FREE VIBRATION OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM Revision B

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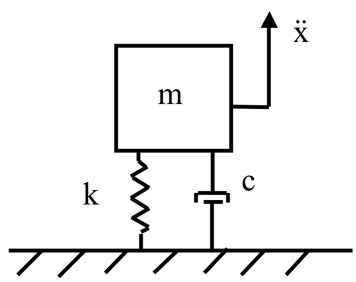
Derivation of the Equation of Motion

Consider a single-degree-of-freedom system.

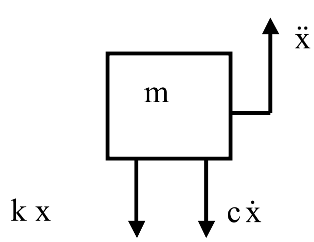


where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{A-1}$$

$$m\ddot{x} = -c\dot{x} - kx \tag{A-2}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{A-3}$$

Divide through by m,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0 \tag{A-4}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = {\omega_n}^2$$

where

ω_n is the natural frequency in (radians/sec),
ξ is the damping ratio.

By substitution,

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{A-5}$$

Now take the Laplace transform.

$$\mathcal{L}\left\{\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x\right\} = \mathcal{L}\{0\} \tag{A-6}$$

$$\begin{aligned} &s^2 X(s) - s\,x(0) - \dot{x}(0) \\ &+ 2\xi\omega_n s X(s) - 2\xi\omega_n x(0) \\ &+ {\omega_n}^2 X(s) \qquad\qquad\qquad = 0 \end{aligned} \tag{A-7}$$

$$\left\{s^2 + 2\xi\omega_n s + {\omega_n}^2\right\}X(s) + \{-1\}\dot{x}(0) + \left\{-s - 2\xi\omega_n\right\}x(0) = 0 \tag{A-8}$$

$$\left\{s^2 + 2\xi\omega_n s + {\omega_n}^2\right\}X(s) = \dot{x}(0) + \left\{s + 2\xi\omega_n\right\}x(0) \tag{A-9}$$

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad \text{(A-10)}$$

Consider the denominator of equation (A-10),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - (\xi\omega_n)^2 \quad \text{(A-11)}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2) \quad \text{(A-12)}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \quad \text{(A-13)}$$

Substitute equation (A-13) into (A-12),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \quad \text{(A-14)}$$

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad \text{(A-15)}$$

$$X(s) = \left\{ \frac{(s + \xi\omega_n)x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad \text{(A-16)}$$

$$X(s) = \left\{ \frac{(s + \xi\omega_n)x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d} \right\}\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad \text{(A-17)}$$

Oscillatory Motion

Now take the inverse Laplace transform using standard tables. Assume that $\xi < 1$. This case is referred to as oscillatory motion.

The resulting displacement is

$$x(t) = \exp(-\xi\omega_n t)\left\{[x(0)]\cos(\omega_d t) + \left[\frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d}\right]\sin(\omega_d t)\right\}, \quad \xi < 1$$

(B-1)

An alternate form is

$$x(t) = \left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\{\omega_d[x(0)]\cos(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\sin(\omega_d t)\}, \quad \xi < 1$$

(B-2)

The velocity is

$$\begin{aligned}\dot{x}(t) = &\left[\frac{-\xi\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\{\omega_d[x(0)]\cos(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\sin(\omega_d t)\} \\ &+ \exp(-\xi\omega_n t)\{-\omega_d[x(0)]\sin(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\cos(\omega_d t)\}, \quad \xi < 1\end{aligned}$$

(B-3)

$$\begin{aligned}\dot{x}(t) = &\exp(-\xi\omega_n t)\left\{-\xi\omega_n[x(0)]\cos(\omega_d t) + \left[\frac{-\xi\omega_n}{\omega_d}\right][\dot{x}(0) + (\xi\omega_n)x(0)]\sin(\omega_d t)\right\} \\ &+ \exp(-\xi\omega_n t)\{-\omega_d[x(0)]\sin(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\cos(\omega_d t)\}, \quad \xi < 1\end{aligned}$$

(B-4)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[-\omega_d[x(0)] + \left[\frac{-\xi\omega_n}{\omega_d}\right][\dot{x}(0) + (\xi\omega_n)x(0)]\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-5)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\left[-\omega_d + \frac{-\xi^2\ \omega_n{}^2}{\omega_d}\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-6)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[\left[-\omega_d{}^2 - \xi^2\omega_n{}^2\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-7)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[\left[-\omega_n{}^2\left(1-\xi^2\right) - \xi^2\omega_n{}^2\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-8)

$$\dot{x}(t) =$$

$$\exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[\left[-\omega_n{}^2 + \xi^2\omega_n{}^2 - \xi^2\omega_n{}^2\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-9)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[-\omega_n{}^2\ x(0)\ - \xi\omega_n\ \dot{x}(0)\right]\sin(\omega_d t)\right\}, \quad \xi < 1$$

(B-10)

Critically Damped Motion

Recall,

$$\omega_d = \omega_n \sqrt{1-\xi^2} \tag{C-1}$$

Consider the special case where

$$\xi = 1 \tag{C-2}$$

The damped natural frequency changes to

$$\omega_d = 0 \tag{C-3}$$

This case is referred to as critically damped motion. Substitute equations (C-2) and (C-3) into equation (A-16),

$$X(s) = \left\{ \frac{(s+\omega_n)x(0)}{(s+\omega_n)^2} \right\} + \left\{ \frac{\dot{x}(0)+\omega_n x(0)}{(s+\omega_n)^2} \right\} \tag{C-4}$$

$$X(s) = \left\{ \frac{x(0)}{s+\omega_n} \right\} + \left\{ \frac{\dot{x}(0)+\omega_n x(0)}{(s+\omega_n)^2} \right\} \tag{C-5}$$

The resulting displacement is found via an inverse Laplace transformation.

$$x(t) = \exp(-\omega_n t)\{ [x(0)] + [\dot{x}(0)+\omega_n x(0)]t \}, \quad \xi = 1 \tag{C-6}$$

Non-oscillatory Motion

Now consider the special case where

$$\xi > 1 \tag{D-1}$$

Recall equation (A-10), restated here as equation (D-2).

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \right\} \tag{D-2}$$

Solve for the roots of the denominator.

$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n{}^2}}{2} \tag{D-3}$$

$$s_{1,2} = \frac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi^2 - 1}}{2} \tag{D-4}$$

$$s_{1,2} = \omega_n\left[-\xi \pm \sqrt{\xi^2 - 1}\right] \tag{D-5}$$

Note that

$$s_1 - s_2 = \omega_n\left[-\xi + \sqrt{\xi^2 - 1}\right] - \omega_n\left[-\xi - \sqrt{\xi^2 - 1}\right] \tag{D-6}$$

$$s_1 - s_2 = \omega_n\left[-\xi + \sqrt{\xi^2 - 1}\right] + \omega_n\left[\xi + \sqrt{\xi^2 - 1}\right] \tag{D-7}$$

$$s_1 - s_2 = 2\omega_n\sqrt{\xi^2 - 1} \tag{D-8}$$

Equation (D-2) can be rewritten as

$$X(s)=\left\{\frac{\dot{x}(0)+\{s+2\xi\omega_n\}x(0)}{[s-s_1][s-s_2]}\right\} \tag{D-9}$$

$$X(s)=\left\{\frac{x(0)s+[\dot{x}(0)+2\xi\omega_n x(0)]}{[s-s_1][s-s_2]}\right\} \tag{D-10}$$

Equation (D-10) can be expanded in terms of partial fractions using the following equation from Reference 1.

$$\left\{\frac{\alpha s+\beta}{(s+\lambda)(s+\sigma)}\right\}=\left\{\frac{1}{\sigma-\lambda}\right\}\left\{\left[\frac{\beta-\alpha\lambda}{s+\lambda}\right]+\left[\frac{\alpha\sigma-\beta}{s+\sigma}\right]\right\} \tag{D-11}$$

The expansion is performed in equation (D-12).

$$\left\{\frac{x(0)s+[\dot{x}(0)+2\xi\omega_n x(0)]}{[s-s_1][s-s_2]}\right\}$$

$$=\left\{\frac{1}{-s_2+s_1}\right\}\left\{\left[\frac{[\dot{x}(0)+2\xi\omega_n x(0)]+x(0)s_1}{s-s_1}\right]+\left[\frac{-x(0)s_2-[\dot{x}(0)+2\xi\omega_n x(0)]}{s-s_2}\right]\right\}$$

(D-12)

$$X(s)=\left\{\frac{1}{-s_2+s_1}\right\}\left\{\left[\frac{[\dot{x}(0)+2\xi\omega_n x(0)]+x(0)s_1}{s-s_1}\right]+\left[\frac{-x(0)s_2-[\dot{x}(0)+2\xi\omega_n x(0)]}{s-s_2}\right]\right\}$$

(D-13)

Take the inverse Laplace transform.

$$x(t)=\left\{\frac{1}{-s_2+s_1}\right\}\{A\exp(s_1 t)+B\exp(s_2 t)\}$$

where

$$A=[\dot{x}(0)+2\xi\omega_n x(0)]+x(0)s_1$$
$$B=-x(0)s_2-[\dot{x}(0)+2\xi\omega_n x(0)]$$

(D-14)

Apply the appropriate terms to equation (D-14).

$$x(t)=\left\{\frac{1}{2\omega_n\sqrt{\xi^2-1}}\right\}\left\{A\exp\left[\omega_n\left[-\xi+\sqrt{\xi^2-1}\right]t\right]+B\exp\left[\omega_n\left[-\xi-\sqrt{\xi^2-1}\right]t\right]\right\}$$

where

$$A=[\dot{x}(0)+2\xi\omega_n x(0)]+x(0)\omega_n\left[-\xi+\sqrt{\xi^2-1}\right]$$
$$B=-x(0)\omega_n\left[-\xi-\sqrt{\xi^2-1}\right]-[\dot{x}(0)+2\xi\omega_n x(0)]$$

(D-15)

Simplify

$$x(t)=\left\{\frac{1}{2\omega_n\sqrt{\xi^2-1}}\right\}\left\{A\exp\left[\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n t\right]+B\exp\left[\left[-\xi-\sqrt{\xi^2-1}\right]\omega_n t\right]\right\}$$

where

$$A=\dot{x}(0)+2\xi\omega_n x(0)-\xi x(0)\omega_n+x(0)\omega_n\sqrt{\xi^2-1}$$
$$B=\xi x(0)\omega_n+x(0)\omega_n\sqrt{\xi^2-1}-\dot{x}(0)-2\xi\omega_n x(0)$$

(D-16)

Simplify again,

$$x(t) = \left\{ \frac{1}{2\omega_n \sqrt{\xi^2 - 1}} \right\} \left\{ A \exp\left[\left[-\xi + \sqrt{\xi^2 - 1} \right] \omega_n t \right] + B \exp\left[\left[-\xi - \sqrt{\xi^2 - 1} \right] \omega_n t \right] \right\}$$

where

$$A = \dot{x}(0) + \omega_n x(0) \left[\xi + \sqrt{\xi^2 - 1} \right]$$

$$B = -\dot{x}(0) + \omega_n x(0) \left[-\xi + \sqrt{\xi^2 - 1} \right]$$

(D-17)

Control Theory

The transfer function denominator forms the characteristic equation, when it is set to zero.

The roots of the characteristic equation are called poles and have a crucial importance. The system is stable if the real part of each root is negative.

The roots of the transfer function numerator are called the zeros.

Again, the transfer function for the single-degree-of-freedom subjected to free vibration is

$$X(s) = \left\{ \frac{\dot{x}(0) + \{ s + 2\xi\omega_n \} x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{E-1}$$

An alternative form is

$$X(s) = \left\{ \frac{\dot{x}(0) + \{ s + 2\xi\omega_n \} x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{E-2}$$

The characteristic equation is thus

$$(s+\xi\omega_n)^2+\omega_d{}^2=0 \tag{E-3}$$

The poles are thus

$$s=-\xi\,\omega_n \;\pm j\,\omega_d \tag{E-4}$$

Or

$$s=-\xi\,\omega_n \;\pm j\,\omega_n\sqrt{1-\xi^2} \tag{E-5}$$

The system is stable as long as $\xi\,\omega_n > 0$.

State Space Model

The governing second-order ODE can be reduced to a pair of first-order ODEs.

$$\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2 x=0 \tag{F-1}$$

Let

$$x_1=x \tag{F-2}$$

$$x_2=\dot{x}_1 \tag{F-3}$$

$$\dot{x}_2+2\xi\omega_n x_2+\omega_n^2 x_1=0 \tag{F-4}$$

The resulting pairs are

$$\dot{x}_1=x_2 \tag{F-5}$$

$$\dot{x}_2=-2\xi\omega_n x_2-\omega_n^2 x_1 \tag{F-6}$$

The pair of equations can be expressed in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{F-7}$$

Solve for the eigenvalues of he coefficient matrix.

$$\det\begin{bmatrix} 0-\lambda & 1 \\ -\omega_n^2 & -2\xi\omega_n - \lambda \end{bmatrix} = 0 \tag{F-8}$$

$$\det\begin{bmatrix} -\lambda & 1 \\ -\omega_n^2 & -2\xi\omega_n - \lambda \end{bmatrix} = 0 \tag{F-9}$$

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n{}^2 = 0 \tag{F-10}$$

The eigenvalues are

$$\lambda = -\xi\,\omega_n \;\; \pm j\,\omega_d \tag{F-11}$$

The eigenvalues are the same as the poles.

The complete solution for equation (F-7) is given in Reference 2.

References

1. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata Publications, 1999.

2. T. Irvine, The State Space Method for Solving Shock and Vibration Problems, Vibrationdata Publications, 2005.