THE GENERALIZED COORDINATE METHOD FOR DISCRETE SYSTEMS SUBJECT TO BASE EXCITATION Revision B

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Two-degree-of-freedom System

Consider a two-degree-of-freedom system subjected to base excitation, as shown in Figure 1. Free-body diagrams are shown in Figure 2.

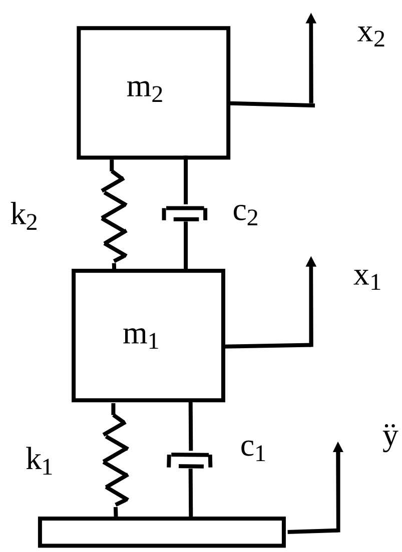

Figure 1.

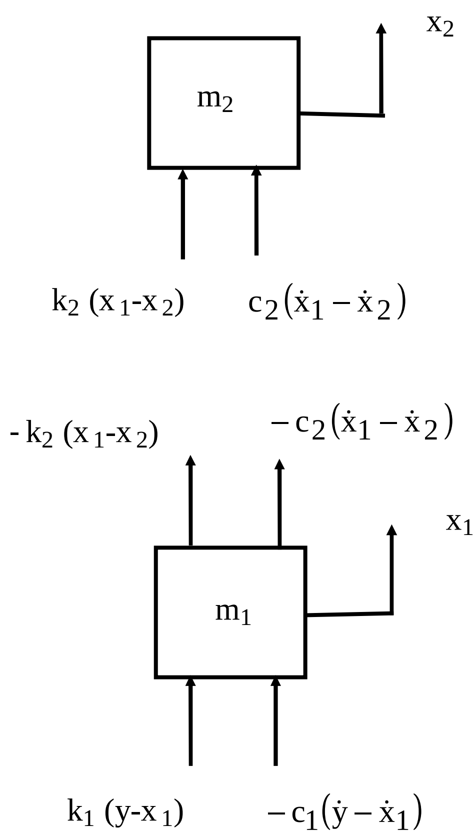

Figure 2.

Determine the equation of motion for mass 2.

$$\sum F = m_2 \ddot{x}_2 \quad (1)$$

$$m_2 \ddot{x}_2 = c_2(\dot{x}_1 - \dot{x}_2) + k_2 (x_1 - x_2) \quad (2)$$

Define relative displacement terms.

$$z_2 = x_2 - y \tag{3}$$

$$x_2 = z_2 + y \tag{4}$$

$$z_1 = x_1 - y \tag{5}$$

$$x_1 = z_1 + y \tag{6}$$

Note that

$$x_1 - x_2 = z_1 - z_2 \tag{7}$$

Substitute the relative displacement terms into equation (2).

$$m_2 \ddot{z}_2 + m_2 \ddot{y} = c_2(\dot{z}_1 - \dot{z}_2) + k_2(z_1 - z_2) \tag{8}$$

$$m_2 \ddot{z}_2 + c_2(-\dot{z}_1 + \dot{z}_2) + k_2(-z_1 + z_2) = -m_2 \ddot{y} \tag{9}$$

$$m_2 \ddot{z}_2 + c_2 \dot{z}_2 - c_2 \dot{z}_1 + k_2 z_2 - k_2 z_1 = -m_2 \ddot{y} \tag{10}$$

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{11}$$

$$m_1 \ddot{x}_1 = -c_2(\dot{x}_1 - \dot{x}_2) + c_1(\dot{y} - \dot{x}_1) - k_2(x_1 - x_2) + k_1(y - x_1) \tag{12}$$

$$m_1 \ddot{x}_1 = -c_2(\dot{x}_1 - \dot{x}_2) + c_1(\dot{y} - \dot{x}_1) - k_2(x_1 - x_2) + k_1(y - x_1) \tag{13}$$

$$m_1 \ddot{z}_1 + m_1 \ddot{y} = -c_2(\dot{z}_1 - \dot{z}_2) + c_1(\dot{y} - \dot{z}_1 - \dot{y}) - k_2(z_1 - z_2) + k_1(y - z_1 - y) \tag{14}$$

$$m_1 \ddot{z}_1 + m_1 \ddot{y} = -c_2(\dot{z}_1 - \dot{z}_2) + c_1(-\dot{z}_1) - k_2(z_1 - z_2) + k_1(-z_1) \tag{15}$$

$$m_1 \ddot{z}_1 + m_1 \ddot{y} = -c_2(\dot{z}_1 - \dot{z}_2) - c_1(\dot{z}_1) - k_2 (z_1 - z_2) - k_1 (z_1) \tag{16}$$

$$m_1 \ddot{z}_1 + c_2(\dot{z}_1 - \dot{z}_2) + c_1(\dot{z}_1) + k_2 (z_1 - z_2) + k_1 (z_1) = -m_1 \ddot{y} \tag{17}$$

$$m_1 \ddot{z}_1 + (c_1 + c_2)\dot{z}_1 - c_2 \dot{z}_2 + (k_1 + k_2) z_1 - k_2 z_2 = -m_1 \ddot{y} \tag{18}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{19}$$

Represent as

$$M\ddot{\bar{z}} + C\dot{\bar{z}} + K\bar{z} = F \tag{20}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{21}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{22}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{23}$$

$$F = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{24}$$

$$\bar{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{25}$$

Consider the undamped, homogeneous form of equation (19).

$$M\,\ddot{\bar{z}} + K\,\bar{z} = \bar{0} \tag{26}$$

Seek a solution of the form

$$\bar{z} = \bar{q}\exp(j\omega t) \tag{27}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{z}} = j\omega\,\bar{q}\exp(j\omega t) \tag{28}$$

$$\ddot{\bar{z}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{29}$$

Substitute these equations into equation (26).

$$-\omega^2 M\,\bar{q}\exp(j\omega t) + K\,\bar{q}\exp(j\omega t) = \bar{0} \tag{30}$$

$$\left\{-\omega^2 M + K\right\}\bar{q}\exp(j\omega t) = \bar{0} \tag{31}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{32}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{33}$$

Equation (28) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{34}$$

$$\det\left\{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{35}$$

$$\det\begin{Bmatrix} (k_1+k_2)-\omega^2 m_1 & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{Bmatrix} = 0 \tag{36}$$

$$\left[(k_1+k_2)-\omega^2 m_1\right]\left[k_2-\omega^2 m_2\right]-{k_2}^2 = 0 \tag{37}$$

$$-\omega^4 m_1 m_2 + \omega^2\left[-m_2(k_1+k_2)-m_1 k_2\right]-{k_2}^2 + k_2(k_1+k_2) = 0 \tag{38}$$

$$-\omega^4 m_1 m_2 + \omega^2\left[-m_2(k_1+k_2)-m_1 k_2\right]-{k_2}^2 + k_1 k_2 + {k_2}^2 = 0 \tag{39}$$

$$-\omega^4 m_1 m_2 + \omega^2\left[-m_2(k_1+k_2)-m_1 k_2\right]+ k_1 k_2 = 0 \tag{40}$$

The eigenvalues are the roots of the polynomial.

$${\omega_1}^2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \tag{41}$$

$${\omega_2}^2 = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{42}$$

where

$$a = m_1 m_2 \tag{43}$$

$$b = \left[-m_2(k_1+k_2)-m_1 k_2\right] \tag{44}$$

$$c = k_1 k_2 \tag{45}$$

The eigenvectors are found via the following equations.

$$\left\{ K - \omega_1^2 \, M \right\} \bar{q}_1 = \bar{0} \tag{46}$$

$$\left\{ K - \omega_2^2 \, M \right\} \bar{q}_2 = \bar{0} \tag{47}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{48}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{49}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \left[\; \bar{q}_1 \mid \bar{q}_2 \; \right] \tag{50}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{51}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T \, M \, \hat{Q} = I \tag{52}$$

$$\hat{Q}^T \, K \, \hat{Q} = \Omega \tag{53}$$

where

subscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{54}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{55}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 1 and 2.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{z} = \hat{Q}\,\bar{\eta} \tag{56}$$

Substitute equation (56) into equation (20).

$$M\hat{Q}\,\ddot{\bar{\eta}} + C\hat{Q}\,\dot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = F \tag{57}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T C\hat{Q}\,\dot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T F \tag{58}$$

The orthogonality relationships yield.

$$I\ \ddot{\bar{\eta}} + \hat{Q}^T C\hat{Q}\,\dot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T F \tag{59}$$

Furthermore, the following assumption is made.

$$\hat{Q}^T C\hat{Q}\,\dot{\bar{\eta}} = \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\,\omega_2 \end{bmatrix} \tag{60}$$

where ξ_i is the modal damping ratio for mode i.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{61}$$

The two equations are now decoupled in terms of the modal coordinate.

$$\ddot{\eta}_1 + 2\xi_1\omega_1\,\dot{\eta}_1 + \omega_1^2\,\eta_1 = [-m_1\hat{v}_1 - m_2\hat{v}_2]\,\ddot{y} \tag{62}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\,\dot{\eta}_2 + \omega_2^2\,\eta_2 = [-m_1\hat{w}_1 - m_2\hat{w}_2]\,\ddot{y} \tag{63}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{z} = \hat{Q}\,\bar{\eta} \tag{64}$$

Thus

$$\bar{z}(0) = \hat{Q}\,\bar{\eta}(0) \tag{65}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\,\bar{z}(0) = \hat{Q}^T M\hat{Q}\,\bar{\eta}(0) \tag{66}$$

Recall

$$\hat{Q}^T M\hat{Q} = I \tag{67}$$

$$\hat{Q}^T M\,\bar{z}(0) = I\,\bar{\eta}(0) \tag{68}$$

$$\hat{Q}^T M\,\bar{z}(0) = \bar{\eta}(0) \tag{69}$$

Finally, the transformed initial displacement matrix is

$$\bar{\eta}(0) = \hat{Q}^T M \, \bar{z}(0) \tag{70}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^T M \, \bar{\dot{z}}(0) \tag{71}$$

A basis for a solution is thus derived.

Harmonic Excitation

Assume that that the net displacement and net velocity values are zero.

Now consider the special case of harmonic base excitation function.

$$\ddot{y}(t) = A \sin(\alpha t) \tag{72a}$$

$$\dot{y}(t) = \frac{-A}{\alpha} \cos(\alpha t) \tag{72b}$$

$$y(t) = \frac{-A}{\alpha^2} \sin(\alpha t) \tag{72c}$$

Solve for the steady-state response.

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 \, \dot{\eta}_1 + \omega_1^2 \, \eta_1 = -A \left[m_1 \hat{v}_1 + m_2 \hat{v}_2 \right] \sin(\alpha t) \tag{73}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 \, \dot{\eta}_2 + \omega_2^2 \, \eta_2 = -A \left[m_1 \hat{w}_1 + m_2 \hat{w}_2 \right] \sin(\alpha t) \tag{74}$$

The solutions are taken from Reference (3).

$$\eta_1(t) = \frac{\left[m_1\,\hat{v}_1 + m_2\,\hat{v}_2\right]A}{\left[\left(\alpha^2 - {\omega_1}^2\right)^2 + (2\,\xi_1\,\alpha\,\omega_1)^2\right]}\left[(2\,\xi_1\,\alpha\omega_1)\cos(\alpha t) + \left(\alpha^2 - {\omega_1}^2\right)\sin(\alpha t)\right]$$

(75a)

$$\eta_2(t) = \frac{\left[m_1\,\hat{w}_1 + m_2\,\hat{w}_2\right]A}{\left[\left(\alpha^2 - {\omega_2}^2\right)^2 + (2\,\xi_2\,\alpha\,\omega_2)^2\right]}\left[(2\,\xi_2\,\alpha\omega_2)\cos(\alpha t) + \left(\alpha^2 - {\omega_2}^2\right)\sin(\alpha t)\right]$$

(75b)

The relative displacements can then be found from

$$\bar{z} = \hat{Q}\,\bar{\eta} \tag{76}$$

$$Q = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{77}$$

$$z_1(t) = \frac{\hat{v}_1\left[m_1\,\hat{v}_1 + m_2\,\hat{v}_2\right]A}{\left[\left(\alpha^2 - {\omega_1}^2\right)^2 + (2\,\xi_1\,\alpha\,\omega_1)^2\right]}\left[(2\,\xi_1\,\alpha\omega_1)\cos(\alpha t) + \left(\alpha^2 - {\omega_1}^2\right)\sin(\alpha t)\right]$$

$$+ \frac{\hat{w}_1\left[m_1\,\hat{w}_1 + m_2\,\hat{w}_2\right]A}{\left[\left(\alpha^2 - {\omega_2}^2\right)^2 + (2\,\xi_2\,\alpha\,\omega_2)^2\right]}\left[(2\,\xi_2\,\alpha\omega_2)\cos(\alpha t) + \left(\alpha^2 - {\omega_2}^2\right)\sin(\alpha t)\right]$$

(78)

$$z_1(t) = 2A\alpha \left\{ \frac{\hat{v}_1 [m_1 \hat{v}_1 + m_2 \hat{v}_2] (\xi_1 \omega_1)}{\left[\left(\alpha^2 - \omega_1{}^2 \right)^2 + (2\xi_1 \alpha \omega_1)^2 \right]} + \frac{\hat{w}_1 [m_1 \hat{w}_1 + m_2 \hat{w}_2] (\xi_2 \omega_2)}{\left[\left(\alpha^2 - \omega_2{}^2 \right)^2 + (2\xi_2 \alpha \omega_2)^2 \right]} \right\} \cos(\alpha t)$$

$$+ A \left\{ \frac{\hat{v}_1 [m_1 \hat{v}_1 + m_2 \hat{v}_2] \left(\alpha^2 - \omega_1{}^2 \right)}{\left[\left(\alpha^2 - \omega_1{}^2 \right)^2 + (2\xi_1 \alpha \omega_1)^2 \right]} + \frac{\hat{w}_1 [m_1 \hat{w}_1 + m_2 \hat{w}_2] \left(\alpha^2 - \omega_2{}^2 \right)}{\left[\left(\alpha^2 - \omega_2{}^2 \right)^2 + (2\xi_2 \alpha \omega_2)^2 \right]} \right\} \sin(\alpha t)$$

(79)

$$z_2(t) = 2A\alpha \left\{ \frac{\hat{v}_2 [m_1 \hat{v}_1 + m_2 \hat{v}_2] (\xi_1 \omega_1)}{\left[\left(\alpha^2 - \omega_1{}^2 \right)^2 + (2\xi_1 \alpha \omega_1)^2 \right]} + \frac{\hat{w}_2 [m_1 \hat{w}_1 + m_2 \hat{w}_2] (\xi_2 \omega_2)}{\left[\left(\alpha^2 - \omega_2{}^2 \right)^2 + (2\xi_2 \alpha \omega_2)^2 \right]} \right\} \cos(\alpha t)$$

$$+ A \left\{ \frac{\hat{v}_2 [m_1 \hat{v}_1 + m_2 \hat{v}_2] \left(\alpha^2 - \omega_1{}^2 \right)}{\left[\left(\alpha^2 - \omega_1{}^2 \right)^2 + (2\xi_1 \alpha \omega_1)^2 \right]} + \frac{\hat{w}_2 [m_1 \hat{w}_1 + m_2 \hat{w}_2] \left(\alpha^2 - \omega_2{}^2 \right)}{\left[\left(\alpha^2 - \omega_2{}^2 \right)^2 + (2\xi_2 \alpha \omega_2)^2 \right]} \right\} \sin(\alpha t)$$

(80)

The absolute displacements can then be found from

$$x_1 = z_1 + y \qquad (81)$$

$$x_2 = z_2 + y \qquad (82)$$

$$x_1(t) = 2A\alpha\left\{\frac{\hat{v}_1\left[m_1\hat{v}_1 + m_2\hat{v}_2\right](\xi_1\omega_1)}{\left[\left(\alpha^2-\omega_1{}^2\right)^2 + (2\xi_1\alpha\omega_1)^2\right]} + \frac{\hat{w}_1\left[m_1\hat{w}_1 + m_2\hat{w}_2\right](\xi_2\omega_2)}{\left[\left(\alpha^2-\omega_2{}^2\right)^2 + (2\xi_2\alpha\omega_2)^2\right]}\right\}\cos(\alpha t)$$

$$+ A\left\{\frac{\hat{v}_1\left[m_1\hat{v}_1 + m_2\hat{v}_2\right]\left(\alpha^2-\omega_1{}^2\right)}{\left[\left(\alpha^2-\omega_1{}^2\right)^2 + (2\xi_1\alpha\omega_1)^2\right]} + \frac{\hat{w}_1\left[m_1\hat{w}_1 + m_2\hat{w}_2\right]\left(\alpha^2-\omega_2{}^2\right)}{\left[\left(\alpha^2-\omega_2{}^2\right)^2 + (2\xi_2\alpha\omega_2)^2\right]}\right\}\sin(\alpha t)$$

$$+ \quad \frac{-A}{\alpha^2}\sin(\alpha t)$$

(83)

$$x_2(t) = 2A\alpha\left\{\frac{\hat{v}_2\left[m_1\hat{v}_1 + m_2\hat{v}_2\right](\xi_1\omega_1)}{\left[\left(\alpha^2-\omega_1{}^2\right)^2 + (2\xi_1\alpha\omega_1)^2\right]} + \frac{\hat{w}_2\left[m_1\hat{w}_1 + m_2\hat{w}_2\right](\xi_2\omega_2)}{\left[\left(\alpha^2-\omega_2{}^2\right)^2 + (2\xi_2\alpha\omega_2)^2\right]}\right\}\cos(\alpha t)$$

$$+ A\left\{\frac{\hat{v}_2\left[m_1\hat{v}_1 + m_2\hat{v}_2\right]\left(\alpha^2-\omega_1{}^2\right)}{\left[\left(\alpha^2-\omega_1{}^2\right)^2 + (2\xi_1\alpha\omega_1)^2\right]} + \frac{\hat{w}_2\left[m_1\hat{w}_1 + m_2\hat{w}_2\right]\left(\alpha^2-\omega_2{}^2\right)}{\left[\left(\alpha^2-\omega_2{}^2\right)^2 + (2\xi_2\alpha\omega_2)^2\right]}\right\}\sin(\alpha t)$$

$$+ \quad \frac{-A}{\alpha^2}\sin(\alpha t)$$

(84)

The velocity and acceleration terms can be found by taking derivatives.

References

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3. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Vibrationdata Publications, 1999.