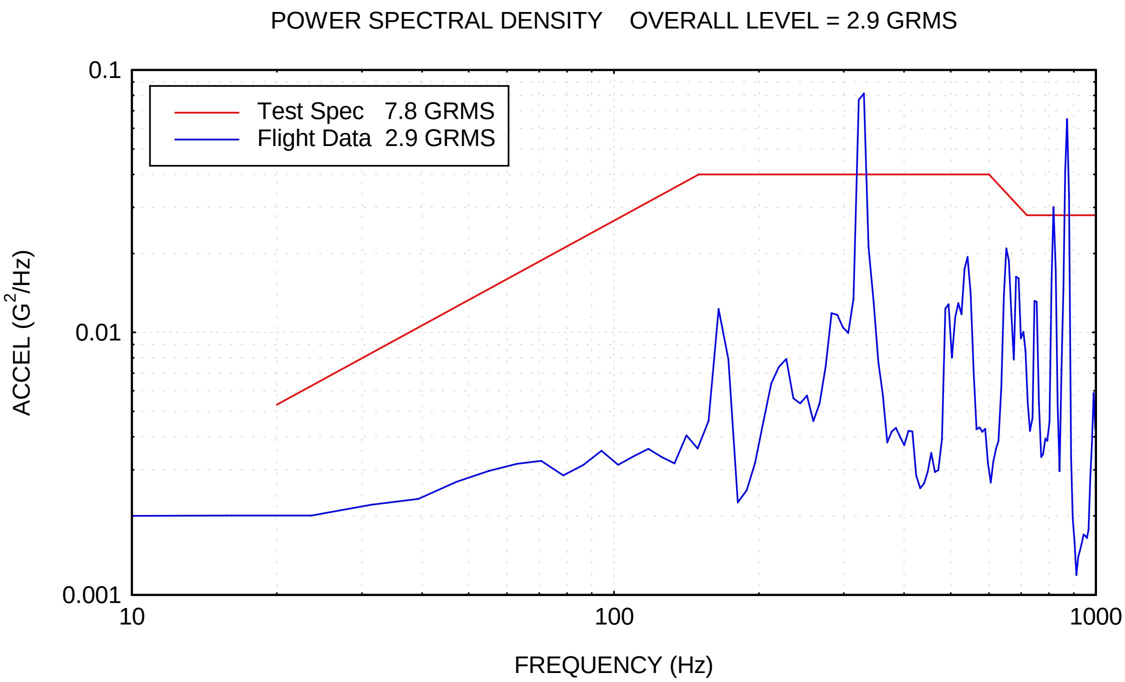

Figure C-1.

The flight data in Figure C-1 is actually flight accelerometer data. The test spec is also the actual specification for component mounted nearby the accelerometer.

The flight data exceeded the test specification at certain frequencies in terms of the respective base input levels. The test specification is still more severe as shown in the following example.

Compare the level in terms of fatigue damage.

Assume that

1. The flight data duration is 30 seconds
2. The flight data had a normal distribution
3. The test specification had a duration of 60 seconds
4. An avionics component is mounted adjacent to the accelerometer location
5. The component has a natural frequency of 325 Hz with an amplification value of Q=10.
6. The fatigue exponent is 4

The natural frequency is deliberately set at 325 Hz because the flight data has a large excursion over the test specification at this frequency.

Again the cumulative fatigue formula is

$$R = \sqrt{\frac{\pi}{2}}\, f_n T \left[b!!\, \sigma^b \right] \tag{C-1}$$

The response parameters for (fn=325 Hz, Q=10) are

Parameter	Flight Data Input	Test Spec Input
Duration	30 sec	60 sec
Overall Response	12.0 GRMS	14.2 GRMS
Maximum Expected Peak	51.3 G	63.3 G
Accumulative Fatigue R	2.03e+09	7.95e+09

The overall response of the system to each input is calculated using the method in Reference 5.

The maximum expected peaks are calculated via Reference 4, Appendix B.

The test spec response is slightly higher than the flight data response in terms of overall level and peak level.

The test spec is nearly four times as severe in terms of accumulative fatigue.