THE FAST FOURIER TRANSFORM (FFT)

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November 15, 1998

INTRODUCTION

The Fourier transform is a method for representing a time history signal in terms of a frequency domain function. Specifically, the Fourier transform represents a signal in terms of its spectral components.

The Fourier transform is a complex exponential transform which is related to the Laplace transform.

The Fourier transform is also referred to as a trigonometric transformation since the complex exponential function can be represented in terms of trigonometric functions. Specifically,

$$\exp[j\omega t] = \cos(\omega t) + j\sin(\omega t) \tag{1a}$$

$$\exp[-j\omega t] = \cos(\omega t) - j\sin(\omega t) \tag{1b}$$

The Fourier transform is often applied to digital time histories. The time histories are sampled from measured analog data.

The transform calculation method, however, requires a relatively high number of mathematical operations. As an alternative, a Fast Fourier transform (FFT) method has been developed to simplify this calculation. The purpose of this tutorial is to present a Fast Fourier transform algorithm.

Background theory is presented prior to the FFT algorithm.

FOURIER TRANSFORM THEORY

Formulas

The Fourier transform X(f) for a continuous time series x(t) is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp[-j2\pi f t]dt \tag{2}$$

where $-\infty < f < \infty$

Thus, the Fourier transform is continuous over an infinite frequency range.
The inverse transform is

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp\left[+j 2\pi f t\right] df \tag{3}$$

Equations (2) and (3) are taken from Reference 1. Note that X(f) has dimensions of [amplitude-time].

Also note that X(f) is a complex function. It may be represented in terms of real and imaginary components, or in terms of magnitude and phase.

The conversion is made as follows for a complex variable V.

$$V = a + jb \tag{4}$$

$$\text{Magnitude } V = \sqrt{a^2 + b^2} \tag{5}$$

$$\text{Phase } V = \arctan(b / a) \tag{6}$$

<u>Example</u>

Consider a sine wave

$$x(t) = A \sin\left[2\pi \hat{f}\, t\right] \tag{7}$$

where

$$-\infty < t < \infty$$

The Fourier transform of the sine wave is

$$X(f) = \left\{\frac{jA}{2}\right\}\left\{-\delta\left(f - \hat{f}\right) + \delta\left(-f - \hat{f}\right)\right\} \tag{8}$$

where δ is the Dirac delta function.

The derivation is given in Appendix A. The Fourier transform is plotted in Figure 1.

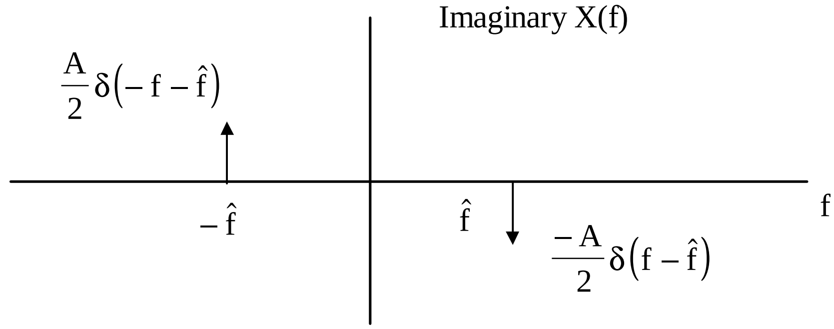

Figure 1. Fourier Transform of Sine Wave

The transform of a sine wave is purely imaginary.

On the other hand, the Fourier transform of a cosine wave is

$$X(f) = \left\{\frac{A}{2}\right\}\left\{\delta(f-\hat{f}) + \delta(-f-\hat{f})\right\} \tag{9}$$

The Fourier transform is plotted in Figure 2.

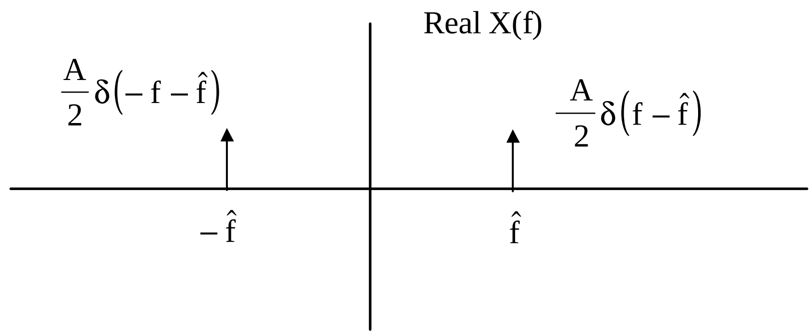

Figure 2. Fourier Transform of Cosine Wave

The transform of a cosine wave is purely real.

Characteristics

The plots in Figures 1 and 2 demonstrate two characteristics of the Fourier transforms of real time history functions:

1. The real Fourier transform is symmetric about the f = 0 line.
2. The imaginary Fourier transform is antisymmetric about the f = 0 line.

DISCRETE FOURIER TRANSFORM

Formulas

The following equation set is taken from Reference 2.

The Fourier transform F(k) for a discrete time series x(n) is

$$F(k) = \frac{1}{N}\sum_{n=0}^{N-1}\left\{x(n)\exp\left(-j\frac{2\pi}{N}nk\right)\right\}, \qquad \text{for } k = 0, 1, \ldots, N-1 \tag{10}$$

where

N is the number of time domain samples,
n is the time domain sample index,
k is the frequency domain index.

Note that the frequency increment Δf is equal to the time domain period T as follows

$$\Delta f = \frac{1}{T} \tag{11}$$

The frequency is obtained from the index parameter k as follows

$$\text{frequency}(k) = k\Delta f \tag{12}$$

Note that F(k) has dimensions of [amplitude]. An alternate form which has dimensions of [amplitude-time] is given in Appendix B.

The corresponding inverse transform is

$$x(n) = \sum_{k=0}^{N-1}\left\{F(k)\exp\left(+j\frac{2\pi}{N}nk\right)\right\}, \qquad \text{for } n = 0, 1, \ldots, N-1 \tag{13}$$

A characteristic of the discrete Fourier transform is that the frequency domain is taken from 0 to (N-1)Δf. The line of symmetry is at a frequency of

$$\left[\frac{N-1}{2}\right]\Delta f \tag{14}$$

Example

The discrete Fourier transform of a sine wave is given in Figure 3.

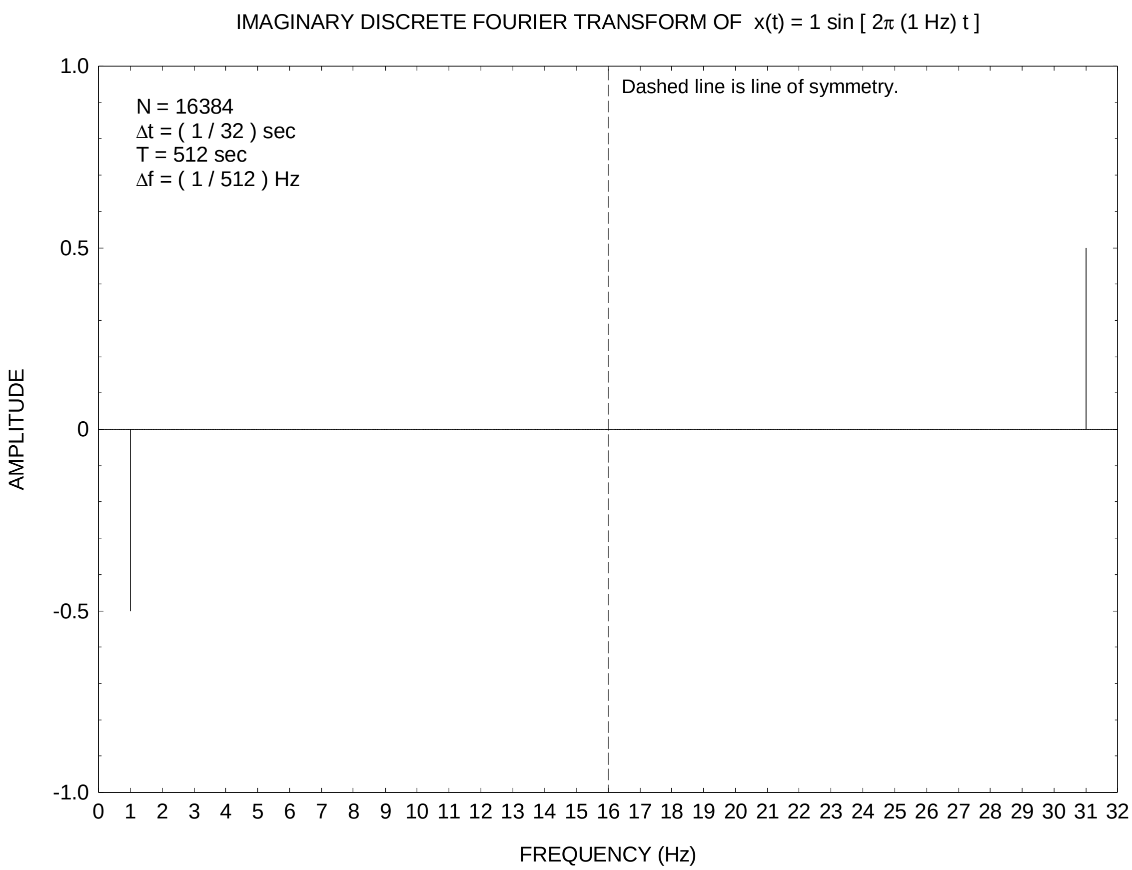

Figure 3. Fourier Transform of a Sine Wave

Note that the sine wave has a frequency of 1 Hz. The total number of cycles is 512, with a resulting period of 512 seconds. Again, the Fourier transform of a sine wave is imaginary and antisymmetric.

Nyquist Frequency

Note that the line of symmetry in Figure 3 marks the Nyquist frequency. The Nyquist frequency is equal to one-half of the sampling rate. Shannon's sampling theorem states that a sampled time signal must not contain components at frequencies above half the Nyquist frequency, from Reference 3.

Spectrum Analyzer Approach

Spectrum analyzer devices typically represent the Fourier transform in terms of magnitude and phase rather than real and imaginary components. Furthermore, spectrum analyzers typically only show one-half the total frequency band due to the symmetry relationship. The spectrum analyzer amplitude may either represent the *half-amplitude* or *the full-amplitude* of the spectral components. Care must be taken to understand the particular convention of the spectrum analyzer.

The full-amplitude Fourier transform would be calculated as

$$\hat{F}(k) = \begin{cases} \left[\frac{1}{N}\right] \sum_{n=0}^{N-1} \left\{ x(n) \right\}, & \text{for } k = 0 \\ \left[\frac{2}{N}\right] \sum_{n=0}^{N-1} \left\{ x(n) \exp\left(-j \frac{2\pi}{N} nk \right) \right\}, & \text{for } k = 1, \ldots, \frac{N}{2} - 1 \end{cases} \tag{15}$$

with N as an even integer.

Note that k=0 is a special case. The Fourier transform at this frequency is already at full-amplitude.

For example, a sine wave with an amplitude of 1 volt and a frequency of 100 Hz would simply have a full-amplitude Fourier magnitude of 1 volt at 100 Hz.

FAST FOURIER TRANSFORM

Number of Data Points

The approach in the following derivation assumes that the number of time history data points is equal to 2^N, where N is an integer.

Weighting Factors

The following derivation is based on Reference 4.

Define a weighting factor W as

$$W = \exp\left(- j \frac{2\pi}{N} \right) \tag{16a}$$

$$W^m = \exp\left(- j \frac{2\pi m}{N} \right) \tag{16b}$$

The discrete Fourier transform becomes

$$F(k) = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ x(n) W^{nk} \right\}, \qquad \text{for } k = 0, 1, \ldots, N-1 \tag{17}$$

The matrix representation is

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(N-1) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W^0 & W^0 & W^0 & \cdots & W^{0(N-1)} \\ W^0 & W^1 & W^2 & \cdots & W^{1(N-1)} \\ W^0 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W^0 & W^{1(N-1)} & W^{2(N-1)} & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{18}$$

Note that the W matrix in equation (13) is symmetric. Also note

$$W^0 = 1 \tag{19}$$

Equation (13) simplifies to

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(N-1) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W^1 & W^2 & \cdots & W^{1(N-1)} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W^{1(N-1)} & W^{2(N-1)} & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{20}$$

Unit Circle

Note that W^m is a point on the unit circle, with an angle m times the size of the angle W. A sample unit circle is shown in Figure 4.

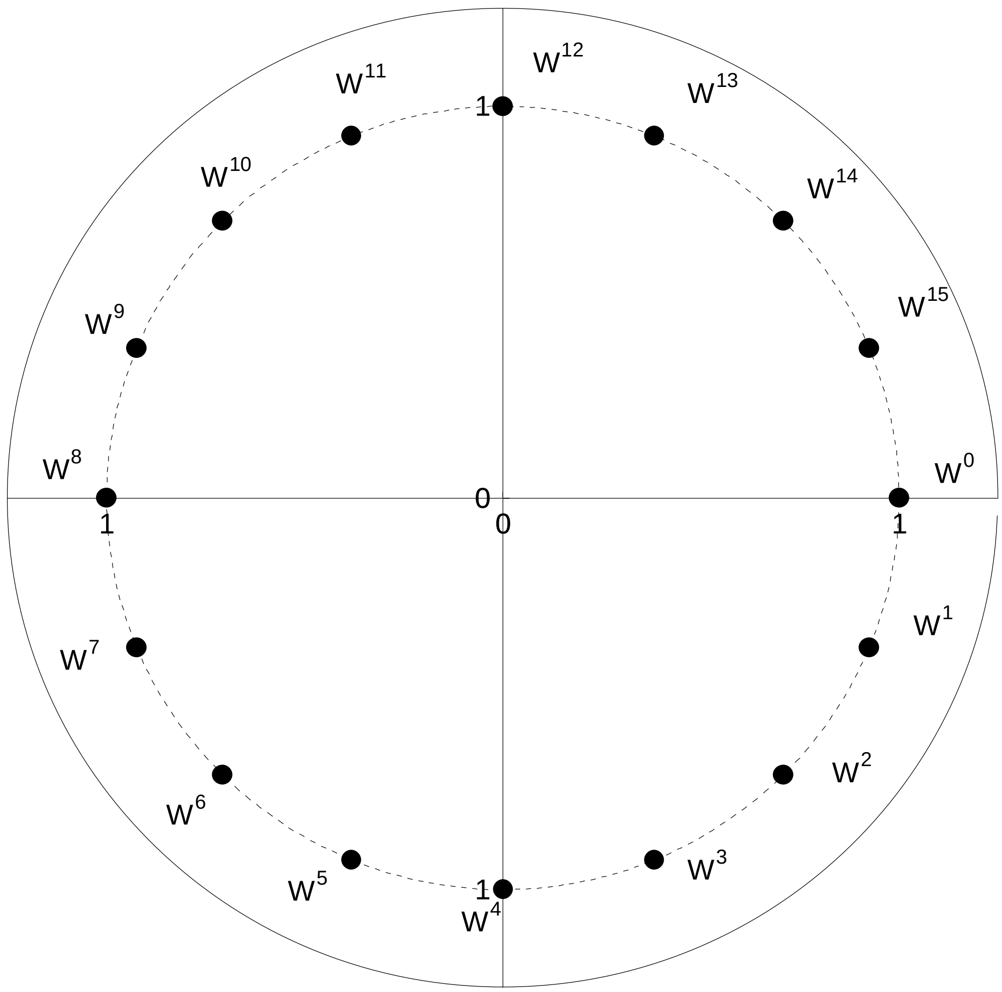

Figure 4. Unit Circle for N=16

Recall that $$W^m = \exp\left(-j\frac{2\pi m}{N}\right)$$

Even and Odd Representation

Recall the Fourier transformation

$$F(k) = \frac{1}{N}\sum_{n=0}^{N-1}\left\{x(n)W^{nk}\right\}, \qquad \text{for } k = 0, 1, \ldots, N-1 \tag{21}$$

Now break the series up into its even and odd terms.

$$F(k) = \frac{1}{N}\left\{ \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n)W^{2nk} \right\} + \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n+1)W^{(2n+1)k} \right\} \right\},$$

$$\text{for } k = 0, 1, \ldots, N-1$$

(22)

Equation (22) can be simplified as shown in the following steps.

$$F(k) = \frac{1}{N}\left\{ \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n)W^{2nk} \right\} + \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n+1)W^{2nk}W^{k} \right\} \right\},$$

$$\text{for } k = 0, 1, \ldots, N-1$$

(23)

$$F(k) = \frac{1}{N}\left\{ \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n)W^{2nk} \right\} + W^{k} \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n+1)W^{2nk} \right\} \right\},$$

$$\text{for } k = 0, 1, \ldots, N-1$$

(24)

$$F(k) = \frac{1}{N}\left\{\ A(k) + W^{\,k} B(k)\ \right\} \tag{25}$$

where

$$A(k) = \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n) W^{\,2nk} \right\}$$

$$B(k) = \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(2n+1) W^{\,2nk} \right\}$$

$$\text{for } k = 0, 1, \ldots, \ N-1$$

The term A(k) is an N/2 transformation over the even indexed data points. The term B(k) is an N/2 point transform over the odd indexed data points.

Note that stepping around the unit circle to pick up the complex coefficients now steps across every other angle. In other words,

$$W^{m} = \exp\left(-\, j \frac{2\pi\, m}{N} \right) \tag{26}$$

$$W^{2m} = \exp\left(-\, j \frac{2\pi\, 2m}{N} \right) \tag{27}$$

$$W^{2m} = \exp\left(-\, j \frac{2\pi\, m}{N/2} \right) \tag{28}$$

Thus, only N/2 angles are required to transform a time history with N points.

Now consider $W^{\left(k+\frac{N}{2}\right)}$, the transform point halfway through the list of output points.

Substituting this argument,

$$F\left(k+\frac{N}{2}\right)=A\left(k+\frac{N}{2}\right)+W^{\left(k+\frac{N}{2}\right)}B\left(k+\frac{N}{2}\right) \tag{29}$$

Note that

$$W^{\left(k+\frac{N}{2}\right)}=\exp\left(-j\frac{2\pi}{N}\left(k+\frac{N}{2}\right)\right) \tag{30}$$

$$W^{\left(k+\frac{N}{2}\right)}=\exp\left(-j\frac{2\pi k}{N}\right)\exp(-j\pi) \tag{31}$$

$$W^{\left(k+\frac{N}{2}\right)}=\exp\left(-j\frac{2\pi k}{N}\right)[\cos(\pi)-j\sin(\pi)] \tag{32}$$

$$W^{\left(k+\frac{N}{2}\right)}=-\exp\left(-j\frac{2\pi k}{N}\right) \tag{33}$$

$$W^{\left(k+\frac{N}{2}\right)}=-W^{k} \tag{34}$$

Recall

$$A(k)=\sum_{n=0}^{\frac{N}{2}-1}\left\{x(2n)W^{2nk}\right\} \tag{35}$$

$$A\left(k+\frac{N}{2}\right)=\sum_{n=0}^{\frac{N}{2}-1}\left\{x(2n)W^{2n\left(k+\frac{N}{2}\right)}\right\} \tag{36}$$

$$A\left(k+\frac{N}{2}\right)=\sum_{n=0}^{\frac{N}{2}-1}\left\{x(2n)W^{2nk}W^{nN}\right\} \tag{37}$$

Note

$$W^{nN}=\exp\left(-j\frac{2\pi\,nN}{N}\right) \tag{38}$$

$$W^{nN}=\exp(-j2\pi\,n) \tag{39}$$

$$W^{nN}=\cos(2\pi\,n)-j\sin(2\pi\,n) \tag{40}$$

The n term is an integer. Thus

$$W^{nN}=1 \tag{41}$$

Substituting equation (36) into equation (32),

$$A\left(k+\frac{N}{2}\right)=\sum_{n=0}^{\frac{N}{2}-1}\left\{x(2n)W^{2nk}\right\} \tag{42}$$

Thus,

$$A(k)=A\left(k+\frac{N}{2}\right) \tag{43}$$

Similarly,

$$B(k)=B\left(k+\frac{N}{2}\right) \tag{44}$$

Substitute equations (34), (43), and (44) into (25).

$$F\left(k+\frac{N}{2}\right)=\frac{1}{N}\left\{\,A(k)-W^{k}B(k)\,\right\} \tag{45}$$

In summary, the Fourier transform is reduced to the following pair of equations

$$F(k) = \frac{1}{N}\left\{ A(k) + W^{k} B(k) \right\} \qquad (46)$$

$$F\left(k + \frac{N}{2}\right) = \frac{1}{N}\left\{ A(k) - W^{k} B(k) \right\} \qquad (47)$$

$$\text{with } k=0, 1, 2, \ldots.. , \frac{N}{2} - 1$$

This pair of equations forms the basis of the FFT algorithm.

The equations form a *butterfly* as shown in Figure 5.

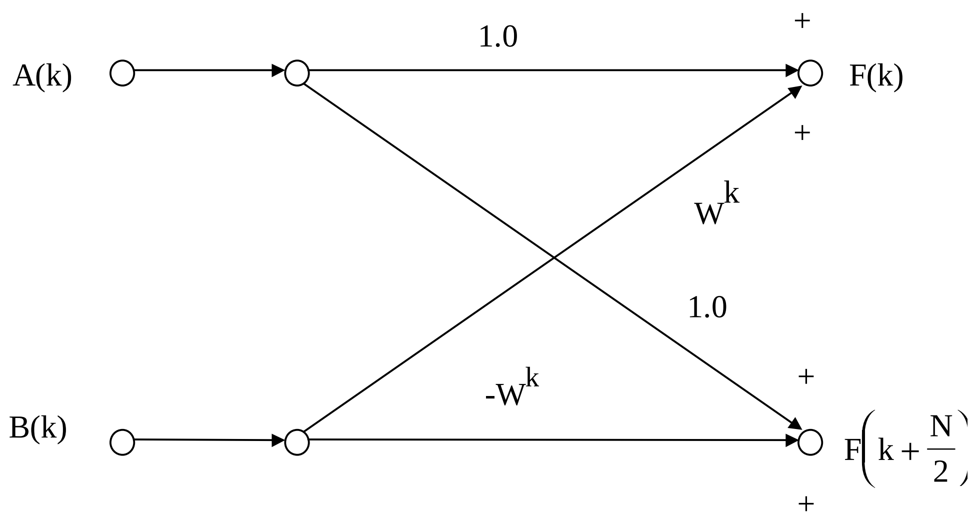

Figure 5. Butterfly Concept

The final division by N is omitted for brevity.

Figure 5 can be simplified as shown in Figure 6.

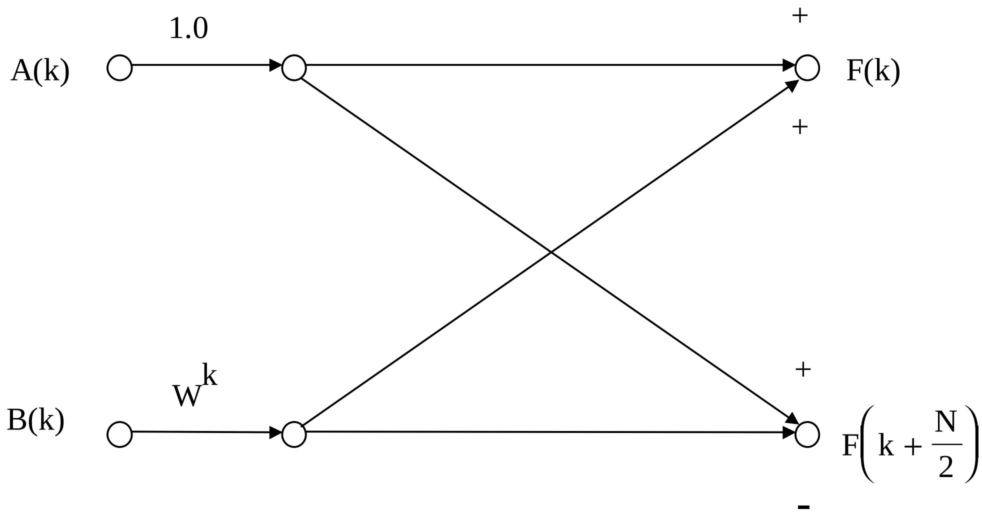

Figure 6. Equivalent Butterfly

The butterfly in Figure 6 significantly reduces the number of calculations required for the Fourier transform, particularly for large data sets.

Binary Reversal

Binary reversal of the time history is necessary to expedite the FFT calculation, as is shown in the following examples. Binary reversal is discussed further in Appendix C.

N=2 Example

Consider the specific case for N=2.

$$A(k) = \sum_{n=0}^{0} \left\{ x(2n) W^{2nk} \right\} \tag{48}$$

$$\text{for } k = 0, 0$$

$$A(0) = x(0) \tag{49}$$

$$B(k) = \sum_{n=0}^{0} \left\{ x(2n+1) W^{2nk} \right\} \tag{50}$$

$$B(0) = x(1) \tag{51}$$

Now substitute equations (49) and (51) into (46),

$$F(0) = \frac{1}{2}\left\{ A(0) + W^{0} B(0) \right\} \tag{52a}$$

$$F(0) = \frac{1}{2}\left\{ x(0) + W^{0} x(1) \right\} \tag{52b}$$

Substitute equations (49) and (51) into (47),

$$F(1) = \frac{1}{2}\left\{ A(1) - W^{0} B(1) \right\} \tag{53a}$$

$$F(1) = \frac{1}{2}\left\{ x(0) - W^{0} x(1) \right\} \tag{53b}$$

Equations (52b) and (53b) are shown in the butterfly diagram in Figure 7.

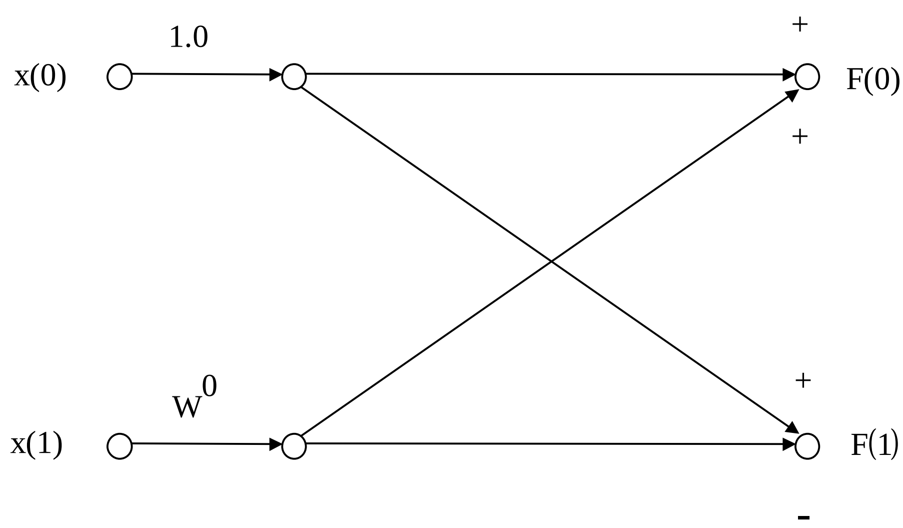

Figure 7. Butterfly for N=2

N=4 Example

Consider the specific case for N=4.

$$A(k) = \sum_{n=0}^{1} \left\{ x(2n) W^{2nk} \right\} \tag{52}$$

for $k = 0, 1$

$$A(0) = x(0) + x(2) \tag{53}$$

$$A(1) = x(0) + x(2) W^2 \tag{54a}$$

$$W^2 = -1 \tag{54b}$$

$$A(1) = x(0) - x(2) \tag{54c}$$

$$B(k) = \sum_{n=0}^{1} \left\{ x(2n+1) W^{2nk} \right\} \tag{55}$$

$$B(0) = x(1) + x(3) \tag{56}$$

$$B(1) = x(1) + x(3) W^2 \tag{57a}$$

$$B(1) = x(1) - x(3) \tag{57b}$$

By substitution

$$F(0) = \frac{1}{4} \left\{ A(0) + W^0 B(0) \right\} \tag{58a}$$

$$F(0) = \frac{1}{4} \left\{ x(0) + x(2) + W^0 \left[x(1) + x(3) \right] \right\} \tag{58b}$$

$$F(1) = \frac{1}{4} \left\{ A(1) + W^1 B(1) \right\} \tag{59a}$$

$$F(1) = \frac{1}{4} \left\{ \left[x(0) - x(2) \right] + W^1 \left[x(1) - x(3) \right] \right\} \tag{59b}$$

$$F(2) = \frac{1}{4}\left\{ A(0) - W^{0}B(0) \right\} \tag{60a}$$

$$F(2) = \frac{1}{4}\left\{ x(0) + x(2) - W^{0}\left[x(1) + x(3)\right] \right\} \tag{60b}$$

$$F(3) = \frac{1}{4}\left\{ A(1) - W^{1} B(1) \right\} \tag{61a}$$

$$F(3) = \frac{1}{4}\left\{ \left[x(0) - x(2)\right] - W^{1}\left[x(1) - x(3)\right] \right\} \tag{61b}$$

Summary

$$F(0) = \frac{1}{4}\left\{ \left[x(0) + x(2)\right] + W^{0}\left[x(1) + x(3)\right] \right\} \tag{62a}$$

$$F(1) = \frac{1}{4}\left\{ \left[x(0) - x(2)\right] + W^{1}\left[x(1) - x(3)\right] \right\} \tag{62b}$$

$$F(2) = \frac{1}{4}\left\{ \left[x(0) + x(2)\right] - W^{0}\left[x(1) + x(3)\right] \right\} \tag{62c}$$

$$F(3) = \frac{1}{4}\left\{ \left[x(0) - x(2)\right] - W^{1}\left[x(1) - x(3)\right] \right\} \tag{62d}$$

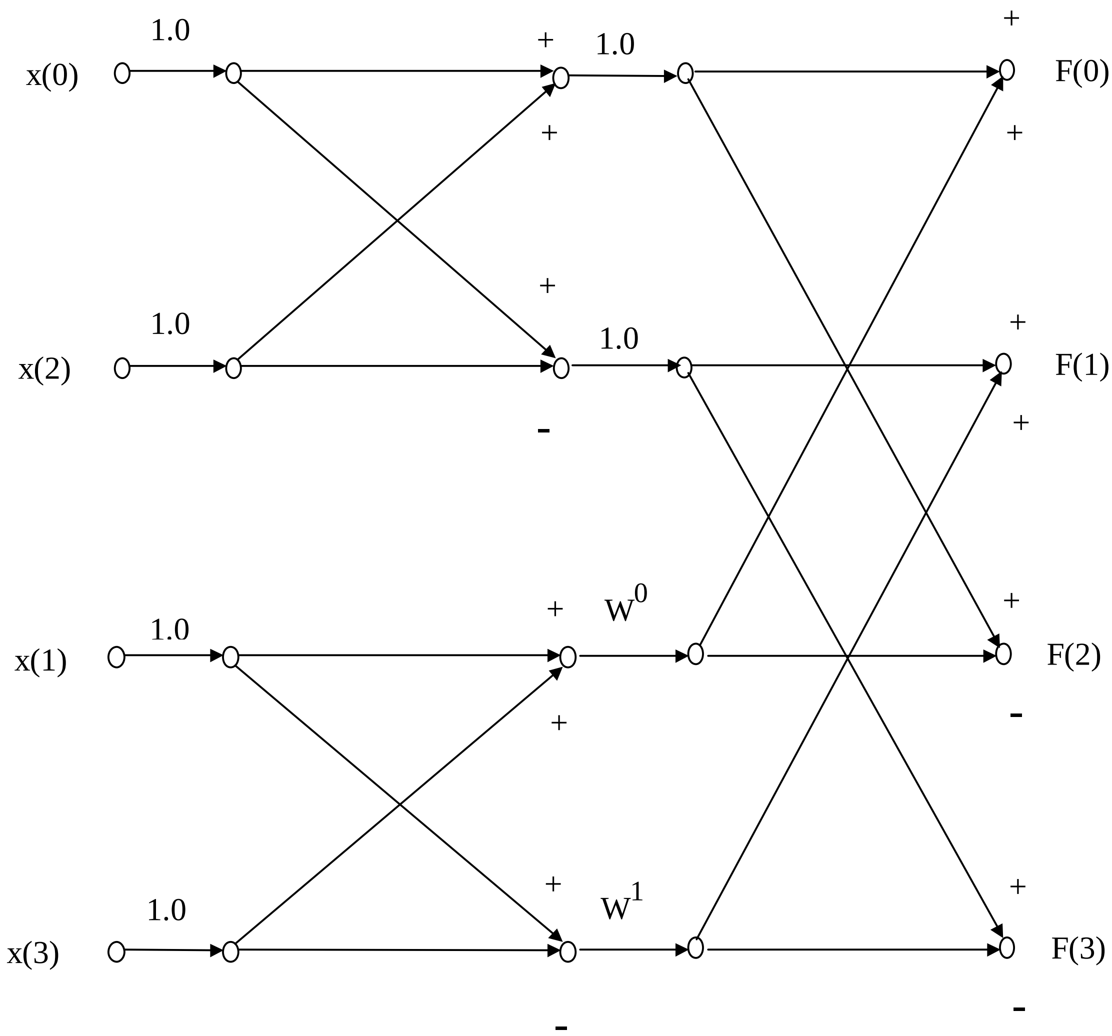

Figure 8. Butterfly for N=4

The four-point Fourier transform is composed of a pair of two-point transforms. The outputs of the two-point transforms are combined to create the four-point transform.

N=8 Example

Consider the specific case for N=8. The unit circle is shown in Figure 9.

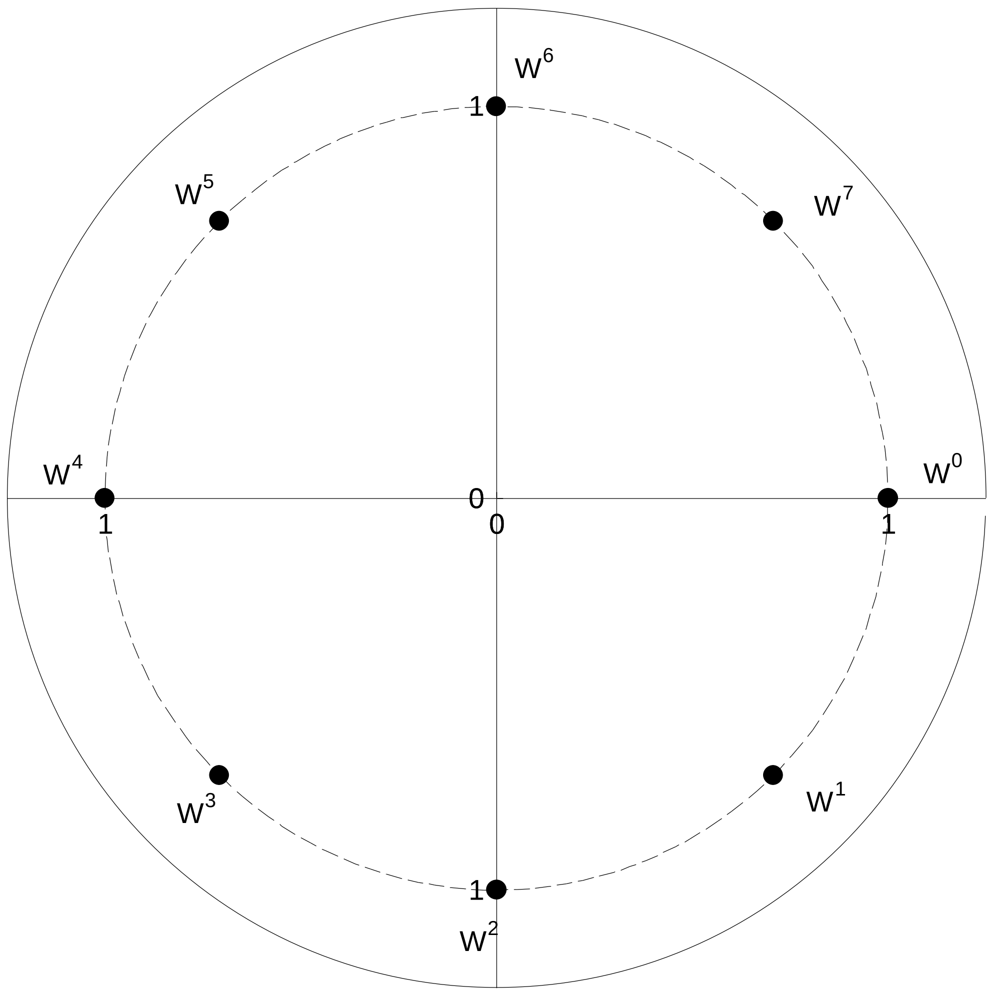

Figure 9. Unit Circle for N=8

Now determine the A coefficients

$$A(k) = \sum_{n=0}^{3} \left\{ x(2n) W^{2nk} \right\} \tag{63}$$

for $k = 0, 1, 2, 3$

$$A(0) = x(0) + x(2) + x(4) + x(6) \tag{64}$$

$$A(1) = x(0) + x(2)W^2 + x(4)W^4 + x(6)W^6 \tag{65}$$

$$A(2) = x(0) + x(2)W^4 + x(4)W^8 + x(6)W^{12} \tag{66}$$

$$A(3) = x(0) + x(2)W^6 + x(4)W^{12} + x(6)W^{18} \tag{67}$$

The unit circle yield the following equivalencies

$$W^4 = -W^0 \tag{68a}$$

$$W^6 = -W^2 \tag{68b}$$

$$W^8 = 1 \tag{68c}$$

$$W^{12} = -W^0 \tag{68d}$$

$$W^{18} = W^2 \tag{68e}$$

By substitution

$$A(0) = x(0) + x(2)W^0 + x(4) + x(6)W^0 \tag{69}$$

$$A(1) = x(0) + x(2)W^2 - x(4) - x(6)W^2 \tag{70}$$

$$A(2) = x(0) - x(2)W^0 + x(4) - x(6)W^0 \tag{71}$$

$$A(3) = x(0) - x(2)W^2 - x(4) + x(6)W^2 \tag{72}$$

Rearrangement

$$A(0) = \left[x(0) + x(4)\right] + W^0\left[x(2) + x(6)\right] \quad (73)$$

$$A(1) = \left[x(0) - x(4)\right] + W^2\left[x(2) - x(6)\right] \quad (74)$$

$$A(2) = \left[x(0) + x(4)\right] - W^0\left[x(2) + x(6)\right] \quad (75)$$

$$A(3) = \left[x(0) - x(4)\right] - W^2\left[x(2) - x(6)\right] \quad (76)$$

Now determine the B coefficients

$$B(k) = \sum_{n=0}^{3}\left\{x(2n+1)W^{2nk}\right\} \quad \text{for } k = 0,1,2,3 \quad (77)$$

$$B(0) = x(1) + x(3) + x(5) + x(7) \quad (78)$$

$$B(1) = x(1) + x(3)W^2 + x(5)W^4 + x(7)W^6 \quad (79)$$

$$B(2) = x(1) + x(3)W^4 + x(5)W^8 + x(7)W^{12} \quad (80)$$

$$B(3) = x(1) + x(3)W^6 + x(5)W^{12} + x(7)W^{18} \quad (81)$$

By substitution

$$B(0) = x(1) + x(3)W^0 + x(5) + x(7)W^0 \quad (82)$$

$$B(1) = x(1) + x(3)W^2 - x(5) - x(7)W^2 \quad (83)$$

$$B(2) = x(1) - x(3)W^0 + x(5) - x(7)W^0 \quad (84)$$

$$B(3) = x(1) - x(3)W^2 - x(5) + x(7)W^2 \quad (85)$$

Rearrangement

$$B(0) = \left[x(1) + x(5)\right] + W^0\left[x(3) + x(7)\right] \quad (86)$$

$$B(1) = \left[x(1) - x(5)\right] + W^2\left[x(3) - x(7)\right] \tag{87}$$

$$B(2) = \left[x(1) + x(5)\right] - W^0\left[x(3) + x(7)\right] \tag{88}$$

$$B(3) = \left[x(1) - x(5)\right] - W^2\left[x(3) - x(7)\right] \tag{89}$$

Recall

$$F(k) = \frac{1}{N}\left\{\ A(k) + W^{\,k} B(k)\ \right\} \tag{90}$$

$$F\left(k + \frac{N}{2}\right) = \frac{1}{N}\left\{\ A(k) - W^{\,k} B(k)\ \right\} \tag{91}$$

with k=0, 1, 2, ... , $\frac{N}{2} - 1$

Thus,

$$F(0) = \frac{1}{8}\left\{\left\{\left[x(0) + x(4)\right] + W^0\left[x(2) + x(6)\right]\right\} + \left\{\left[x(1) + x(5)\right] + W^0\left[x(3) + x(7)\right]\right\}\right\} \tag{92}$$

$$F(1) = \frac{1}{8}\left\{\left\{\left[x(0) - x(4)\right] + W^2\left[x(2) - x(6)\right]\right\} + W^1\left\{\left[x(1) - x(5)\right] + W^2\left[x(3) - x(7)\right]\right\}\right\} \tag{93}$$

$$F(2) = \frac{1}{8}\left\{\left\{\left[x(0) + x(4)\right] - W^0\left[x(2) + x(6)\right]\right\} + W^2\left\{\left[x(1) + x(5)\right] - W^0\left[x(3) + x(7)\right]\right\}\right\} \tag{94}$$

$$F(3) = \frac{1}{8}\left\{\left\{\left[x(0) - x(4)\right] - W^2\left[x(2) - x(6)\right]\right\} + W^3\left\{\left[x(1) - x(5)\right] - W^2\left[x(3) - x(7)\right]\right\}\right\} \tag{95}$$

$$F(4) = \frac{1}{8}\left\{\left\{\left[x(0) + x(4)\right] + W^0\left[x(2) + x(6)\right]\right\} - W^0\left\{\left[x(1) + x(5)\right] + W^0\left[x(3) + x(7)\right]\right\}\right\} \tag{96}$$

$$F(5) = \frac{1}{8}\left\{\left\{\left[x(0) - x(4)\right] + W^2\left[x(2) - x(6)\right]\right\} - W^1\left\{\left[x(1) - x(5)\right] + W^2\left[x(3) - x(7)\right]\right\}\right\} \tag{97}$$

$$F(6) = \frac{1}{8}\left\{\left\{\left[x(0) + x(4)\right] - W^0\left[x(2) + x(6)\right]\right\} - W^2\left\{\left[x(1) + x(5)\right] - W^0\left[x(3) + x(7)\right]\right\}\right\} \tag{98}$$

$$F(7) = \frac{1}{8}\left\{\left\{\left[x(0) - x(4)\right] - W^2\left[x(2) - x(6)\right]\right\} - W^3\left\{\left[x(1) - x(5)\right] - W^2\left[x(3) - x(7)\right]\right\}\right\} \tag{99}$$

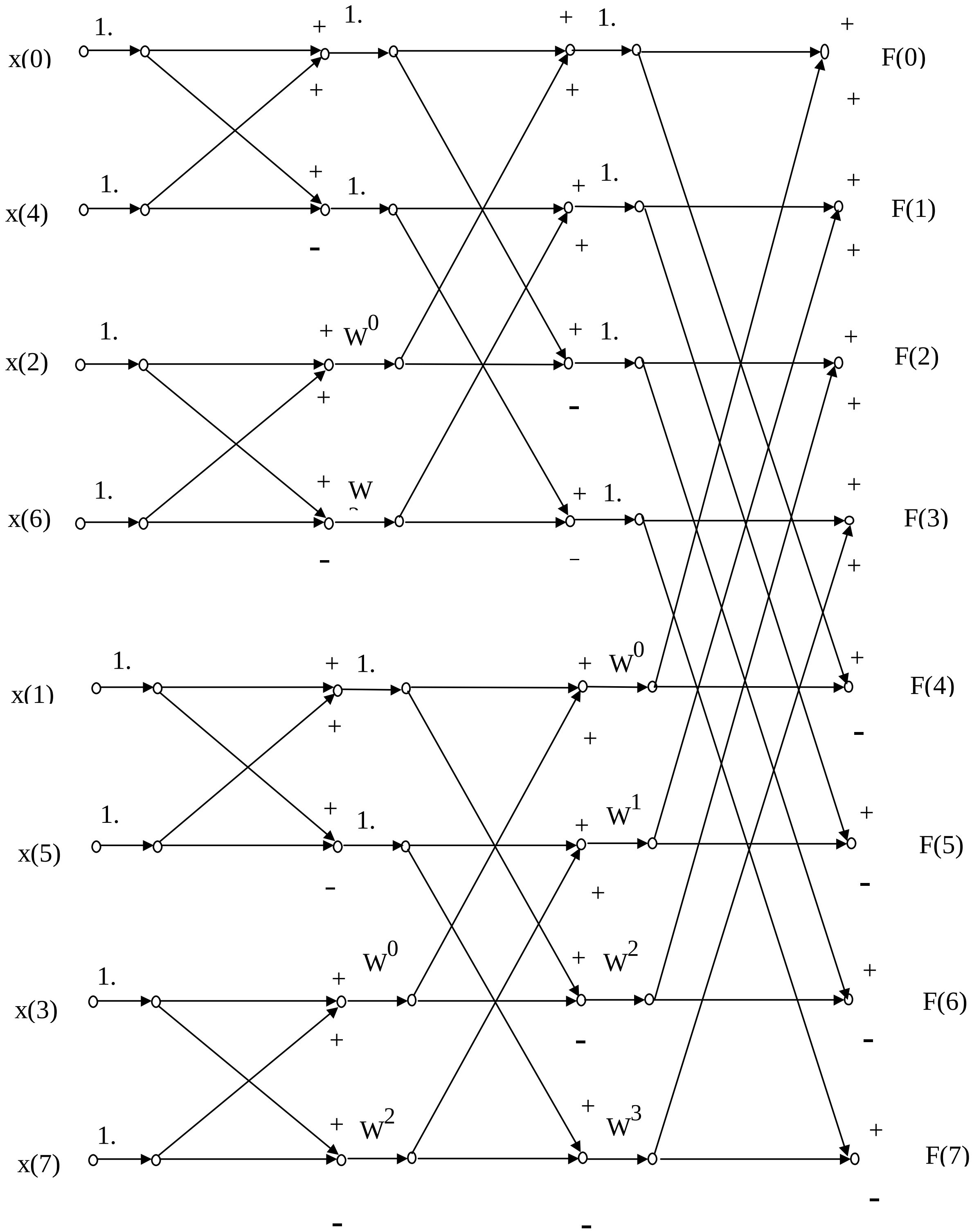

Figure 10. Butterfly for N=8

The eight-point Fourier transform is composed of a pair of four-point transforms. In turn, each four-point transform is composed of a pair of two-point transforms.

Data Sets with Higher N values

This concept can be extended to higher data sets. Again, the time history points must be represented in reverse binary order.

As an example, a 16-point transform would consist of a pair of a pair of eight-point transforms. Each 8-point transform would consist of a pair of four point-transforms, and so on.

Representing a butterfly diagram for a 16-point transforms is cumbersome. Nevertheless, the weight factors are summarized in Table 1.

Table 1. Weighting Factors for 16-point Transform

Stage 1	Stage 2	Stage 3	Stage 4
1	1	1	1
W^0	1	1	1
1	W^0	1	1
W^0	W^4	1	1
1	1	W^0	1
W^0	1	W^2	1
1	W^0	W^4	1
W^0	W^4	W^6	1
1	1	1	W^0
W^0	1	1	W^1
1	W^0	1	W^2
W^0	W^4	1	W^3
1	1	W^0	W^4
W^0	1	W^2	W^5
1	W^0	W^4	W^6
W^0	W^4	W^6	W^7

Savings

A full description of the efficiency of the FFT relative to the conventional Fourier transform is given in Reference 4.

As an example, consider a series with N=1024. Table 2 gives the number of operations per each method.

Table 2. Computation Workload for N=1024		
	FFT	Conventional Fourier Transform
Multiplication steps	16,384	4.2 million
Addition steps	28,672	4.2 million

Note that the addition and multiplication steps in Table 2 are based on real numbers. A complex multiplication requires 4 real multiplication steps and 2 real addition steps. Again, these real steps are accounted for in Table 2.

Also note that the 4.2 million number in Table 2 is approximate. The exact number is 4,194,304.

POWER SPECTRAL DENSITY FUNCTION

Dimensions

The power spectral density function has dimensions of [$\text{amplitude}^2 \cdot \text{time}$].

Formal Definition

Recall the Fourier transform X(f) for a continuous time series x(t)

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp\left[-j2\pi f t\right]dt \tag{100}$$

where $-\infty < f < \infty$

The power spectral density S(f) is defined as

$$S(f) = \lim_{T \to \infty} \frac{1}{T} X(f)\, X^*(f) \tag{101}$$

where $-\infty < f < \infty$

Note that the * symbol denotes complex conjugate.

Furthermore, the power spectral density function can be defined as the Fourier transform of the autocorrelation function per the Wiener-Khintchine equations, as noted in Reference 1.

Practical Application

Recall the double-amplitude spectrum analyzer version of the Fourier transform,

$$\hat{F}(k)=\begin{cases}\left[\frac{1}{N}\right]\sum_{n=0}^{N-1}\left\{x(n)\right\}, & \text{for } k=0\\ \left[\frac{2}{N}\right]\sum_{n=0}^{N-1}\left\{x(n)\exp\left(-j\frac{2\pi}{N}nk\right)\right\}, & \text{for } k=1,\ldots,\frac{N}{2}-1\end{cases}$$

with N as an even integer.

(102)

The single-sided power spectral density function $PSD(f_k)$ for a discrete series is

$$PSD(f_k)=\begin{cases}\left[\frac{\hat{F}(0)\hat{F}^*(0)}{\Delta f}\right], & \text{for } k=0\\ \left[\frac{1}{2}\right]\left[\frac{\hat{F}(k)\hat{F}^*(k)}{\Delta f}\right], & \text{for } k=1,\ldots,\frac{N}{2}-1\end{cases} \tag{103}$$

Recall that the frequency increment Δf is equal to the time domain period T as follows

$$\Delta f=\frac{1}{T} \tag{104}$$

Recall that the frequency is obtained from the index parameter k as follows

$$\text{frequency}(k)=k\Delta f \tag{105}$$

The ½ factor in equation (103) is required to convert $[\text{amplitude peak}]^2$ to $[\text{amplitude RMS}]^2$, per the convention of a power spectral density function.

The k=0 case does not require this peak-to-RMS conversion. Note that the RMS amplitude is equal to the peak amplitude for a signal with zero frequency. This signal is often called a DC signal.

FURTHER PROCESSING CONCEPTS

Discrete Fourier transforms calculated from finite data records can suffer from an error called *leakage*. This error causes energy to be smeared into adjacent frequency bands.

The leakage error is reduced by applying a window to the data. Typically, the window is applied to a segment of the data. The segments are taken with an overlap in order to recover statistical degrees of freedom lost as a result of the window. These concepts are explained in References 3 through 5.

REFERENCES:

1. W. Thomson, Theory of Vibration with Applications, 2nd Ed, Prentice-Hall, 1981.
2. GenRad TSL25 Time Series Language for 2500-Series Systems, Santa Clara, California, 1981.
3. R. Randall, Frequency Analysis 3rd edition, Bruel & Kjaer, 1987.
4. F. Harris, Trigonometric Transforms, Scientific-Atlanta, Technical Publication DSP-005, San Diego, CA.
5. T. Irvine, Statistical Degrees of Freedom, 1995.

APPENDIX A

Consider a sine wave

$$x(t) = A \sin\left[2\pi \hat{f}\, t\right] \tag{A-1}$$

where

$$-\infty < t < \infty$$

The Fourier transform is calculated indirectly, by considering the inverse transform. Note that the sine wave is a special case in this regard.

Recall

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp\left[+j 2\pi f\, t\right] df \tag{A-2}$$

Thus

$$A \sin\left[2\pi \hat{f}\, t\right] = \int_{-\infty}^{\infty} X(f) \exp\left[+\, j\, 2\pi f\, t\right] df \tag{A-3}$$

$$A \sin\left[2\pi \hat{f}\, t\right] = \int_{-\infty}^{\infty} X(f) \left\{\cos\left[2\pi f\, t\right] + j \sin\left[2\pi f\, t\right]\right\} df \tag{A-4}$$

Let

$$X(f) = P(f) + j\, Q(f) \tag{A-5}$$

where

P(f) and Q(f) are both real coefficients

and

$$-\infty < f < \infty.$$

$$A \sin\left[2\pi \hat{f}\, t\right] = \int_{-\infty}^{\infty} \left\{P(f) + j\, Q(f)\right\}\left\{\cos\left[2\pi f\, t\right] + j \sin\left[2\pi f\, t\right]\right\} df \tag{A-6}$$

$$\begin{aligned} A \sin\left[2\pi \hat{f}\, t\right] &= \int_{-\infty}^{\infty} \left\{P(f) \cos\left[2\pi f\, t\right] - Q(f) \sin\left[2\pi f\, t\right]\right\} df \\ &\quad + j \int_{-\infty}^{\infty} \left\{P(f) \sin\left[2\pi f\, t\right] + Q(f) \cos\left[2\pi f\, t\right]\right\} df \end{aligned} \tag{A-7}$$

Equation (A-7) can be broken into two parts

$$A\sin\left[2\pi\hat{f}\,t\right]=\int_{-\infty}^{\infty}\left\{P(f)\cos\left[2\pi f\,t\right]-Q(f)\sin\left[2\pi f\,t\right]\right\}df \qquad \text{(A-8)}$$

$$0=j\int_{-\infty}^{\infty}\left\{P(f)\sin\left[2\pi f\,t\right]+Q(f)\cos\left[2\pi f\,t\right]\right\}df \qquad \text{(A-9)}$$

Consider equation (A-8)

$$A\sin\left[2\pi\hat{f}\,t\right]=\int_{-\infty}^{\infty}\left\{P(f)\cos\left[2\pi f\,t\right]-Q(f)\sin\left[2\pi f\,t\right]\right\}df \qquad \text{(A-10)}$$

Now assume

$$P(f)=0 \qquad \text{(A-11)}$$

With this assumption,

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}Q(f)\sin\left[2\pi f\,t\right]df \qquad \text{(A-12)}$$

Now let

$$Q(f)=q_1(f)+q_2(f) \qquad \text{(A-13)}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}\left[q_1(f)+q_2(f)\right]\sin\left[2\pi f\,t\right]df \qquad \text{(A-14)}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}\left[q_1(f)\right]\sin\left[2\pi f\,t\right]dt-\int_{-\infty}^{\infty}\left[q_2(f)\right]\sin\left[2\pi f\,t\right]df \qquad \text{(A-15)}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}\left[q_1(f)\right]\sin\left[2\pi f\,t\right]dt+\int_{-\infty}^{\infty}\left[q_2(f)\right]\sin\left[-2\pi f\,t\right]df \qquad \text{(A-16)}$$

Equation (A-14) is satisfied by the pair of equations

$$q_1(f) = -\frac{A}{2}\delta\left(f - \hat{f}\right) \tag{A-17}$$

$$q_2(f) = \frac{A}{2}\delta\left(-f - \hat{f}\right) \tag{A-18}$$

where δ is the Dirac delta function.

By substitution,

$$Q(f) = \frac{-A}{2}\delta\left(f - \hat{f}\right) + \frac{A}{2}\delta\left(-f - \hat{f}\right) \tag{A-19}$$

Verification must be made that equation (A-9) is satisfied. Recall

$$0 = j\int_{-\infty}^{\infty}\left\{P(f)\sin\left[2\pi f t\right] + Q(f)\cos\left[2\pi f t\right]\right\}df \tag{A-20}$$

$$0 \stackrel{?}{=} j\int_{-\infty}^{\infty}\left\{0\sin\left[2\pi f t\right] + \left\{\frac{-A}{2}\delta\left(f - \hat{f}\right) + \frac{A}{2}\delta\left(-f - \hat{f}\right)\right\}\cos\left[2\pi f t\right]\right\}df \tag{A-21}$$

$$0 \stackrel{?}{=} j\left\{\frac{-A}{2}\cos\left[2\pi\hat{f}\ t\right] + \frac{A}{2}\cos\left[-2\pi\hat{f}\ t\right]\right\} \tag{A-22}$$

$$0 \stackrel{?}{=} j\left\{\frac{-A}{2}\cos\left[2\pi\hat{f}\ t\right] + \frac{A}{2}\cos\left[2\pi\hat{f}\ t\right]\right\} \tag{A-23}$$

$$0 = 0 \tag{A-24}$$

Recall the time domain function

$$x(t) = A\sin\left[2\pi\hat{f}\ t\right] \tag{A-25}$$

where

$$-\infty < t < \infty$$

The Fourier transform is thus

$$X(f) = \frac{-jA}{2}\delta\left(f - \hat{f}\right) + \frac{jA}{2}\delta\left(-f - \hat{f}\right) \tag{A-26}$$

$$X(f) = \left\{\frac{jA}{2}\right\}\left\{-\delta\left(f - \hat{f}\right) + \delta\left(-f - \hat{f}\right)\right\} \tag{A-27}$$

APPENDIX B

An alternate form of the discrete Fourier Transform is

$$\hat{F}(k) = \Delta t \sum_{n=0}^{N-1} \left\{ x(n) \exp\left(-j \frac{2\pi}{N} nk \right) \right\}, \qquad \text{for } k = 0, 1, \ldots, N-1 \tag{B-1}$$

$\hat{F}(k)$ has dimensions of [amplitude-time].

The corresponding inverse transform is

$$x(n) = \Delta f \sum_{n=0}^{N-1} \left\{ \hat{F}(k) \exp\left(+j \frac{2\pi}{N} nk \right) \right\}, \qquad \text{for } n = 0, 1, \ldots, N-1 \tag{B-2}$$

These alternate equations are based on the following reference:

MAC/RAN IV Applications Manual, Revision 2, University Software Systems, Los Angeles, California, 1991.

APPENDIX C

Binary Reversal

Consider a set of two numbers. The numbers are shown in Figure C-1, along with the binary forms.

Table C-1. Binary Values for N=2			
Number	Binary	Reverse Binary	Numbers in Reverse Binary Order
0	0	0	0
1	1	1	1

The analysis is repeated for a set of four numbers in Table C-2.

Table C-2. Binary Values for N=4			
Number	Binary	Reverse Binary	Numbers in Reverse Binary Order
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

The analysis is shown for a set of eight numbers in Table C-3.

Table C-3. Binary Values for N=8			
Number	Binary	Reverse Binary	Numbers in Reverse Binary Order
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7