A signal may be represented in terms of a Fourier series.

\[ f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \] (1)

The period of the signal is \( T \).

The coefficients are

\[ a_0 = \left(\frac{2}{T}\right) \int_{-T/2}^{T/2} f(t) \, dt \] (2)

\[ a_n = \left(\frac{2}{T}\right) \int_{-T/2}^{T/2} f(t) \cos\left(\frac{2\pi nt}{T}\right) \, dt \] (3)

\[ b_n = \left(\frac{2}{T}\right) \int_{-T/2}^{T/2} f(t) \sin\left(\frac{2\pi nt}{T}\right) \, dt \] (4)

Consider a rectangular wave with an infinite number of cycles. The one-cycle segment of the rectangular wave centered about \( t=0 \) is

\[ f(t) = \begin{cases} -1 & \text{for } -T/2 < t < 0 \\ 0 & \text{for } t = 0 \\ 1 & \text{for } 0 < t < T/2 \end{cases} \] (5)
The coefficients for a rectangular wave are

\[ a_n = -\left( \frac{2}{T} \right) \int_{-T/2}^{0} \cos\left( \frac{2\pi nt}{T} \right) dt + \left( \frac{2}{T} \right) \int_{0}^{T/2} \cos\left( \frac{2\pi nt}{T} \right) dt \]  

(6)

\[ a_n = -\frac{1}{\pi n} \sin\left( \frac{2\pi nt}{T} \right) \bigg|_{-T/2}^{0} + \frac{1}{\pi n} \sin\left( \frac{2\pi nt}{T} \right) \bigg|_{0}^{T/2} \]  

(7)

\[ a_n = 0 \]  

(8)

\[ b_n = -\left( \frac{2}{T} \right) \int_{-T/2}^{0} \sin\left( \frac{2\pi nt}{T} \right) dt + \left( \frac{2}{T} \right) \int_{0}^{T/2} \sin\left( \frac{2\pi nt}{T} \right) dt \]  

(9)

\[ b_n = \frac{1}{\pi n} \cos\left( \frac{2\pi nt}{T} \right) \bigg|_{-T/2}^{0} - \frac{1}{\pi n} \cos\left( \frac{2\pi nt}{T} \right) \bigg|_{0}^{T/2} \]  

(10)

\[ b_n = \frac{1}{\pi n} [1 - \cos(-n\pi)] - \frac{1}{\pi n} [\cos(n\pi) - 1] \]  

(11)

\[ b_n = \frac{2}{\pi n} \left[ 1 - \cos(n\pi) \right] \]  

(12)

Thus, the Fourier transform for a rectangular wave is

\[ f(t) = 2 \sum_{n=1}^{N} \frac{1}{n\pi} \left[ 1 - \cos(n\pi) \right] \sin\left( \frac{2\pi nt}{T} \right), \]  

where \( N \to \infty \)  

(13)

Let \( T=1 \) seconds. Equation (13) is plotted for several \( N \) values in Figure 1. It is plotted for \( N=10000 \) in Figure 2.
Figure 1.

Note: color plot.
Figure 2.
The first twenty spectral lines are shown. Note that the even components have an amplitude of zero.
Matlab Program

disp(' '); disp(' rectangular.m   ver 1.0  June 13, 2005'); disp(' by Tom Irvine'); disp(' '); disp(' This program calculates the Fourier series of a rectangular wave. '); disp('');
% clear all;
% disp(' Enter frequency (Hz) '); fn=input(' '); 
% disp(' Enter N limit '); LIMIT=input(' '); 
% num=4000; T=1/fn; dt=T/num; tpi=2.*pi;
% for(i=1:(num+1))
  t=(i-1)*dt; f(i)=0.; TT=tpi*t/T; 
  for(m=1:LIMIT) n=LIMIT-m+1; f(i)=f(i)+(1/n)*(1.-cos(n*pi))*sin(n*TT); end end
  time(i)=t; 
end f=2.*f/pi; 
% for(n=1:20) spectralf(n)=n/T; spectrala(n)=(2/(n*pi))*(1.-cos(n*pi)); end
% figure(1); plot(time,f); xlabel('Time(sec)')
ylabel('f(t)');
out1=sprintf('Rectangular Wave - Fourier Series N=%d',LIMIT);
title(out1);
grid on;
set(gca,'MinorGridLineStyle','none','GridLineStyle',':','XScale','lin','YScale','lin');

disp(' ');%disp(' Plot Fourier magnitude? ');
choice=input('1=yes 2=no ');
if(choice==1)
    figure(2);
    bar(spectralf,spectrala,0.4);
    title('Rectangular Wave - Fourier Magnitude ');
    xlabel('Frequency (Hz)');
    ylabel('Amplitude ');
end