THE TIME-DOMAIN RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A UNIT STEP FORCE
Revision B

By Tom Irvine
Email: tomirvine@aol.com

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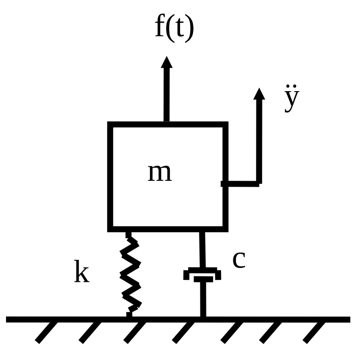
Consider a single-degree-of-freedom system.

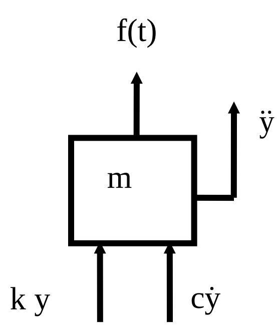


Where

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
y	=	displacement of the mass
f(t)	=	applied force

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{y} \tag{1}$$

$$m\ddot{y} = -c\dot{y} - ky + f(t) \tag{2}$$

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{3}$$

Divide through by m,

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right)y = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n{}^2 \tag{6}$$

where

ω_n is the natural frequency in (radians/sec)
ξ is the damping ratio

By substitution,

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}f(t) \tag{7}$$

Now apply a unit step force.

$$f(t) = u(t) \tag{8}$$

The governing equation becomes

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}u(t) \tag{9}$$

Now consider that the system undergoes oscillation with $\xi < 1$.
(Other damping cases are considered in the appendices.)

Take the Laplace transform of each side.

$$L\left\{\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y\right\} = L\left\{\frac{1}{m} u(t)\right\} \tag{10}$$

$$\begin{aligned} & s^2 Y(s) - s y(0) - y'(0) \\ & + 2\xi\omega_n s Y(s) - 2\xi\omega_n y(0) \\ & + \omega_n^2 Y(s) = \frac{1}{m s} \end{aligned} \tag{11}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) - \left\{s + 2\xi\omega_n\right\} y(0) - y'(0) = \frac{1}{m s} \tag{12}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) = \frac{1}{m s} + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \tag{13}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 - \left(\xi\omega_n\right)^2 + \omega_n^2 \tag{14}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_n^2\left(1 - \xi^2\right) \tag{15}$$

Let

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{16}$$

Substitute equation (16) into (15).

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{17}$$

$$\left\{(s + \xi\omega_n)^2 + \omega_d^2\right\} Y(s) = \frac{1}{ms} + \{s + 2\xi\omega_n\} y(0) + y'(0) \tag{18}$$

$Y(s) =$

$$\frac{1}{ms}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\}$$
$$+\left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0)$$
$$\tag{19}$$

Divide the right-hand-side of equation (19) into two parts.

$$Y(s) = Y_1(s) + Y_2(s) \tag{20}$$

where

$$Y_1(s) = \frac{1}{ms}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \tag{21}$$

$$Y_2(s) = \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0) \tag{22}$$

Consider $Y_1(s)$ from equation (21).

$$Y_1(s) = \frac{1}{m\,s}\left\{\frac{1}{(s+\xi\omega_n)^2 + \omega_d^{\,2}}\right\} \tag{23}$$

Expand into partial fractions using Reference 1.

$$\begin{aligned}\left\{\frac{1}{m\,s}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} &= \frac{1}{m\,\omega_n^2}\left\{\frac{1}{s}\right\} \\ &\quad - \left(\frac{1}{m\,\omega_n^2}\right)\left\{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^{\,2}}\right\} \\ &\quad - \left(\frac{1}{m\,\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\left\{\frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^{\,2}}\right\}\end{aligned} \tag{24}$$

The inverse Laplace transform per Reference 1 is

$$y_1(t) = \frac{1}{m\,\omega_n^2}\,u(t) - \frac{1}{m\,\omega_n^2}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right],\ t \ge 0 \tag{25}$$

$$y_1(t) = \frac{1}{m\,\omega_n^2}\left\{u(t) - \exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\},\ t \ge 0 \tag{26}$$

The inverse Laplace transform from Reference 2 for the natural response is

$$y_2(t) = [y(0)]\exp(-\xi\omega_n t)\cos(\omega_d t) + \left[\frac{\xi\omega_n y(0) + y'(0)}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{27}$$

$$\begin{aligned} y_2(t) &= y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ &\quad + y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \end{aligned} \tag{28}$$

The final displacement solution is obtained by adding equations (26) and (28).

$$y(t) = y_1(t) + y_2(t) \tag{29}$$

$$\begin{aligned} y(t) &= y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ &\quad + y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \\ &\quad + \frac{u(t)}{m\,\omega_n^2}\left\{1 - \exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\} \end{aligned} \tag{30}$$

$$y(t) = y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+ y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$

$$+ \frac{u(t)}{k}\left\{1 - \exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\}, \ \xi < 1$$

(31)

<u>References</u>

1. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Revision G, Vibrationdata, 2011.
2. T. Irvine, Table of Laplace Transforms, Revision I, Vibrationdata, 2011.
3. T. Irvine, Free Vibration of a Single-Degree-of-Freedom System, Revision B, Vibrationdata, 2005.

APPENDIX A

<u>Critically Damped Case</u>, $\xi = 1$

Recall equation (13).

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}Y(s) = \frac{1}{ms} + \left\{s + 2\xi\omega_n\right\}y(0) + y'(0) \tag{A-1}$$

By substitution,

$$\left\{s^2 + 2\omega_n s + \omega_n^2\right\}Y(s) = \frac{1}{ms} + \left\{s + 2\omega_n\right\}y(0) + y'(0) \tag{A-2}$$

$$\left(s+\omega_n\right)^2 Y(s) = \frac{1}{ms} + \{s+2\omega_n\}y(0) + y'(0) \tag{A-3}$$

$$Y(s) = \frac{1}{ms\left(s+\omega_n\right)^2} + \frac{\{s+2\omega_n\}y(0)}{\left(s+\omega_n\right)^2} + \frac{y'(0)}{\left(s+\omega_n\right)^2} \tag{A-4}$$

$$\begin{aligned} y(t) = {} & \frac{u(t)}{m\omega_n^2}\left\{1-\left[1+\omega_n t\right]\exp(-\omega_n t)\right\} \\ & + y(0)\left\{\left[1-(\omega_n t)\right]\exp(-\omega_n t) + 2\omega_n t\exp(-\omega_n t)\right\} + y'(0)\left[t\exp(-\omega_n t)\right] \end{aligned} \tag{A-5}$$

$$y(t) = \frac{u(t)}{k}\left\{1-\left[1+\omega_n t\right]\exp(-\omega_n t)\right\} + y(0)\left[1+(\omega_n t)\right]\exp(-\omega_n t) + y'(0)\left[t\exp(-\omega_n t)\right] \tag{A-6}$$

APPENDIX B

<u>Non-Oscillating Case</u>, $\xi > 1$

Recall

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) = \frac{1}{ms} + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \qquad \text{(B-1)}$$

$$Y(s) = \frac{1}{ms\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}} + \frac{\left\{s + 2\xi\omega_n\right\} y(0) + y'(0)}{\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}} \qquad \text{(B-2)}$$

The roots of the denominator are

$$s_{1,2} = \omega_n\left[-\xi \pm \sqrt{\xi^2 - 1}\right] \qquad \text{(B-3)}$$

Also note that

$$s_1 - s_2 = 2\omega_n\sqrt{\xi^2 - 1} \qquad \text{(B-4)}$$

$$Y(s) = \frac{1}{ms(s - s_1)(s - s_2)} + \frac{\left\{s + 2\xi\omega_n\right\} y(0) + y'(0)}{(s - s_1)(s - s_2)} \qquad \text{(B-5)}$$

Divide the right-hand-side of equation (B-5) into two parts.

$$Y(s) = Y_1(s) + Y_2(s) \qquad \text{(B-6)}$$

$$Y_1(s) = \frac{1}{m s (s - s_1)(s - s_2)} \tag{B-7}$$

$$Y_2(s) = \frac{\{s + 2\xi\omega_n\} y(0) + y'(0)}{(s - s_1)(s - s_2)} \tag{B-8}$$

$$Y_1(s) = \frac{1}{m\omega_n^2 s} - \frac{1}{m\omega_n^2}\left\{\frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} \tag{B-9}$$

$$Y_1(s) = \frac{1}{m\omega_n^2 s} - \frac{1}{m\omega_n^2}\left\{\frac{s + 2\xi\omega_n}{(s - s_1)(s - s_2)}\right\} \tag{B-10}$$

$$\begin{aligned} y_1(t) = \frac{u(t)}{m\omega_n^2} &- \frac{u(t)}{m\omega_n^2 (s_2 - s_1)}\{-s_1 \exp(s_1 t) + s_2 \exp(s_2 t)\} \\ &+ \frac{2\xi\omega_n u(t)}{m\omega_n^2 (s_2 - s_1)}\{\exp(s_1 t) - \exp(s_2 t)\} \end{aligned} \tag{B-11}$$

$$\begin{aligned} y_1(t) &= \frac{u(t)}{k} \\ &- \frac{u(t)}{k(s_2 - s_1)}\{-s_1 \exp(s_1 t) + s_2 \exp(s_2 t)\} \\ &+ \frac{2\xi\omega_n u(t)}{k(s_2 - s_1)}\{\exp(s_1 t) - \exp(s_2 t)\} \end{aligned} \tag{B-12}$$

$$y_1(t) = \frac{u(t)}{k}$$

$$+ \frac{u(t)}{k(s_2 - s_1)} \{(+ s_1 + 2\xi\omega_n)\} \exp(s_1 t)$$

$$+ \frac{u(t)}{k(s_2 - s_1)} \{(- s_2 - 2\xi\omega_n)\} \exp(s_2 t)$$

(B-13)

The natural response is

$$y_2(t) = \left\{ \frac{1}{2\omega_n \sqrt{\xi^2 - 1}} \right\} \left\{ A \exp\left[\left[-\xi + \sqrt{\xi^2 - 1} \right] \omega_n t \right] + B \exp\left[\left[-\xi - \sqrt{\xi^2 - 1} \right] \omega_n t \right] \right\}$$

(B-14)

where

$$A = y'(0) + \omega_n y(0) \left[\xi + \sqrt{\xi^2 - 1} \right]$$

$$B = - y'(0) + \omega_n y(0) \left[- \xi + \sqrt{\xi^2 - 1} \right]$$

The displacement is

$$y(t) = \quad y_1(t) + y_2(t) \qquad \text{(B-15)}$$