THE TIME-DOMAIN RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A SINUSOIDAL FORCE
Revision B

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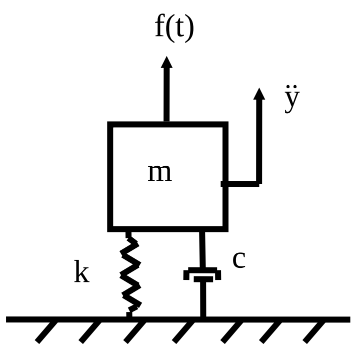
Consider a single-degree-of-freedom system.

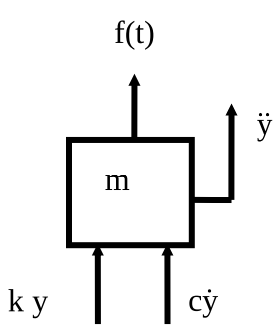


where

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
y	=	displacement of the mass
f(t)	=	applied force

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{y} \tag{1}$$

$$m\ddot{y} = -c\dot{y} - ky + f(t) \tag{2}$$

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{3}$$

Divide through by m,

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right)y = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n{}^2 \tag{6}$$

where

ω_n is the natural frequency in (radians/sec)
ξ is the damping ratio

By substitution,

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}f(t) \tag{7}$$

Now assume a sinusoidal force function.

$$f(t) = f_o \sin(\omega t) \tag{8}$$

The governing equation becomes.

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 x = \frac{1}{m}f_o \sin(\omega t) \tag{9}$$

The right-hand-side can be rewritten as

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{\omega_n^2}{k} f_o \sin(\omega t) \tag{10}$$

Take the Laplace transform of each side.

$$L\left\{ \ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y \right\} = L\left\{ \frac{\omega_n^2}{k} f_o \sin(\omega t) \right\} \tag{11}$$

$$\begin{aligned} & s^2 Y(s) - s y(0) - y'(0) \\ & + 2\xi\omega_n s Y(s) - 2\xi\omega_n y(0) \\ & + \omega_n^2 Y(s) = \frac{\omega_n^2}{k} f_o \left\{ \frac{\omega}{s^2 + \omega^2} \right\} \end{aligned} \tag{12}$$

$$\left\{ s^2 + 2\xi\omega_n s + \omega_n^2 \right\} Y(s) - \left\{ s + 2\xi\omega_n \right\} y(0) - y'(0) = \frac{\omega_n^2}{k} f_o \left\{ \frac{\omega}{s^2 + \omega^2} \right\} \tag{13}$$

$$\left\{ s^2 + 2\xi\omega_n s + \omega_n^2 \right\} Y(s) = \frac{\omega_n^2}{k} f_o \left\{ \frac{\omega}{s^2 + \omega^2} \right\} + \left\{ s + 2\xi\omega_n \right\} y(0) + y'(0) \tag{14}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 - (\xi\omega_n)^2 + \omega_n^2 \tag{15}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 \left(1 - \xi^2\right) \tag{16}$$

Let

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{17}$$

Substitute equation (17) into (16).

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{18}$$

$$\left\{(s + \xi\omega_n)^2 + \omega_d^2\right\} Y(s) = \frac{\omega_n^2}{k} f_o \left\{\frac{\omega}{s^2 + \omega^2}\right\} + \{s + 2\xi\omega_n\} y(0) + y'(0) \tag{19}$$

$Y(s) =$

$$\frac{\omega_n^2}{k} f_o \left\{\frac{\omega}{s^2 + \omega^2}\right\}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\}$$
$$+ \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0) \tag{20}$$

Divide the right-hand-side of equation (20) into two parts.

$$Y(s) = Y_1(s) + Y_2(s) \tag{21}$$

where

$$Y_1(s) = \frac{\omega_n^2}{k} f_o \left\{\frac{\omega}{s^2 + \omega^2}\right\}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \tag{22}$$

$$Y_2(s) = \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0) \tag{23}$$

Consider $Y_1(s)$ from equation (22).

$$Y_1(s) = \frac{\omega\,\omega_n^2}{k} f_o \left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{24}$$

Expand into partial fractions.

$$\left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} = \left\{ \frac{\lambda s + \rho}{s^2 + \omega^2} \right\} + \left\{ \frac{\sigma s + \phi}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{25}$$

Take the inverse Laplace transform of the first term on the right-hand-side of equation (25).

$$L^{-1}\left\{ \frac{\lambda s + \rho}{s^2 + \omega^2} \right\} = L^{-1}\left\{ \lambda\left[\frac{s}{s^2 + \omega^2} \right] + \rho\left[\frac{1}{s^2 + \omega^2} \right] \right\} \tag{26}$$

$$L^{-1}\left\{ \frac{\lambda s + \rho}{s^2 + \omega^2} \right\} = L^{-1}\left\{ \lambda\left[\frac{s}{s^2 + \omega^2} \right] + \left[\frac{\rho}{\omega} \right]\left[\frac{\omega}{s^2 + \omega^2} \right] \right\} \tag{27}$$

The inverse transform is obtained from standard tables.

$$L^{-1}\left\{ \frac{\lambda s + \rho}{s^2 + \omega^2} \right\} = \lambda \cos(\omega t) + \left[\frac{\rho}{\omega} \right] \sin(\omega t) \tag{28}$$

Now take the inverse Laplace transform of the second term on the right-hand-side of equation (25).

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=L^{-1}\left\{\frac{\sigma s}{(s+\xi\omega_n)^2+\omega_d^{\,2}}+\frac{\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}$$

(29)

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]+\left[\frac{\phi}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]\right\}$$

(30)

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]-[\sigma]\left[\frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]+\left[\frac{\phi}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]\right\}$$

(31)

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]-\left[\frac{\xi\omega_n\sigma}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]+\left[\frac{\phi}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]\right\}$$

(32)

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]\right\}$$

(33)

The inverse transform is obtained from standard tables.

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}= e^{-\xi\omega_n t}\left\{\sigma\cos(\omega_d t)+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\}$$

(34)

In summary,

$$L^{-1}\left\{\left[\frac{1}{s^2+\omega^2}\right]\left[\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

$$=\lambda\cos(\omega t)+\left[\frac{\rho}{\omega}\right]\sin(\omega t)+e^{-\xi\omega_n t}\left\{\sigma\cos(\omega_d t)+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\}$$

(35)

Assemble equations (28) and (35).

$$y_1(t) \quad =\left\{\frac{\omega\,\omega_n^2 f_o}{k}\right\}\left\{\lambda\cos(\omega t)+\left[\frac{\rho}{\omega}\right]\sin(\omega t)+e^{-\xi\omega_n t}\left\{\sigma\cos(\omega_d t)+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\}\right\}$$

(36)

Solve for the coefficients. Recall the partial fraction expansion.

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right\} \quad =\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^2}\right\} \tag{37}$$

Change the denominator to an equivalent form via equation (18).

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \quad =\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \tag{38}$$

$$1 = \{\lambda s + \rho\}\{s^2 + 2\xi\omega_n s + \omega_n^2\} + \{\sigma s + \phi\}\{s^2 + \omega^2\} \tag{39}$$

$$\begin{aligned} 1 &= \lambda s^3 + (\rho + 2\xi\omega_n \lambda) s^2 + (2\xi\omega_n \rho + \lambda\omega_n^2) s + (\rho\omega_n^2) \\ &\quad + \sigma s^3 + \phi s^2 + \sigma\omega^2 s + \phi\omega^2 \end{aligned} \tag{40}$$

$$\begin{aligned} 1 = \; & \\ & [\lambda + \sigma] s^3 \\ & + [\rho + 2\xi\omega_n \lambda + \phi] s^2 \\ & + [2\xi\omega_n \rho + \lambda\omega_n^2 + \sigma\omega^2] s \\ & + [\rho\omega_n^2 + \phi\omega^2] \end{aligned} \tag{41a}$$

Equation (41a) implies four separate equations.

$$\lambda + \sigma = 0 \tag{41b}$$

$$\rho + 2\xi\omega_n \lambda + \phi = 0 \tag{41c}$$

$$2\xi\omega_n \rho + \lambda\omega_n^2 + \sigma\omega^2 = 0 \tag{41d}$$

$$\rho\omega_n^2 + \phi\omega^2 = 1 \tag{41e}$$

Equations (41b) through (41e) can be assembled into matrix form.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2\xi\omega_n & 1 & 0 & 1 \\ \omega_n^2 & 2\xi\omega_n & \omega^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{42}$$

Gaussian elimination is used to simplify the coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2\xi\omega_n & 1 \\ 0 & 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{43}$$

Equation (43) can be reduced to a 3 x 3 matrix.

$$\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{44}$$

Complete the solution using Cramer’s rule.

$$\det\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \omega^2\left(\omega^2-\omega_n^2\right) + \omega^2(2\xi\omega_n)^2 - \omega_n^2\left(\omega^2-\omega_n^2\right) \tag{45}$$

$$\det\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2 \tag{46}$$

$$\rho = \frac{1}{\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \det\begin{bmatrix} 0 & -2\xi\omega_n & 1 \\ 0 & \omega^2-\omega_n^2 & 0 \\ 1 & 0 & \omega^2 \end{bmatrix} \tag{47}$$

$$\rho = \frac{-\left(\omega^2 - \omega_n^2\right)}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{48}$$

$$\sigma = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \det\begin{bmatrix} 1 & 0 & 1 \\ 2\xi\omega_n & 0 & 0 \\ \omega_n^2 & 1 & \omega^2 \end{bmatrix} \tag{49}$$

$$\sigma = \frac{2\xi\omega_n}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{50}$$

Recall equation (41b).

$$\lambda = -\sigma \tag{51}$$

$$\lambda = \frac{-2\xi\omega_n}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{52}$$

$$\phi = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \det\begin{bmatrix} 1 & -2\xi\omega_n & 0 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ \omega_n^2 & 0 & 1 \end{bmatrix} \tag{53}$$

$$\phi = \frac{\omega^2 - \omega_n^2 + (2\xi\omega_n)^2}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{54}$$

The coefficients are summarized in equation (55).

$$\begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \begin{bmatrix} -2\xi\omega_n \\ -\left(\omega^2 - \omega_n^2\right) \\ 2\xi\omega_n \\ \omega^2 - \omega_n^2 + (2\xi\omega_n)^2 \end{bmatrix} \tag{55}$$

Recall equation (23).

$$Y_2(s) = \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0) \tag{56}$$

$$Y_2(s) = \left\{\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{\left[\frac{\xi\omega_n}{\omega_d} y(0) + \frac{1}{\omega_d} y'(0)\right]\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \tag{57}$$

The inverse Laplace transform from standard tables is

$$y_2(t) = [y(0)]\exp(-\xi\omega_n t)\cos(\omega_d t) + \left[\frac{\xi\omega_n y(0) + y'(0)}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{58}$$

$$\begin{aligned} y_2(t) &= y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ &+ y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \end{aligned} \tag{59}$$

The final displacement solution is obtained by adding equations (36) and (59).

$$y(t) = y_1(t) + y_2(t) \tag{60}$$

$$\begin{aligned} y(t) = {} & y(0)\, e^{-\xi\omega_n t}\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ & + y'(0)\left[\frac{1}{\omega_d}\right] e^{-\xi\omega_n t}\sin(\omega_d t) \\ & + \left\{\frac{\omega\,\omega_n^2 f_o}{k}\right\}\left\{\lambda\cos(\omega t) + \left[\frac{\rho}{\omega}\right]\sin(\omega t)\right\} \\ & + \left\{\frac{\omega\,\omega_n^2 f_o}{k}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\sigma\cos(\omega_d t) + \left[\frac{\phi - \xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\} \end{aligned} \tag{61}$$

Simplify the expression.

$$\begin{aligned} y(t) = {} & y(0)\, e^{-\xi\omega_n t}\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ & + y'(0)\left[\frac{1}{\omega_d}\right] e^{-\xi\omega_n t}\sin(\omega_d t) \\ & + \left\{\frac{\omega_n^2 f_o}{k}\right\}\{\omega\lambda\cos(\omega t) + \rho\sin(\omega t)\} \\ & + \left\{\frac{\omega\,\omega_n^2 f_o}{\omega_d k}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\omega_d\sigma\cos(\omega_d t) + [\phi - \xi\omega_n\sigma]\sin(\omega_d t)\right\} \end{aligned} \tag{62}$$

Now consider the term.

$$\left[\phi - \xi\omega_n\sigma\right] = \frac{\omega^2 - \omega_n^2 + (2\xi\omega_n)^2 - 2(\xi\omega_n)^2}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{63}$$

$$\left[\phi - \xi\omega_n\sigma\right] = \frac{\omega^2 - \omega_n^2 + 2(\xi\omega_n)^2}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{64}$$

$$\left[\phi - \xi\omega_n\sigma\right] = \frac{\omega^2 + \omega_n^2\left[-1 + 2\xi^2\right]}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2} \tag{65}$$

Substitute the coefficients from equations (55) and (65) into equation (62). The displacement is

$$
\begin{aligned}
y(t) = {} & y(0)\, e^{-\xi\omega_n t}\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\
& + y'(0)\left[\frac{1}{\omega_d}\right] e^{-\xi\omega_n t}\sin(\omega_d t) \\
& + \frac{1}{k}\left\{\frac{\omega_n^{\,2} f_o}{\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\,\omega_n)^2}\right\}\left\{2\xi\omega_n\,\omega\cos(\omega t)-\left(\omega^2-\omega_n^2\right)\sin(\omega t)\right\} \\
& + \frac{1}{\omega_d k}\left\{\frac{\omega\,\omega_n^{\,2} f_o}{\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\,\omega_n)^2}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{2\xi\omega_n\omega_d\cos(\omega_d t)\right\} \\
& + \frac{1}{\omega_d k}\left\{\frac{\omega\,\omega_n^{\,2} f_o}{\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\,\omega_n)^2}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\left[\omega^2+\omega_n^2\left[-1+2\xi^2\right]\right]\sin(\omega_d t)\right\}
\end{aligned}
\tag{66}
$$

As an aside, consider the steady-state component, $y_{ss}(t)$.

$$
y_{ss}(t) = \left\{\frac{\omega_n^{\,2} f_o}{k\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\,\omega_n)^2\right]}\right\}\left\{2\xi\omega_n\,\omega\cos(\omega t)-\left(\omega^2-\omega_n^2\right)\sin(\omega t)\right\}
\tag{67}
$$

The magnitude is

$$|y_{ss}(t)| = \left\{ \frac{\omega_n^2 f_o}{k\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \right\} \sqrt{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \tag{68}$$

$$|y_{ss}(t)| = \left\{ \frac{\omega_n^2 f_o}{k} \right\} \frac{1}{\sqrt{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2}} \tag{69}$$

$$|y_{ss}(t)| = \left\{ \frac{\omega_n^2 f_o}{k} \right\} \frac{1}{\sqrt{\omega_n^4 \left\{ \left(-\left(\frac{\omega^2}{\omega_n^2} \right) + 1 \right)^2 + \left(2\xi \left(\frac{\omega}{\omega_n} \right) \right)^2 \right\}}} \tag{70}$$

$$|y_{ss}(t)| = \left\{ \frac{f_o}{k} \right\} \frac{1}{\sqrt{\left(1 - \left(\frac{\omega^2}{\omega_n^2} \right) \right)^2 + \left(2\xi \left(\frac{\omega}{\omega_n} \right) \right)^2}} \tag{71}$$

The steady-state magnitude is

$$|y_{ss}(t)| = \left\{ \frac{f_o}{k} \right\} \frac{1}{\sqrt{\left(1 - \rho^2\right)^2 + (2\xi\rho)^2}} \tag{72}$$

where

$$\rho = \frac{\omega}{\omega_n} \tag{73}$$

Equation (72) is also the transfer function magnitude.

Furthermore, equation (67) implies a phase angle relationship.

$$\theta = \arctan\left[\frac{2\xi\omega_n\omega}{-\left(\omega^2 - \omega_n^2\right)}\right] \tag{74}$$

$$\theta = \arctan\left[\frac{2\xi\omega_n\omega}{\left(\omega_n^2 - \omega^2\right)}\right] \tag{75}$$

$$\theta = \arctan\left[\frac{\omega_n^2\left(2\xi\frac{\omega}{\omega_n}\right)}{\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}\right)}\right] \tag{76}$$

$$\theta = \arctan\left[\frac{2\xi\rho}{1-\rho^2}\right] \tag{77}$$

The magnitude and phase angle relationships are derived by an alternative approach in Reference 1.

Reference

1. T. Irvine, The Steady-state Response of a Single-degree-of-Freedom System to a Harmonic Force, Vibrationdata, 1999.