AN INTRODUCTION TO THE FILTERING OF DIGITAL SIGNALS
Revision A

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INTRODUCTION

Vibration Signals

Vibration signals appear in a variety of forms.

Many vibration signals are composed of a broad frequency spectrum of energy components, where the amplitude may vary with frequency. A broad spectrum is a characteristic of *random vibration.*

This broad spectrum is evident in a power spectral density of the random vibration signal. A power spectral density is a frequency-domain function. The random characteristic would also be evident in an instantaneous time history of the signal.

In some cases, the vibration signal may have a sine-*on-random* character. A certain solid rocket motor, for example, has a chamber resonance which sweeps downward from 550 Hz to 450 Hz. This sinusoidal resonance is superimposed on broadband random vibration.

Filtering

Filtering is a tool for resolving signals. Filtering can be performed on either analog or digital signals. In addition, filtering can be used for a number of purposes.

For example, analog signals are typically routed through a lowpass filter prior to analog-to-digital conversion. The lowpass filter in this case is designed to prevent an error called *aliasing.* This is an error whereby high frequency spectral components are added to lower frequencies. Further information on aliasing is given in Reference 1.

Another purpose of filtering is to clarify resonant behavior by attenuating the energy at frequencies away from the resonance.

This report is concerned with the clarification purpose. The fundamentals of filtering are discussed. Sample filtering equations are derived.

DIGITAL FILTER DESIGN

A useful tool for designing digital filters is the *z-transform.* The two-sided z-transform X(z) of a time history sequence x_k is defined as

$$X(z) = \sum_{k=-\infty}^{\infty} x_k z^{-k} \tag{1}$$

The z-transform method is used to derive a transfer function H(z). This transfer function relates the output Y(z) to the input X(z) as follows

$$H(z) = \frac{Y(z)}{X(z)} \tag{2}$$

Digital filters are based on this transfer function, as shown in the block diagram in Figure 1. Note that x_k and y_k are the time domain input and output, respectively.

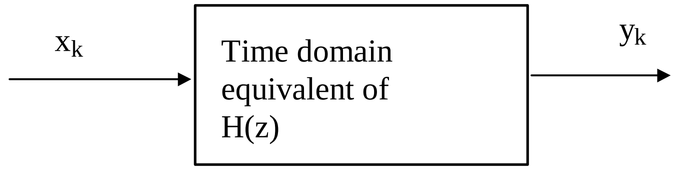

Figure 1. Filter Block Diagram

The transfer function can be represented by a series of a_n and b_n coefficients as follows

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_L z^{-L}}{1 + a_1 z^{-1} + \cdots + a_L z^{-L}} \tag{3}$$

The coefficients are constants which determine the system response. Note the H(z) defines the direct form transfer function for an Lth-order, linear, time-invariant digital system.

Note that

$$z = \exp\left(j\omega T\right) \tag{4}$$

The T variable is the time step. By substitution,

$$H(\omega) = \frac{b_0 + b_1\left[\exp\left(j\omega T\right)\right]^{-1} + \cdots + b_L\left[\exp\left(j\omega T\right)\right]^{-L}}{1 + a_1\left[\exp\left(j\omega T\right)\right]^{-1} + \cdots + a_L\left[\exp\left(j\omega T\right)\right]^{-L}} \tag{5}$$

With some manipulation, equations (1) through (3) can be used to derive the time domain equivalent of H (z) as

$$y_k = \left\{\sum_{n=0}^{L} b_n x_{k-n}\right\} - \left\{\sum_{n=1}^{L} a_n y_{k-n}\right\} \tag{6}$$

Equation (6) is recursive because the output at any time index k depends on the output at previous times.

FILTER IMPULSE RESPONSE CLASS

Digital filters are classified according to their impulse response: infinite impulse response (IIR) and finite impulse response (FIR).

Each class can be implemented in the time domain. The theory underlying each of these classes is discussed in Reference 2.

The familiar Bessel, Butterworth, and Chebyshev filters are all examples of IIR filters. Signal processing software typically uses this class of filters, perhaps since IIR filter algorithms involve less computation.

This report will focus on the IIR class.

IIR FILTER DESIGN

Phase Correction

Ideally, a filter should provide linear phase response. This is particularly desirable if shock response spectra calculations are required. IIR filters, however, do not have a linear phase response, for reasons discussed in Reference 2. A number of methods are available, however, to correct the phase response. One method is based on time reversals and multiple filtering as shown in Figure 2.

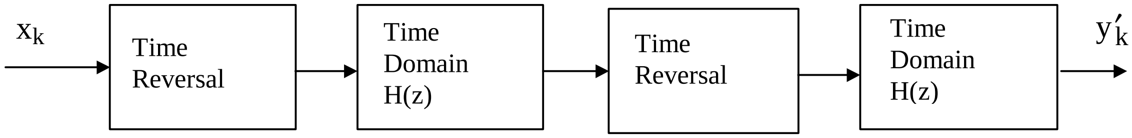

Figure 2. Phase Correction Method

Further information about refiltering is given in Reference 3.

IIR Filter Types

There are many types of IIR filters. A complete description is beyond the scope of this report.

The Bessel filter is discussed briefly. The Butterworth filter is discussed in detail.

Bessel Filters

Bessel filters use the direct-form transfer function as shown in Figure 1. Other filters, however, use a cascade approach, as explained later in this report.

An analog Bessel filter has a nearly linear phase response. This property translates only approximately into to the digital version, however.

Bessel filter transfer functions tend to have a very gradual roll-off beyond the cut-off frequency. In this sense, a Bessel filter may be a poor choice for an anti-aliasing filter.

The Bessel transfer function is given for three cases in Figure 3.

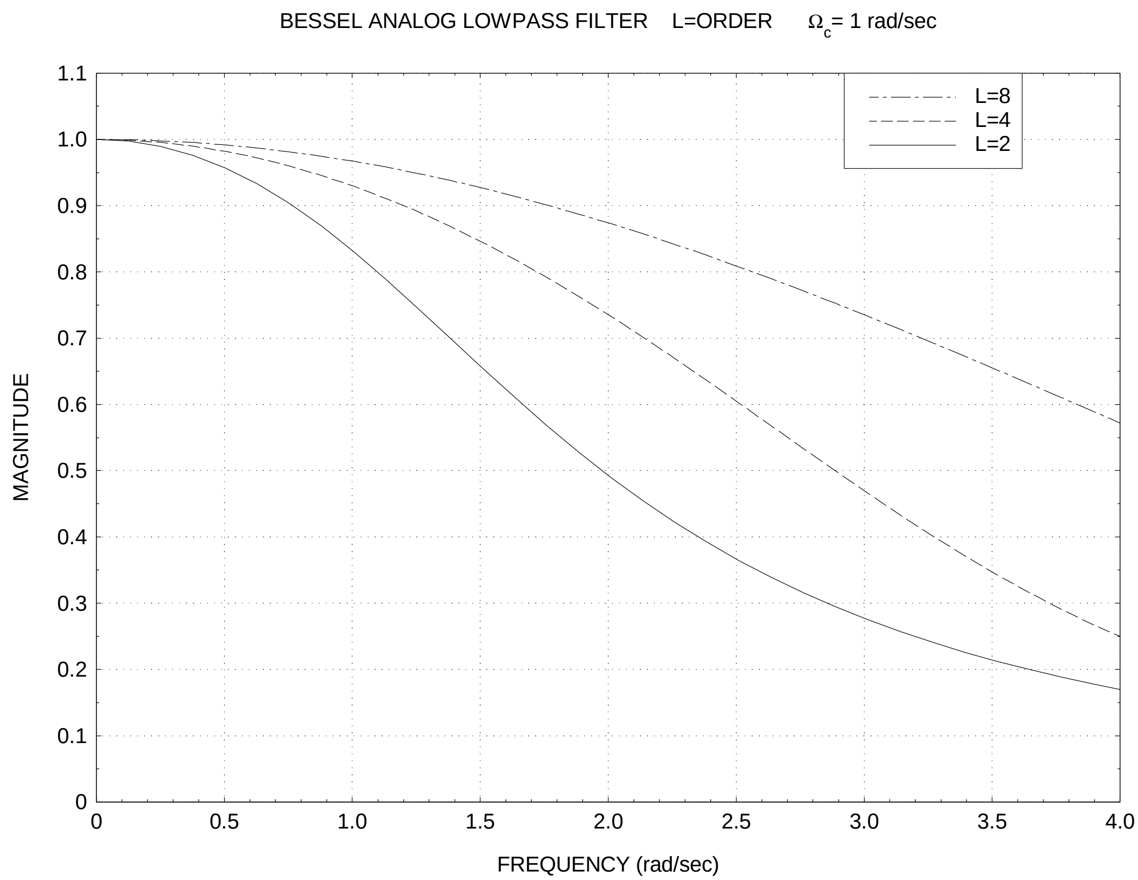

Figure 3. Bessel Filter Transfer Magnitude

A characteristic of Bessel filters is that a lower order provides more attenuation above the cutoff frequency.

Butterworth Filters

Analog Transfer Function Magnitude

Butterworth filters are often used as anti-aliasing filters. For example, the Pegasus vehicle has on-board analog anti-aliasing filters which are 4-pole Butterworth.

The Lth-order lowpass analog Butterworth filter magnitude response is

$$\left| P_L(\Omega) \right| = \frac{1}{\sqrt{1+\Omega^{2L}}}, \quad L \geq 1 \tag{7}$$

Equation (7) is normalized to have a cutoff frequency Ω_c equal to 1 radian/sec.

A characteristic of equation (7) is that all curves pass through the coordinate $\left(1, \frac{1}{\sqrt{2}}\right)$, regardless of the order. Note that other filter types do not necessarily have this same characteristic.

Equation (7) is graphed in Figure 4 for three cases.

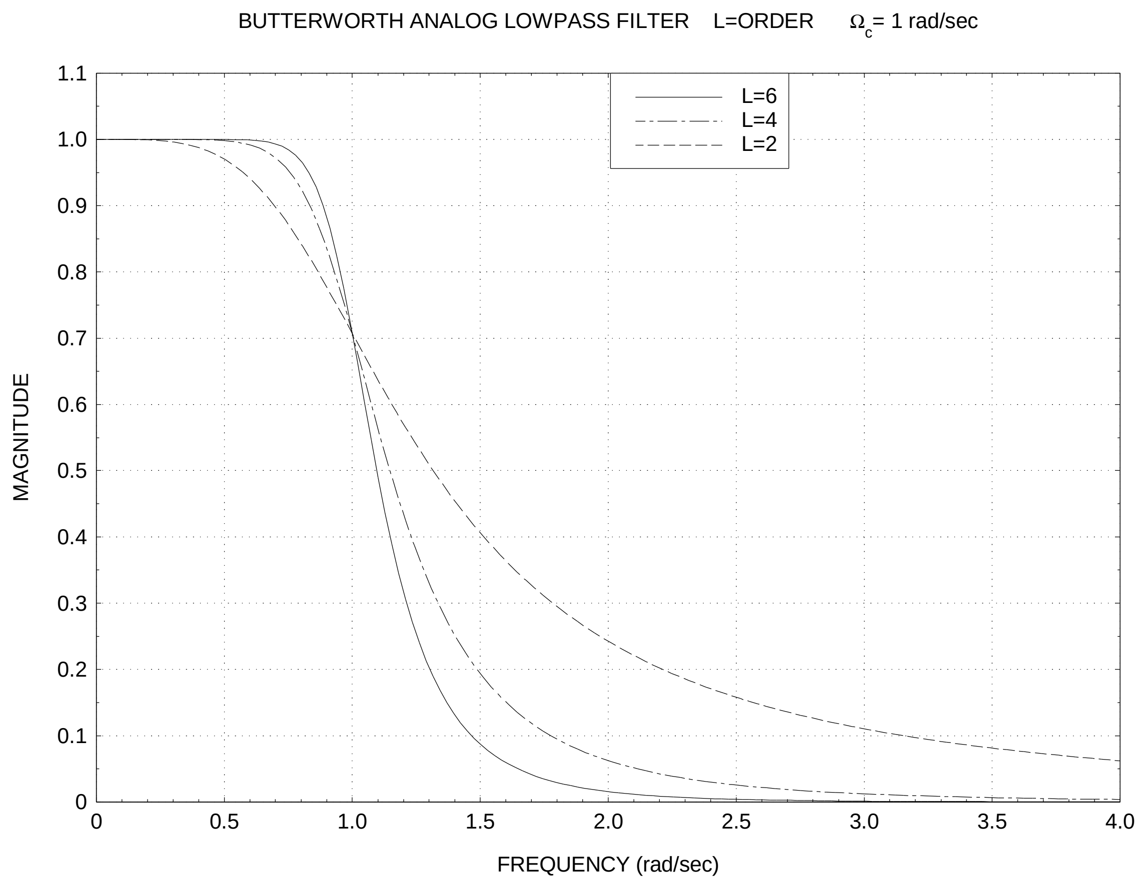

Figure 4. Butterworth Filter Transfer Magnitude

An Lth-order filter has a total of 2L poles. Only the L poles in the left half of the s-plane are used, however, to form a stable filter.

The poles s_k are given by

$$s_k = \exp\left(\frac{j\,(2k + L - 1)\pi}{2L}\right), \quad 1 \le k \le 2L \tag{8}$$

$$s_k = \cos\left(\frac{(2k + L - 1)\pi}{2L}\right) + j\sin\left(\frac{(2k + L - 1)\pi}{2L}\right), \qquad 1 \le k \le 2L \tag{9}$$

Again, only the poles in the left half s-plane are used. Effectively, only the poles for $1 \le k \le L$ are used.

Note that the same pole equations are used for both lowpass and highpass filter designs.

The poles are inserted into the following transfer function

$$H(s) = \frac{1}{(s - s_1)(s - s_2)\ldots(s - s_L)} \tag{10}$$

The conventional implementation is to apply the filter in a cascade manner rather than fully expanding the denominator in equation (10). Each section of the cascade is a second order-section $H_k(s)$ given by

$$H_k(s) = \frac{1}{(s - s_k)(s - s_{L+1-k})} \tag{11}$$

Note that the sections are arranged to match the complex conjugate pairs of the poles.

The analog transfer function for a lowpass Butterworth filter with even order can now be written as

$$H(s) = \prod_{k=1}^{L/2} H_k(s) \tag{12}$$

Sixth-order Lowpass Butterworth Example

A sixth-order lowpass Butterworth filter has the poles given in Table 1. Only the poles on the left half of the s-plane are given.

Table 1. Sixth-order Lowpass Butterworth Filter	
k	s_k Pole
1	$\cos\left(\frac{7\pi}{12}\right) + j\sin\left(\frac{7\pi}{12}\right)$
2	$\cos\left(\frac{9\pi}{12}\right) + j\sin\left(\frac{9\pi}{12}\right)$
3	$\cos\left(\frac{11\pi}{12}\right) + j\sin\left(\frac{11\pi}{12}\right)$
4	$\cos\left(\frac{13\pi}{12}\right) + j\sin\left(\frac{13\pi}{12}\right)$
5	$\cos\left(\frac{15\pi}{12}\right) + j\sin\left(\frac{15\pi}{12}\right)$
6	$\cos\left(\frac{17\pi}{12}\right) + j\sin\left(\frac{17\pi}{12}\right)$

Note the following complex conjugate pairings:

$$\begin{aligned} s_4 &= s_3 * \\ s_5 &= s_2 * \\ s_6 &= s_1 * \end{aligned} \tag{13}$$

Apply the poles into equation (13).

$$H_1(s) = \frac{1}{(s - s_1)(s - s_6)} \tag{14}$$

$$H_1(s) = \frac{1}{(s - s_1)(s - s_1 *)} \tag{15}$$

$$H_1(s) = \frac{1}{\left\{s - \left[\cos\left(\frac{7\pi}{12}\right) + j\sin\left(\frac{7\pi}{12}\right)\right]\right\}\left\{s - \left[\cos\left(\frac{7\pi}{12}\right) - j\sin\left(\frac{7\pi}{12}\right)\right]\right\}} \tag{16}$$

$$H_1(s) = \frac{1}{\left\{s - \cos\left(\frac{7\pi}{12}\right) - j\sin\left(\frac{7\pi}{12}\right)\right\}\left\{s - \cos\left(\frac{7\pi}{12}\right) + j\sin\left(\frac{7\pi}{12}\right)\right\}} \tag{17}$$

$$H_1(s) = \frac{1}{\left\{s - \cos\left(\frac{7\pi}{12}\right)\right\}^2 + \left\{\sin\left(\frac{7\pi}{12}\right)\right\}^2} \tag{18}$$

$$H_1(s) = \frac{1}{s^2 - 2\cos\left(\frac{7\pi}{12}\right)s + 1} \tag{19}$$

Similarly,

$$H_2(s) = \frac{1}{s^2 - 2\cos\left(\frac{9\pi}{12}\right)s + 1} \tag{20}$$

$$H_3(s) = \frac{1}{s^2 - 2\cos\left(\frac{11\pi}{12}\right)s + 1} \tag{21}$$

Normalized Frequency Parameter

Now consider a generic stage.

$$H_g(s) = \frac{1}{s^2 - \alpha s + 1} \tag{22}$$

Define a frequency parameter Ω_c as

$$\Omega_c = \tan(\pi f_0 T) \tag{23}$$

Note that T is the time segment duration. It is thus the inverse of the sampling rate. Furthermore, f_0 is the filter cutoff frequency.

Apply the frequency parameter to the generic transfer function.

$$\hat{H}_g(s) = H_g(s)\Big|_{s=s/\Omega_c} = H_g\left(\frac{s}{\Omega_c}\right) \tag{24}$$

$$\hat{H}_g(s) = \frac{1}{\left(\frac{s}{\Omega_c}\right)^2 - \alpha\left(\frac{s}{\Omega_c}\right) + 1} \tag{25}$$

$$\hat{H}_g(s) = \frac{\Omega_c{}^2}{s^2 - \alpha\Omega_c s + \Omega_c} \tag{26}$$

Z-transform of Sixth-Order Lowpass Butterworth Filter

The bilinear transform is defined by

$$s = \frac{z-1}{z+1} \tag{27}$$

The purpose of this function is to transform an analog filter into the z-domain. The frequency transformation in equation (23) actually follows from the bilinear transformation in equation (27). The derivation is given in Appendix A.

Substitute the bilinear transform into the transfer function in equation (26).

$$\hat{H}_g(s) = \frac{\Omega_c{}^2}{\left[\frac{z-1}{z+1}\right]^2 - \alpha\Omega_c\left[\frac{z-1}{z+1}\right] + \Omega_c} \tag{28}$$

$$\hat{H}_g(s) = \frac{\Omega_c{}^2[z+1]^2}{[z-1]^2 - \alpha\Omega_c[z-1][z+1] + \Omega_c[z+1]^2} \tag{29}$$

$$\hat{H}_g(s) = \frac{\Omega_c{}^2\left[z^2 + 2z + 1\right]}{\left[z^2 - 2z + 1\right] - \alpha\Omega_c\left[z^2 - 1\right] + \Omega_c\left[z^2 + 2z + 1\right]} \tag{30}$$

$$\hat{H}_g(s) = \frac{\Omega_c{}^2 z^2 + 2\Omega_c{}^2 z + \Omega_c{}^2}{z^2 - 2z + 1 - \alpha\Omega_c z^2 + \alpha\Omega_c + \Omega_c{}^2 z^2 + 2\Omega_c{}^2 z + \Omega_c{}^2} \tag{31}$$

$$\hat{H}_g(s) = \frac{\Omega_c{}^2 z^2 + 2\Omega_c{}^2 z + \Omega_c{}^2}{\left[-\alpha\Omega_c + \Omega_c{}^2 + 1\right]z^2 + \left[+2\Omega_c{}^2 - 2\right]z + \left[\Omega_c{}^2 + \alpha\Omega_c + 1\right]} \tag{32}$$

$$\hat{H}_g(s) = \frac{\Omega_c{}^2 z^2 + 2\Omega_c{}^2 z + \Omega_c{}^2}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]z^2 + 2\left[\Omega_c{}^2 - 1\right]z + \left[\Omega_c{}^2 + \alpha\Omega_c + 1\right]} \tag{33}$$

$$\hat{H}_g(s) = \frac{\left\{\frac{\Omega_c{}^2}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]}\right\}z^2 + \left\{\frac{2\Omega_c{}^2}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]}\right\}z + \left\{\frac{\Omega_c{}^2}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]}\right\}}{z^2 + \left\{\frac{2\left[\Omega_c{}^2 - 1\right]}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]}\right\}z + \left\{\frac{\left[\Omega_c{}^2 + \alpha\Omega_c + 1\right]}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]}\right\}} \tag{34}$$

Recall equation (3).

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_L z^{-L}}{1 + a_1 z^{-1} + \cdots + a_L z^{-L}} \tag{35}$$

Set L=2.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{36}$$

Multiply through by z^2,

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \tag{37}$$

$$b_0 = \frac{\Omega_c{}^2}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]} \tag{38}$$

$$b_1 = \frac{2\Omega_c^2}{\left[\Omega_c^2 - \alpha\Omega_c + 1\right]} \tag{39}$$

$$b_2 = b_0 \tag{40}$$

$$a_1 = \frac{2\left[\Omega_c^2 - 1\right]}{\left[\Omega_c^2 - \alpha\Omega_c + 1\right]} \tag{41}$$

$$a_2 = \frac{\left[\Omega_c^2 + \alpha\Omega_c + 1\right]}{\left[\Omega_c^2 - \alpha\Omega_c + 1\right]} \tag{42}$$

The coefficients can be inserted into equation (7). The resulting recursive equation for a filter section is

$$y_k = \left[b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2}\right] - \left[a_1 y_{k-1} + a_2 y_{k-2}\right] \tag{43}$$

Equation (43) represents one of three cascade stages for a sixth-order filter. Note that there is a unique set of coefficients for each of these stages.

Equation (43) is applied six times if refiltering is used for phase correction, again assuming a sixth-order filter.

Highpass Butterworth Filter

Recall the generic transfer function for a lowpass filter stage

$$H_g(s) = \frac{1}{s^2 - \alpha s + 1} \tag{44}$$

The lowpass filter H can be transformed into a highpass filter J by changing s to 1/s.

$$J_g(s) = \frac{1}{\left(\frac{1}{s^2}\right) - \left(\frac{\alpha}{s}\right) + 1} \tag{45}$$

$$J_g(s) = \frac{s^2}{1 - \alpha s + s^2} \qquad (46)$$

$$J_g(s) = \frac{\left(\frac{s}{\Omega_c}\right)^2}{1 - \alpha\left(\frac{s}{\Omega_c}\right) + \left(\frac{s}{\Omega_c}\right)^2} \qquad (47)$$

$$J_g(s) = \frac{s^2}{\Omega_c{}^2 - \alpha\Omega_c s + s^2} \qquad (48)$$

Recall the bilinear transform

$$s = \frac{z-1}{z+1} \qquad (49)$$

By substitution,

$$\hat{J}_g(s) = \frac{\left[\frac{z-1}{z+1}\right]^2}{(\Omega_c)^2 - \alpha\Omega_c\left[\frac{z-1}{z+1}\right] + \left[\frac{z-1}{z+1}\right]^2} \qquad (50)$$

$$\hat{J}_g(s) = \frac{[z-1]^2}{\Omega_c{}^2[z+1]^2 - \alpha\Omega_c[z+1][z-1] + [z-1]^2} \qquad (51)$$

$$\hat{J}_g(s) = \frac{\left[z^2 - 2z + 1\right]}{\Omega_c{}^2\left[z^2 + 2z + 1\right] - \alpha\Omega_c\left[z^2 - 1\right] + \left[z^2 - 2z + 1\right]} \qquad (52)$$

$$\hat{J}_g(s) = \frac{\left[z^2 - 2z + 1\right]}{\left[\Omega_c{}^2 - \alpha\Omega_c + 1\right]z^2 + \left[2\Omega_c - 2\right]z + \left[\Omega_c{}^2 + \alpha\Omega_c + 1\right]} \tag{53}$$

$$\hat{J}_g(s) = \frac{\left[\left\{\frac{1}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]}\right\}z^2 + \left\{\frac{-2}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]}\right\}z + \left\{\frac{1}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]}\right\}\right]}{z^2 + \left\{\frac{\left[2\Omega_c{}^2 - 2\right]}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]}\right\}z + \left\{\frac{\left[1 + \alpha\Omega_c + \Omega_c{}^2\right]}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]}\right\}} \tag{54}$$

Recall the z-transform

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \tag{55}$$

Thus the highpass filter coefficients are thus

$$a_1 = \frac{2\left[-1 + \Omega_c{}^2\right]}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]} \tag{56}$$

$$a_2 = \frac{\left[1 + \alpha\Omega_c + \Omega_c{}^2\right]}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]} \tag{57}$$

$$b_0 = \frac{1}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]} \tag{58}$$

$$b_1 = \frac{-2}{\left[1 - \alpha\Omega_c + \Omega_c{}^2\right]} \tag{59}$$

$$b_1 = -2\, b_0 \tag{60}$$

$$b_2 = b_0 \tag{61}$$

Again, the coefficients are inserted into the following filter equation for an individual stage

$$y_k = \left[b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2}\right] - \left[a_1 y_{k-1} + a_2 y_{k-2}\right] \tag{62}$$

Equation (62) represents one of three cascade stages for a sixth-order filter. Note that there is a unique set of coefficients for each of these stages.

Numerical Stability

The following criteria are taken from Reference 4. The criteria apply to a cascade filter implementation. Note that the stability of each stage must be evaluated separately.

The filter tends to become unstable if the product $\left(f_0 T\right)$ is very small. This comes about because the filter weights require more digits for a very small $\left(f_0 T\right)$ product.

Recall the z-transform

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \tag{63}$$

Define stability coordinates

$$(x, y) = (-a1, -a2) \tag{64}$$

The coordinates must fall inside the triangle shown in Figure 5. The coordinate (0.6, -0.4) is used as an example in Figure 5.

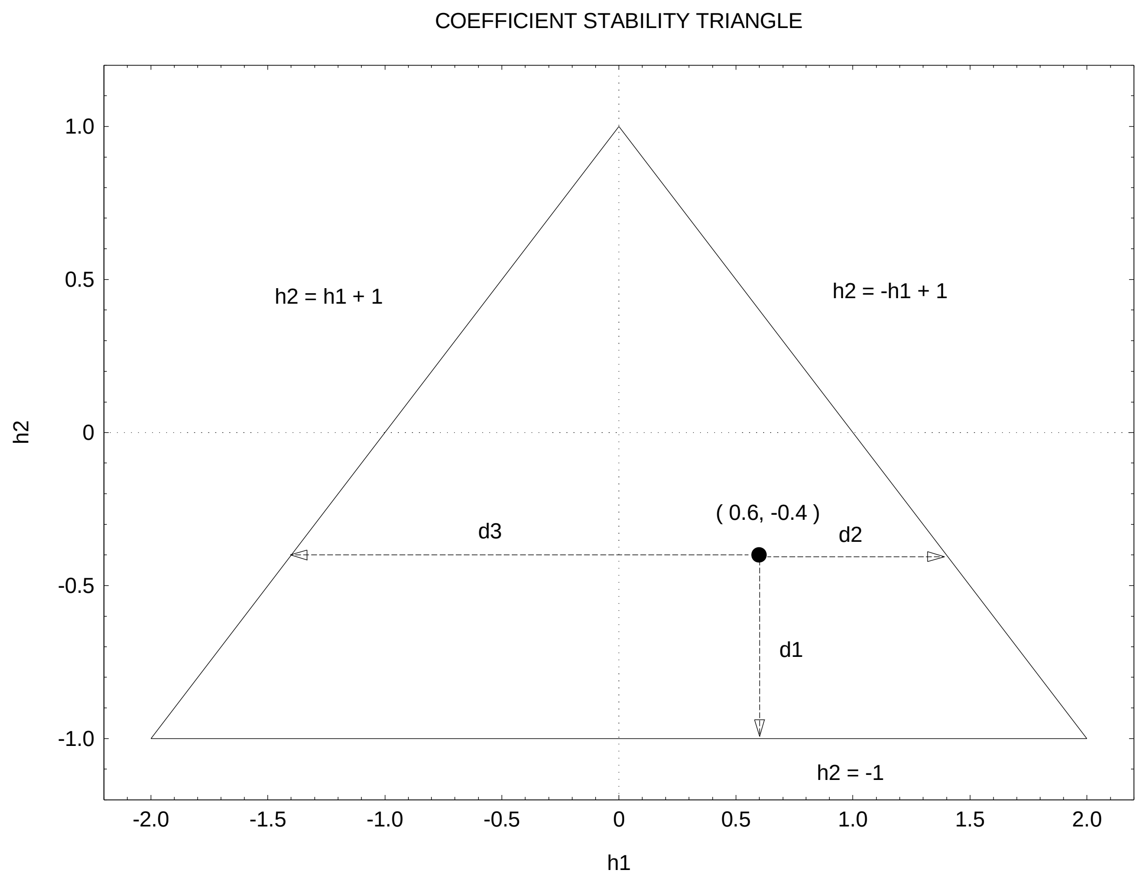

Figure 5. Coefficient Stability Triangle

The filter is stable is the coordinate pair is within the triangle. The filter becomes unstable if the pair is on a border or outside the triangle.

Three distance parameters are shown in Figure 5. For this example,

d1 = 0.6
d2 = 0.8
d3 = 2.0

The key distance is the smallest of the three values, denoted as d. For this example, d = 0.6.

The stability criteria for d are shown in Table 2.

Table 2. Stability Criteria	
Size of d	Stability Comment
$d > 5.0(10^{-6})$	Good Stability
$5.0(10^{-6}) \geq d > 0$	Marginally Unstable
$d \leq 0$	Unstable

Actually, the criteria in Table 2 apply to computers with word lengths greater than 32 bit. The $5.0(10^{-6})$ threshold should be increased for computers with smaller word lengths.

Also note that the cutoff frequency must be less than the Nyquist frequency, which is one-half of the sampling rate. The filter becomes unstable in the limiting case where the cutoff frequency is equal to the Nyquist frequency.

Time History Example

A synthesized time history is shown in Figure 6.

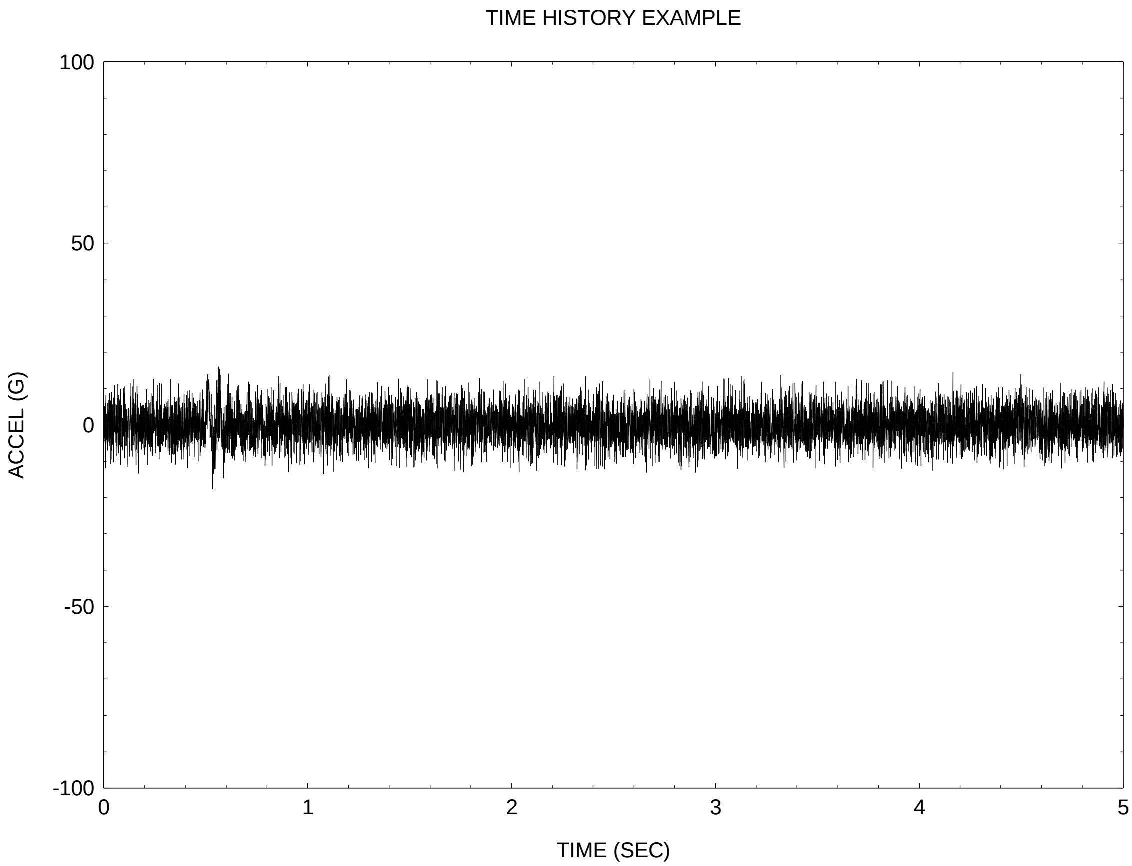

Figure 6. Synthesized Time History, Raw Data

The sample rate is 2000 samples per second. The time increment is thus 0.0005 seconds. The signal is largely white noise, but a possible transient event appears near 0.6 seconds.

The raw data was lowpass filtered at 30 Hz using a 6th order Butterworth filter with refiltering for phase correction. The resulting time history clarifies the transient, as shown in Figure 7.

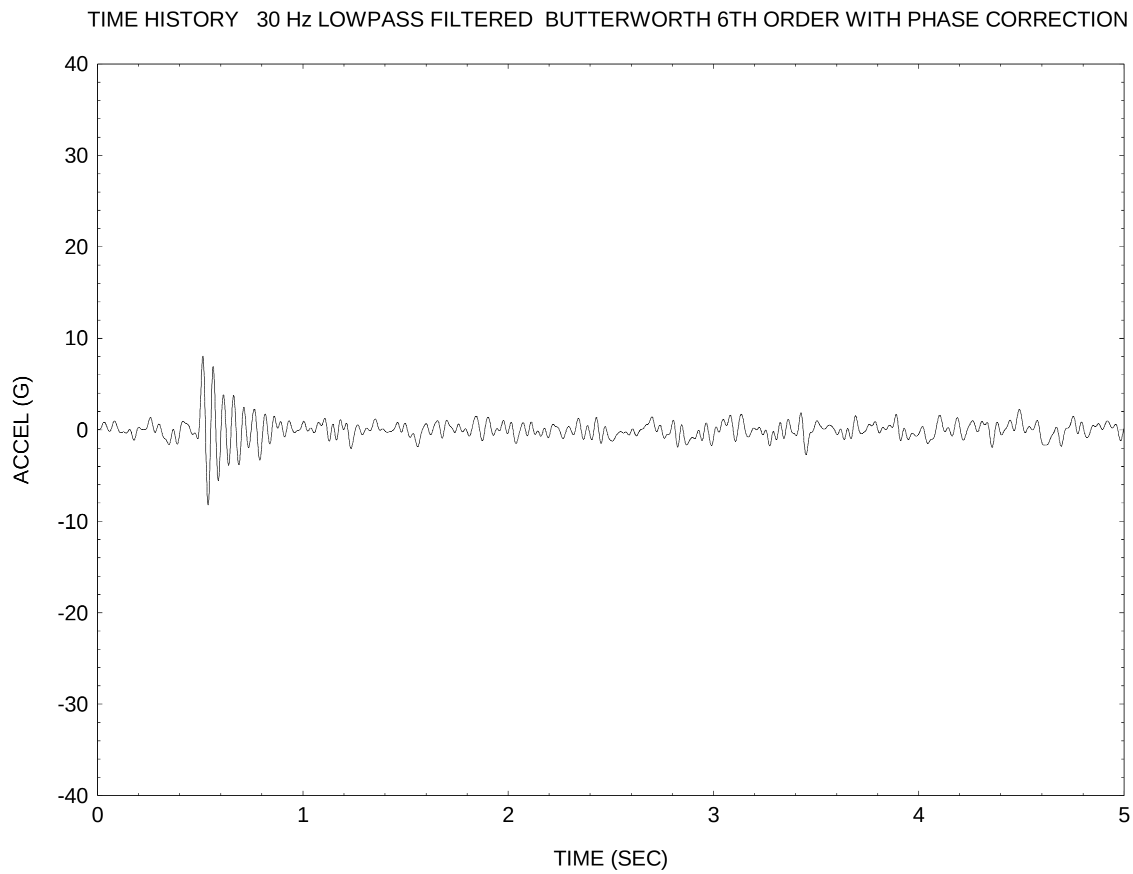

Figure 7. Filtered Time History

The transient is a 20 Hz sinusoid with 5% damping.

The digital filter transfer function is shown in Figure 8. The transfer function includes the refiltering effects.

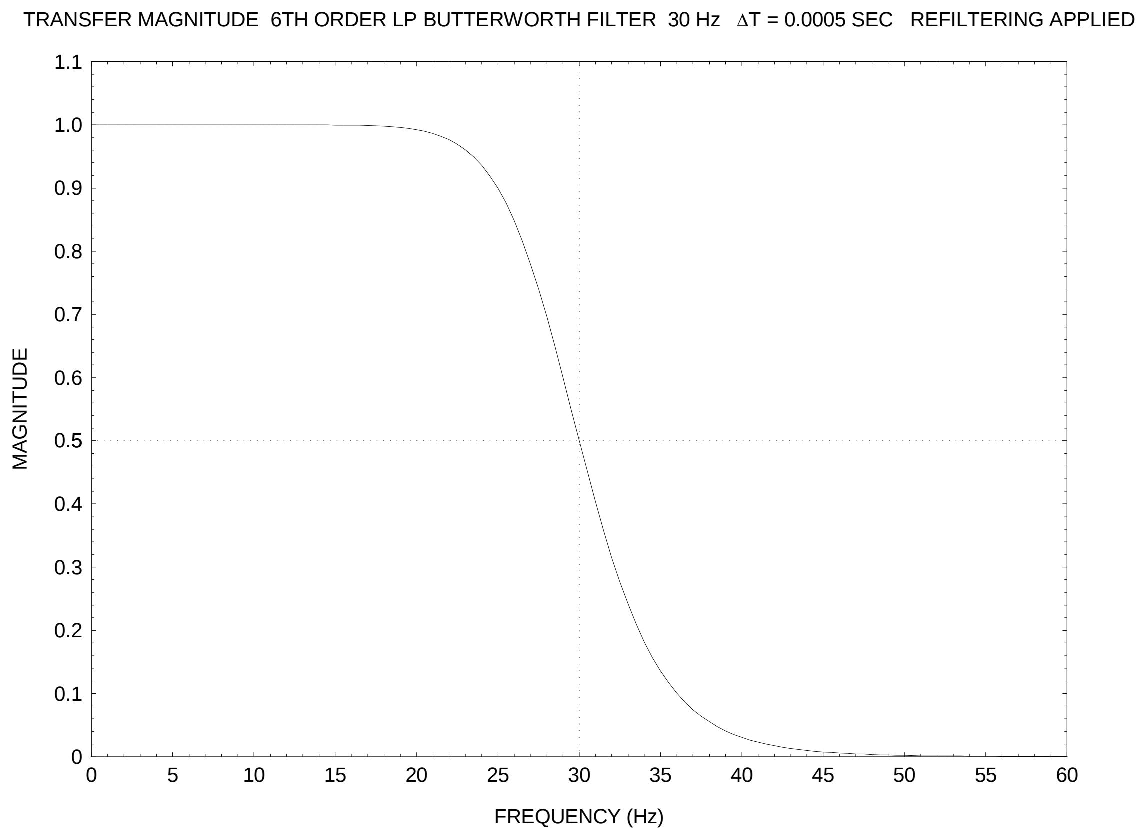

Figure 8. Transfer Magnitude

Note that refiltering decreases the -3 dB point to -6 dB at the cutoff frequency of 30 Hz.

The poles are those shown in Table 1. The filter coefficients are shown in Table 3.

Table 3. Filter Coefficients

Stage	Denominator	Numerator
1	a(1)= -1.943779 a(2)= .9524443	b(0)= .2166255E-02 b(1)= .4332509E-02 b(2)= .2166255E-02
2	a(1)= -1.866892 a(2)= .8752146	b(0)= .2080568E-02 b(1)= .4161135E-02 b(2)= .2080568E-02
3	a(1)= -1.825209 a(2)= .8333458	b(0)= .2034114E-02 b(1)= .4068227E-02 b(2)= .2034114E-02

The stability results are shown in Figure 9. The plot is focused on the lower right corner of the triangle. Each of the three coordinates is close to the right leg of the triangle. Nevertheless, each coordinate is sufficiently inside the boundary to yield good stability.

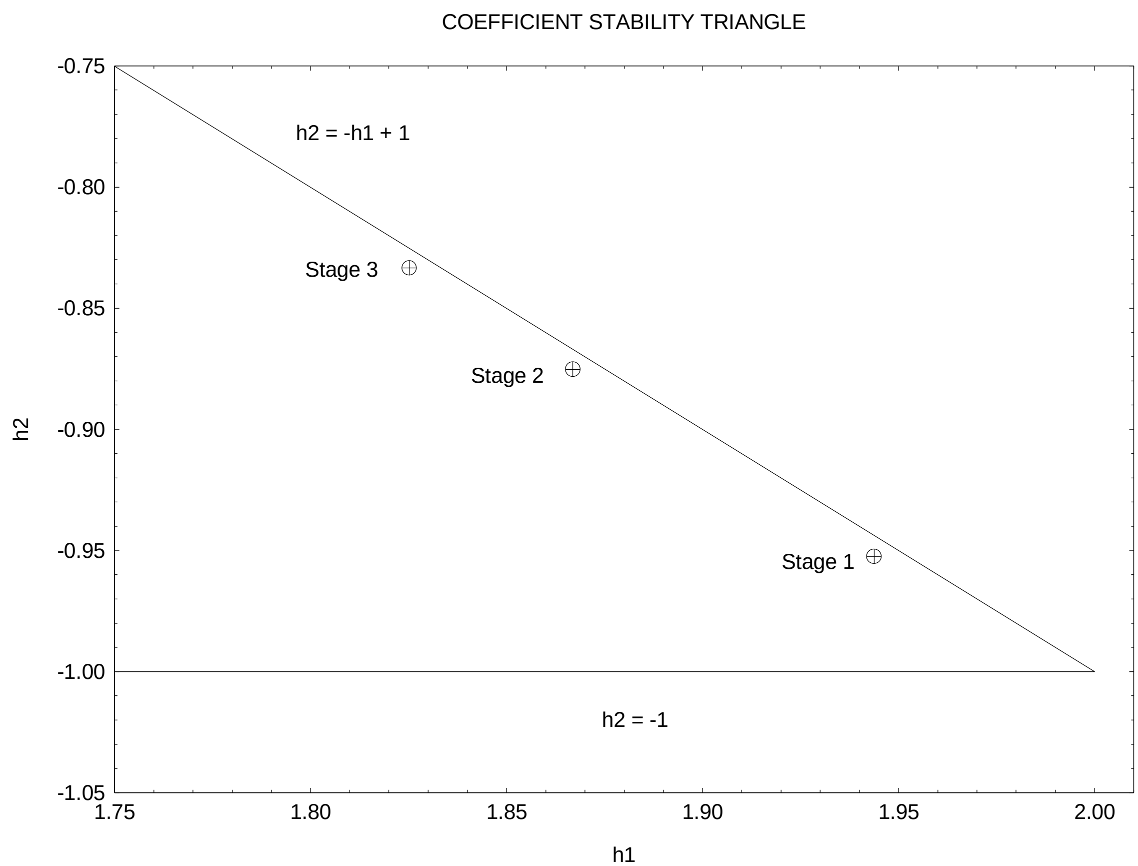

Figure 9. Stability Results

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3. T. Irvine, Notes on Refiltering for Phase Correction, 1998.
4. MAC/RAN IV Applications Manual, Revision 2, University Software Systems, Los Angeles, California, 1991.

APPENDIX A

Consider the points at $s = j\,\Omega_0$ on the s-plane and $z = \exp\left[j\omega_0\right]$ on the z-plane.

By substitution in to equation (27),

$$j\Omega_0 = \frac{\exp\left[j\omega_0\right] - 1}{\exp\left[j\omega_0\right] + 1} \tag{A-1}$$

$$j\Omega_0 = \frac{\cos\left[\omega_0\right] + j\sin\left[\omega_0\right] - 1}{\cos\left[\omega_0\right] + j\sin\left[\omega_0\right] + 1} \tag{A-2}$$

$$\Omega_0 = \frac{\cos\left[\omega_0\right] + j\sin\left[\omega_0\right] - 1}{j\cos\left[\omega_0\right] - \sin\left[\omega_0\right] + j} \tag{A-3}$$

$$\Omega_0 = \frac{\cos\left[\omega_0\right] - 1 + j\sin\left[\omega_0\right]}{-\sin\left[\omega_0\right] + j\left\{\cos\left[\omega_0\right] + 1\right\}} \tag{A-4}$$

$$\Omega_0 = \left\{\frac{\cos\left[\omega_0\right] - 1 + j\sin\left[\omega_0\right]}{-\sin\left[\omega_0\right] + j\left\{\cos\left[\omega_0\right] + 1\right\}}\right\}\left\{\frac{-\sin\left[\omega_0\right] - j\left\{\cos\left[\omega_0\right] + 1\right\}}{-\sin\left[\omega_0\right] - j\left\{\cos\left[\omega_0\right] + 1\right\}}\right\} \tag{A-5}$$

$$\Omega_0 = \left\{\frac{\left\{\cos\left[\omega_0\right] - 1\right\}\left\{-\sin\left[\omega_0\right] - j\left\{\cos\left[\omega_0\right] + 1\right\}\right\} + \left\{j\sin\left[\omega_0\right]\right\}\left\{-\sin\left[\omega_0\right] - j\left\{\cos\left[\omega_0\right] + 1\right\}\right\}}{\sin^2\left[\omega_0\right] + \left\{\cos\left[\omega_0\right] + 1\right\}^2}\right\} \tag{A-6}$$

$$\Omega_0 = \left\{\frac{\left\{-\cos\left[\omega_0\right]\sin\left[\omega_0\right] + \sin\left[\omega_0\right] - j\left\{\cos^2\left[\omega_0\right] - 1\right\}\right\} + \left\{\cos\left[\omega_0\right]\sin\left[\omega_0\right] + \sin\left[\omega_0\right] - j\sin^2\left[\omega_0\right]\right\}}{\sin^2\left[\omega_0\right] + \cos^2\left[\omega_0\right] + 2\cos\left[\omega_0\right] + 1}\right\} \tag{A-7}$$

$$\Omega_0 = \left\{ \frac{2\sin[\omega_0]}{2\cos[\omega_0] + 2} \right\} \tag{A-8}$$

$$\Omega_0 = \left\{ \frac{\sin[\omega_0]}{\cos[\omega_0] + 1} \right\} \tag{A-9}$$

$$\Omega_0 = \tan\left[\frac{\omega_0}{2} \right] \tag{A-10}$$

$$\Omega_0 = \tan[\pi f_0 T] \tag{A-11}$$