THE STEADY-STATE FREQUENCY RESPONSE FUNCTION OF A FOUR-DEGREE-OF-FREEDOM SYSTEM TO HARMONIC FORCE EXCITATION

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Introduction

The Frequency Response Function (FRF) method is demonstrated by an example. Consider the system in Figure 1.

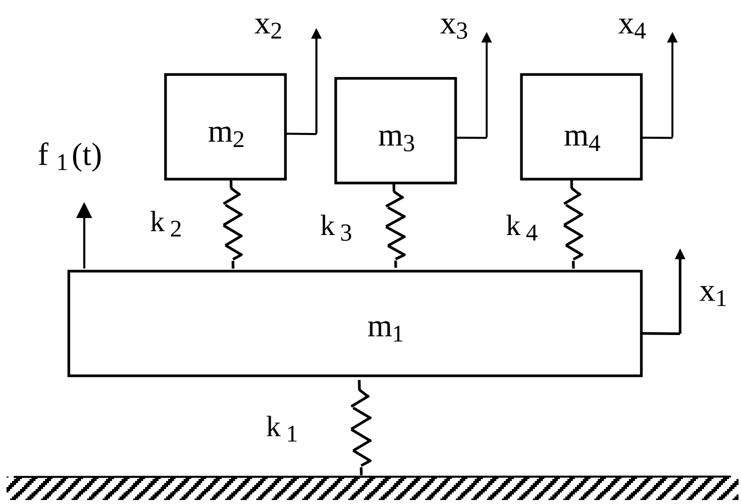

Figure 1.

The system also has damping, but it is modeled as modal damping.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

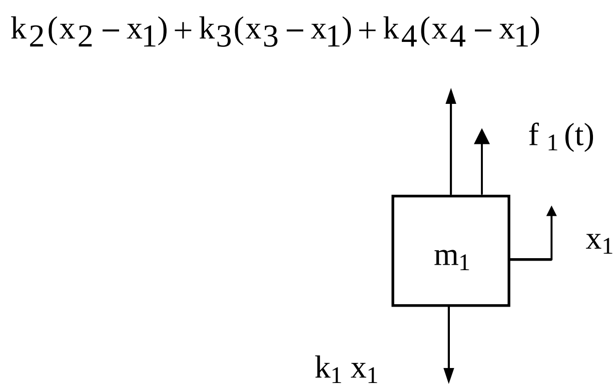

Figure 2.

Determine the equation of motion for mass 1.

$$\Sigma\, F = m_1\, \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = k_2(x_2 - x_1) + k_3(x_3 - x_1) + k_4(x_4 - x_1) - k_1 x_1 + f_1(t) \tag{2}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) - k_3(x_3 - x_1) - k_4(x_4 - x_1) = f_1(t) \tag{3}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_2 + k_3 x_1 - k_3 x_3 + k_4 x_1 - k_4 x_4 = f_1(t) \tag{4}$$

$$m_1 \ddot{x}_1 + \left(k_1 + k_2 + k_3 + k_4\right) x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4 = f_1(t) \tag{5}$$

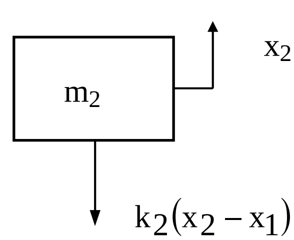

Figure 3.

Derive the equation of motion for mass 2.

$$\Sigma\, F = m_2\, \ddot{x}_2 \tag{6}$$

$$m_2\, \ddot{x}_2 = -k_2 (x_2 - x_1) \tag{7}$$

$$m_2\, \ddot{x}_2 + k_2\, x_2 - k_2 x_1 = 0 \tag{8}$$

Similarly,

$$m_3\, \ddot{x}_3 + k_3\, x_3 - k_3 x_1 = 0 \tag{9a}$$

$$m_4\, \ddot{x}_4 + k_4\, x_4 - k_4 x_1 = 0 \tag{9b}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & -k_2 & -k_3 & -k_4 \\ -k_2 & k_2 & 0 & 0 \\ -k_3 & 0 & k_3 & 0 \\ -k_4 & 0 & 0 & k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

<u>Decoupling</u>

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\, \bar{\ddot{x}} + K\, \bar{x} = \bar{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & -k_2 & -k_3 & -k_4 \\ -k_2 & k_2 & 0 & 0 \\ -k_3 & 0 & k_3 & 0 \\ -k_4 & 0 & 0 & k_4 \end{bmatrix} \tag{13}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{14}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\,\ddot{\bar{x}} + K\,\bar{x} = \bar{0} \tag{16}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(\,j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega\,\bar{q}\exp(\,j\omega t) \tag{18}$$

$$\ddot{\bar{x}} = -\omega^2\,\bar{q}\exp(\,j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M\,\bar{q}\exp(\,j\omega t) + K\,\bar{q}\exp(\,j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M\,\bar{q} + K\,\bar{q}\right\}\exp(\,j\omega t) = \bar{0} \tag{21}$$

$$\left\{-\omega^2 M \bar{q} + K\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{23}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{25}$$

$$\det\left\{\begin{bmatrix} k_1+k_2+k_3+k_4 & -k_2 & -k_3 & -k_4 \\ -k_2 & k_2 & 0 & 0 \\ -k_3 & 0 & k_3 & 0 \\ -k_4 & 0 & 0 & k_4 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\right\} = 0 \tag{26}$$

The resulting eigenvalues are ω_1 through ω_4.

The eigenvectors are found via the following equations.

$$\left\{K - \omega_i^{\,2} M\right\}\bar{q}_i = \bar{0}\ , \quad i = 1, 2, 3, 4 \tag{27}$$

where

$$\bar{q}_i = \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \\ q_{i4} \end{bmatrix}\ , \quad i = 1, 2, 3, 4 \tag{28}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\ \bar{q}_1 \ \ | \ \ \bar{q}_2 \ \ | \ \ \bar{q}_3 \ \ | \ \ \bar{q}_4\ ] \tag{29}$$

$$Q = \begin{bmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ q_{12} & q_{22} & q_{32} & q_{42} \\ q_{13} & q_{23} & q_{33} & q_{43} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix} \tag{30}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{31}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{32}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} & \hat{q}_{31} & \hat{q}_{41} \\ \hat{q}_{12} & \hat{q}_{22} & \hat{q}_{32} & \hat{q}_{42} \\ \hat{q}_{13} & \hat{q}_{23} & \hat{q}_{33} & \hat{q}_{43} \\ \hat{q}_{14} & \hat{q}_{24} & \hat{q}_{34} & \hat{q}_{44} \end{bmatrix} \tag{33a}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{13} & \hat{q}_{14} \\ \hat{q}_{21} & \hat{q}_{22} & \hat{q}_{23} & \hat{q}_{24} \\ \hat{q}_{31} & \hat{q}_{32} & \hat{q}_{33} & \hat{q}_{34} \\ \hat{q}_{41} & \hat{q}_{42} & \hat{q}_{43} & \hat{q}_{44} \end{bmatrix} \tag{33b}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 5 and 6.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\overline{x} = \hat{Q}\ \overline{\eta} \tag{34}$$

The displacement terms are

$$x_i = \hat{q}_{i1}\,\eta_1 + \hat{q}_{i2}\,\eta_2 + \hat{q}_{i3}\,\eta_3 + \hat{q}_{i4}\,\eta_4 \ , \quad i = 1, 2, 3, 4 \tag{35}$$

The velocity terms are

$$\dot{x}_i = \hat{q}_{i1}\,\dot{\eta}_1 + \hat{q}_{i2}\,\dot{\eta}_2 + \hat{q}_{i3}\,\dot{\eta}_3 + \hat{q}_{i4}\,\dot{\eta}_4 \tag{36}$$

The acceleration terms are

$$\ddot{x}_i = \hat{q}_{i1}\,\ddot{\eta}_1 + \hat{q}_{i2}\,\ddot{\eta}_2 + \hat{q}_{i3}\,\ddot{\eta}_3 + \hat{q}_{i4}\,\ddot{\eta}_4 \tag{37}$$

Substitute equation (34) into the equation of motion, equation (11).

$$M\hat{Q}\ \overline{\ddot{\eta}} + K\hat{Q}\ \overline{\eta} = \overline{F} \tag{38}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T\ M\hat{Q}\ \overline{\ddot{\eta}} + \hat{Q}^T K\hat{Q}\ \overline{\eta} = \hat{Q}^T\overline{F} \tag{39}$$

The orthogonality relationships yield

$$I\ \overline{\ddot{\eta}} + \Omega\ \overline{\eta} = \hat{Q}^T\overline{F} \tag{40}$$

For the sample problem, equation (40) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \\ \ddot{\eta}_4 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 \\ 0 & 0 & 0 & \omega_4^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{13} & \hat{q}_{14} \\ \hat{q}_{21} & \hat{q}_{22} & \hat{q}_{23} & \hat{q}_{24} \\ \hat{q}_{31} & \hat{q}_{32} & \hat{q}_{33} & \hat{q}_{34} \\ \hat{q}_{41} & \hat{q}_{42} & \hat{q}_{43} & \hat{q}_{44} \end{bmatrix} \begin{bmatrix} f_1(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

Note that the four equations are decoupled in terms of the modal coordinate.

Now assume modal damping by adding an uncoupled damping matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \\ \ddot{\eta}_4 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 & 0 & 0 \\ 0 & 2\xi_2\omega_2 & 0 & 0 \\ 0 & 0 & 2\xi_3\omega_3 & 0 \\ 0 & 0 & 0 & 2\xi_4\omega_4 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \end{bmatrix}$$

$$+ \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 \\ 0 & 0 & 0 & \omega_4^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{13} & \hat{q}_{14} \\ \hat{q}_{21} & \hat{q}_{22} & \hat{q}_{23} & \hat{q}_{24} \\ \hat{q}_{31} & \hat{q}_{32} & \hat{q}_{33} & \hat{q}_{34} \\ \hat{q}_{41} & \hat{q}_{42} & \hat{q}_{43} & \hat{q}_{44} \end{bmatrix} \begin{bmatrix} f_1(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

Equation (42) yields four equations

$$\ddot{\eta}_i + 2\xi_i\,\omega_i + \omega_i^2\,\eta_i = \hat{q}_{i1}\, f_1(t) \quad , \qquad i = 1, 2, 3, 4 \tag{43}$$

Now assume a harmonic base input.

$$f_1(t) = A \exp(j\omega t) \tag{44}$$

Assume a harmonic modal displacement at the same frequency as the applied force.

$$\eta_i = \psi_i \exp(j\omega t) \tag{45}$$

$$\dot{\eta}_i = j\,\omega_i\,\psi_i\ \exp(j\omega t) \tag{46}$$

$$\ddot{\eta}_i = -\omega_i^2\,\psi_i\ \exp(j\omega t) \tag{47}$$

By substitution,

$$\left\{-\omega^2 + j\,2\,\xi_i\,\omega_i\,\omega + \omega_i^2\right\}\psi_i \exp(j\omega t) = \hat{q}_{i1}\ A\exp(j\omega t) \tag{48}$$

$$\left\{\left[\omega_i^2 - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}\psi_i \exp(j\omega t) = \hat{q}_{i1}\ A\exp(j\omega t) \tag{49}$$

$$\eta_i = \psi_i \exp(j\omega t) = \frac{\hat{q}_{i1}\ A\exp(j\omega t)}{\left\{\left[\omega_i^2 - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}} \tag{50}$$

The modal velocity is

$$\dot{\eta}_i = j\frac{\omega\,\hat{q}_{i1}\ A\exp(j\omega t)}{\left\{\left[\omega_i^2 - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}} \tag{51}$$

The modal acceleration is

$$\ddot{\eta}_i = -\frac{\omega^2\,\hat{q}_{i1}\ A\exp(j\omega t)}{\left\{\left[\omega_i^2 - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}} \tag{52}$$

Recall

$$\ddot{x}_i = \hat{q}_{i1}\,\ddot{\eta}_1 + \hat{q}_{i2}\,\ddot{\eta}_2 + \hat{q}_{i3}\,\ddot{\eta}_3 + \hat{q}_{i4}\,\ddot{\eta}_4 \tag{53}$$

$$\ddot{x}_i(t) = \frac{\hat{q}_{i1}{}^2}{\left\{\left[\omega_1{}^2 - \omega^2\right] + j\,2\,\xi_1\,\omega_1\,\omega\right\}}\,\omega^2 A \exp(j\omega t)$$

$$+ \frac{\hat{q}_{i2}{}^2}{\left\{\left[\omega_2{}^2 - \omega^2\right] + j\,2\,\xi_2\,\omega_2\,\omega\right\}}\,\omega^2 A \exp(j\omega t)$$

$$+ \frac{\hat{q}_{i3}{}^2}{\left\{\left[\omega_3{}^2 - \omega^2\right] + j\,2\,\xi_3\,\omega_3\,\omega\right\}}\,\omega^2 A \exp(j\omega t)$$

$$+ \frac{\hat{q}_{i4}{}^2}{\left\{\left[\omega_4{}^2 - \omega^2\right] + j\,2\,\xi_4\,\omega_4\,\omega\right\}}\,\omega^2 A \exp(j\omega t)$$

(54)

The Fourier transform equation is

$$\hat{X}_i(f) = \int_{-\infty}^{\infty} \ddot{x}_i(t) \exp[-\,j\omega t]\,dt \qquad (55)$$

Take the Fourier transform of each side of equation (54).

$$\hat{X}_i(\omega) / F_1(\omega) = \frac{\hat{q}_{i1}{}^2\ \omega^2}{\left\{\left[\omega_1{}^2 - \omega^2\right] + j\,2\,\xi_1\,\omega_1\,\omega\right\}}$$

$$+ \frac{\hat{q}_{i2}{}^2\ \omega^2}{\left\{\left[\omega_2{}^2 - \omega^2\right] + j\,2\,\xi_2\,\omega_2\,\omega\right\}}$$

$$+ \frac{\hat{q}_{i3}{}^2\ \omega^2}{\left\{\left[\omega_3{}^2 - \omega^2\right] + j\,2\,\xi_3\,\omega_3\,\omega\right\}}$$

$$+ \frac{\hat{q}_{i4}{}^2\ \omega^2}{\left\{\left[\omega_4{}^2 - \omega^2\right] + j\,2\,\xi_4\,\omega_4\,\omega\right\}}$$

(56)

References

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2. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Revision A, Vibrationdata, 1999.

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5. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.

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APPENDIX A

EXAMPLE 1

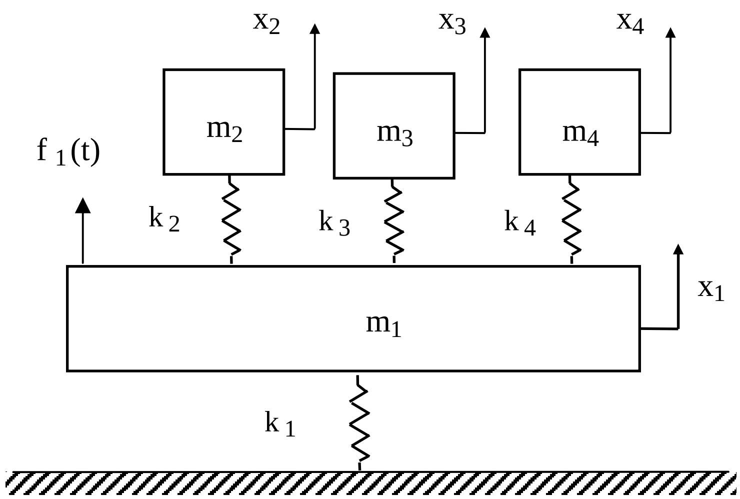

Figure A-1.

Consider the system in Figure A-1. Assign the values in Table A-1. The natural frequencies, mode shapes and frequency response function curves are found using Matlab script: four_dof_force_frf.m.

Table A-1. Parameters	
Variable	Value
m_1	5.0 kg
m_2	1.0 kg
m_3	2.0 kg
m_4	3.0 kg
k_1	500,000 N/m
k_2	200,000 N/m
k_3	250,000 N/m
k_4	300,000 N/m
$\| f_1 \|$	100 N

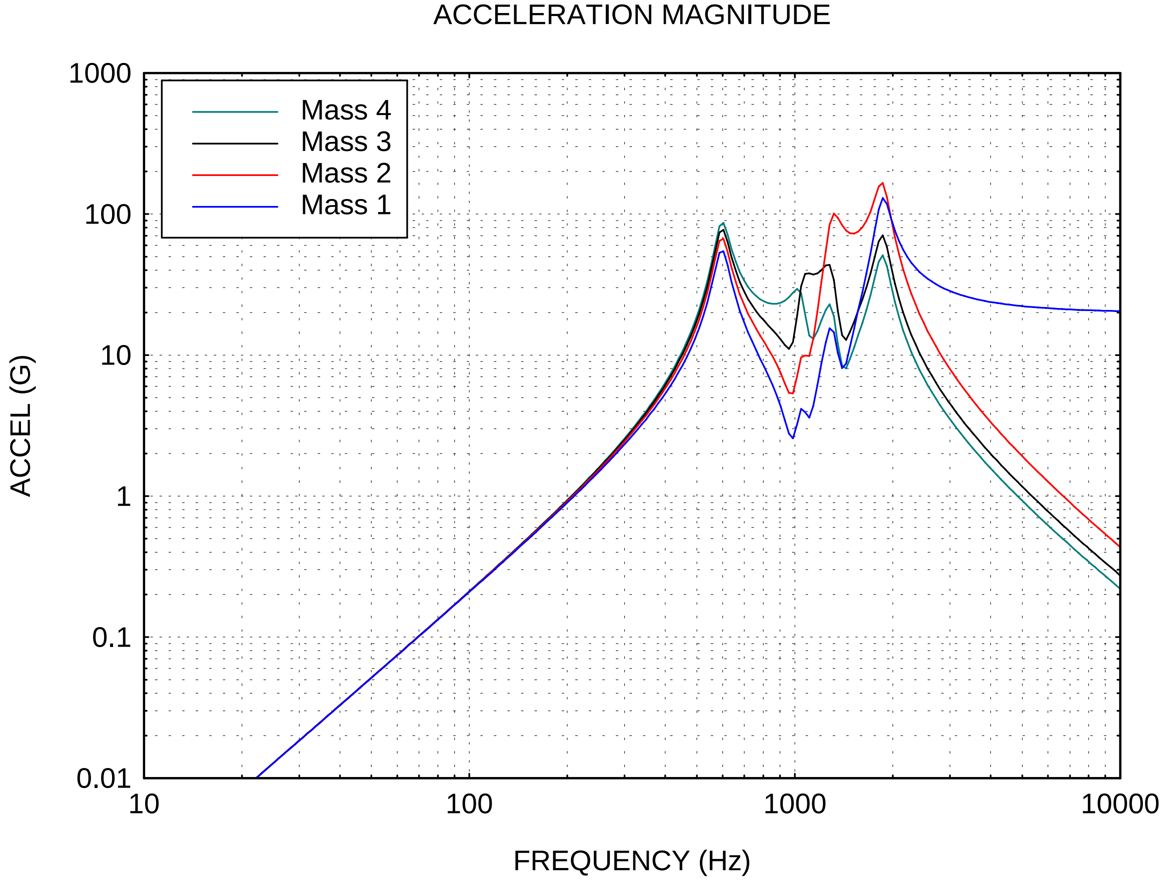

Figure A-2.

Peak Values

Mass 1:		Mass 3:	
604.1 Hz	54.42 G	604.1 Hz	77.44 G
1046 Hz	4.15 G	1108 Hz	37.98 G
1280 Hz	15.47 G	1280 Hz	43.72 G
1863 Hz	130.2 G	1863 Hz	70.77 G
Mass 2:		Mass 4:	
604.1 Hz	66.87 G	604.1 Hz	86.53 G
1076 Hz	9.918 G	1016 Hz	29.53 G
1318 Hz	100.8 G	1280 Hz	22.9 G
1863 Hz	166.1 G	1863 Hz	51.17 G

```
The mass matrix is

m =

    0.0130         0         0         0
         0    0.0026         0         0
         0         0    0.0052         0
         0         0         0    0.0078


 The stiffness matrix is

k =

     1250000     -200000     -250000     -300000
     -200000      200000           0           0
     -250000           0      250000           0
     -300000           0           0      300000


  Natural Frequencies =

     596.2 Hz
      1050 Hz
      1299 Hz
      1859 Hz

  Modes Shapes (column format) =

        4.647        0.9781          -2.23         -7.048
        5.679         2.244         -16.25          9.197
        6.552         10.04           5.86           3.86
        7.301        -7.627          3.073          2.783

 Participation Factors =

    0.1656
   0.01123
  -0.01674
  -0.02584
```

```
Effective Modal Mass =

   10.59    lbm
   0.0487   lbm
   0.1081   lbm
   0.2578   lbm

Total Modal Mass =      11.0000 lbm
```