RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM
SUBJECTED TO A HALF-SINE PULSE APPLIED FORCE

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

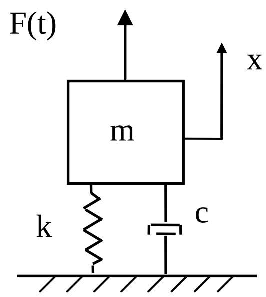

Figure 1.

where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass
F(t)	is the applied force

A free-body diagram is shown in Figure 2.

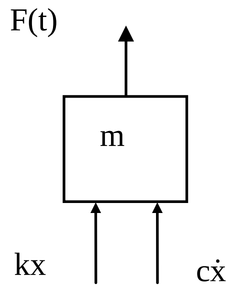

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = -c\dot{x} - kx + F(t) \tag{2}$$

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{3}$$

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)F(t) \tag{4}$$

By convention

$$(c/m) = 2\xi\omega_n$$

$$(k/m) = \omega_n{}^2$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n{}^2 x = \left(\frac{1}{m}\right)F(t) \tag{5}$$

Half-Sine Pulse

Consider the pulse given by equation (6).

$$F(t) = \begin{cases} \hat{F}\sin(\beta t), & 0 \le t \le T \\ \\ 0, & t > T \end{cases}$$

$$\text{where} \quad \beta = \frac{\pi}{T} \tag{6}$$

The equation of motion becomes

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = (\hat{F}/m)\sin(\beta t), \quad 0 \le t \le T \tag{7}$$

Now take the Laplace transform.

$$L\{\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x\} = L\{(\hat{F}/m)\sin(\beta t)\} \tag{8}$$

$$\begin{aligned} &s^2 X(s) - sX(0) - \dot{x}(0) \\ &+ 2\xi\omega_n sX(s) - 2\xi\omega_n x(0) \\ &+ \omega_n^2 X(s) = \quad (\hat{F}/m)\left(\frac{\beta}{s^2 + \beta^2}\right) \end{aligned} \tag{9}$$

$$\{s^2 + 2\xi\omega_n s + \omega_n^2\}X(s) - \{s + 2\xi\omega_n\}x(0) - \dot{x}(0) = (\hat{F}/m)\left(\frac{\beta}{s^2 + \beta^2}\right) \tag{10}$$

$$\{s^2 + 2\xi\omega_n s + \omega_n^2\}X(s) = \{s + 2\xi\omega_n\}x(0) + \dot{x}(0) + (\hat{F}/m)\left(\frac{\beta}{s^2 + \beta^2}\right) \tag{11}$$

$$X(s) = \frac{\left(s + 2\xi\omega_n\right)x(0) + \dot{x}(0)}{\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)} + \left(\hat{F}/m\right)\left(\frac{\beta}{s^2 + \beta^2}\right)\left(\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right) \tag{12}$$

The displacement is found via an inverse Laplace transform using the methods in References 1 through 4.

$$x(t) = \exp\left(-\xi\omega_n t\right)\left\{x(0)\cos\left(\omega_d t\right) + \left\{\frac{\dot{x}(0) + \left(\xi\omega_n\right)x(0)}{\omega_d}\right\}\sin\left(\omega_d t\right)\right\}$$

$$-\frac{\hat{F}/m}{\left[\left(\beta^2 - \omega_n{}^2\right)^2 + \left(2\xi\beta\omega_n\right)^2\right]}\left[\left(2\xi\beta\omega_n\right)\cos\left(\beta t\right) + \left(\beta^2 - \omega_n{}^2\right)\sin\left(\beta t\right)\right]$$

$$+\frac{\frac{\hat{F}}{m}\frac{\beta}{\omega_d}\left[\exp\left(-\xi\omega_n t\right)\right]}{\left[\left(\beta^2 - \omega_n{}^2\right)^2 + \left(2\xi\beta\omega_n\right)^2\right]}\left[\left(2\xi\omega_n\omega_d\right)\cos\left(\omega_d t\right) + \left(\beta^2 - \omega_n{}^2\left(1 - 2\xi^2\right)\right)\sin\left(\omega_d t\right)\right],$$

$$0 \le t \le T$$

(13)

The solution for t > T is the free vibration solution.

$$x(t) = \exp\left(-\xi\omega_n \tau\right)\left\{x(T)\cos\left(\omega_d \tau\right) + \left\{\frac{\dot{x}(T) + \left(\xi\omega_n\right)x(T)}{\omega_d}\right\}\sin\left(\omega_d \tau\right)\right\},$$

where

$$\tau = t - T$$

(14)

References

1. T. Irvine, Table of Laplace Transforms, Vibrationdata, 1999.
2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata, 1999.
3. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Rev A, Vibrationdata, 1999.
4. T. Irvine, The Response of a Single-degree-of-Freedom System Subjected to a Combined Force and Base Excitation, Vibrationdata, 2010.

APPENDIX A

Example

```
>> half_sine_force

 half_sine_force.m
 ver 1.0  December 14, 2012
 by Tom Irvine  Email: tomirvine@aol.com

 This program calculates the vibration of a
 single-degree-of-freedom system subjected to
 a half-sine force.

 select units
 1=English  2=metric 1

 Enter the amplitude (lbf)
 10

 Enter the duration (sec)
 0.011

 input initial velocity (in/sec) 0

 input initial displacement (in) 0

 input mass (lbm) 1

 Enter the natural frequency (Hz)  75
         stiffness =   575.3 lbf/in

 static deflection =0.001738 in


 Enter damping format:  1= damping ratio   2= Q  2

 Enter the amplification factor (typically Q=10) 10
```

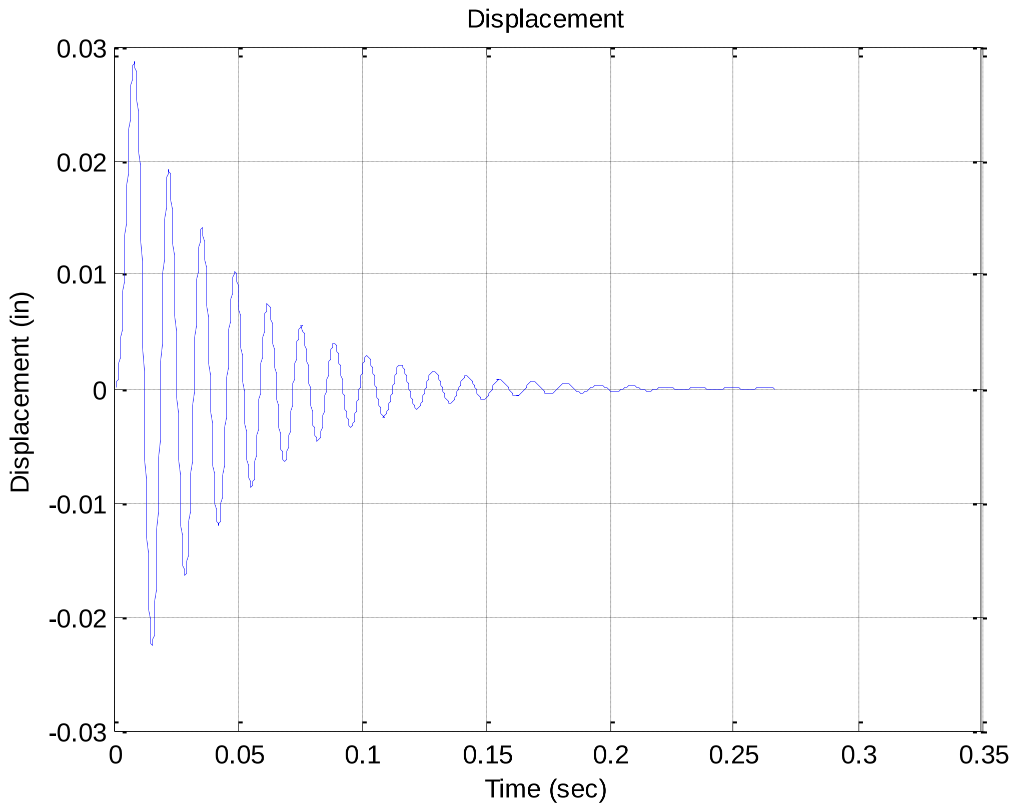
Displacement
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
Displacement (in)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Time (sec)

Figure A-1.

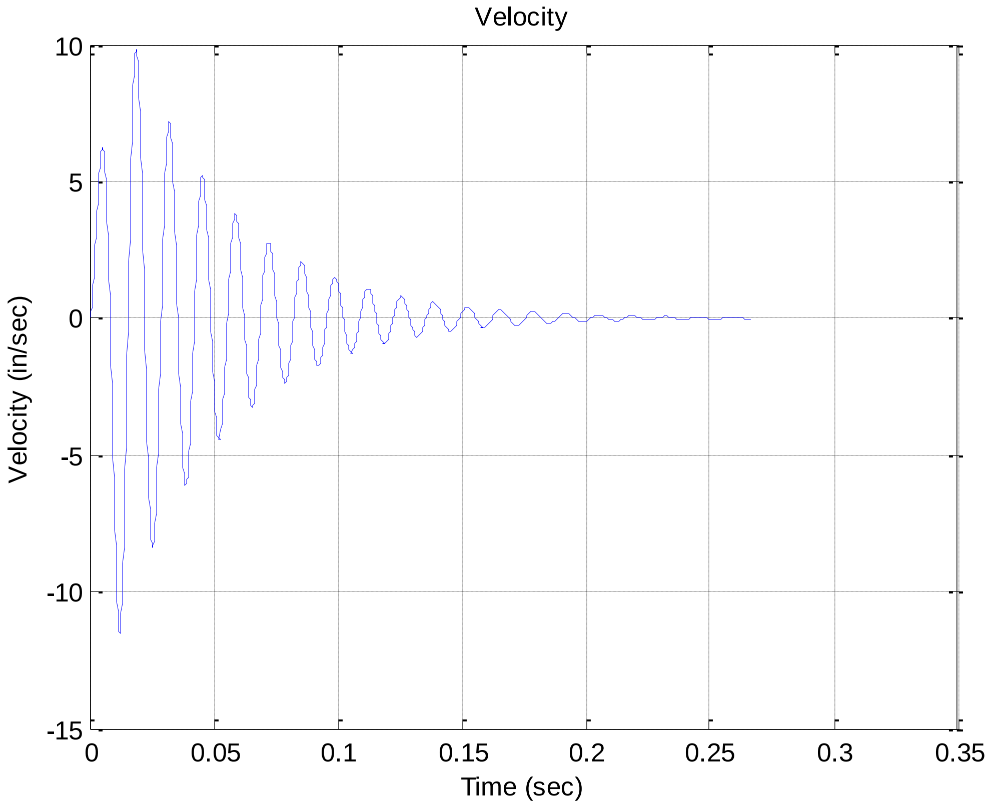
Velocity
10
5
0
-5
-10
-15
Velocity (in/sec)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Time (sec)

Figure A-2.

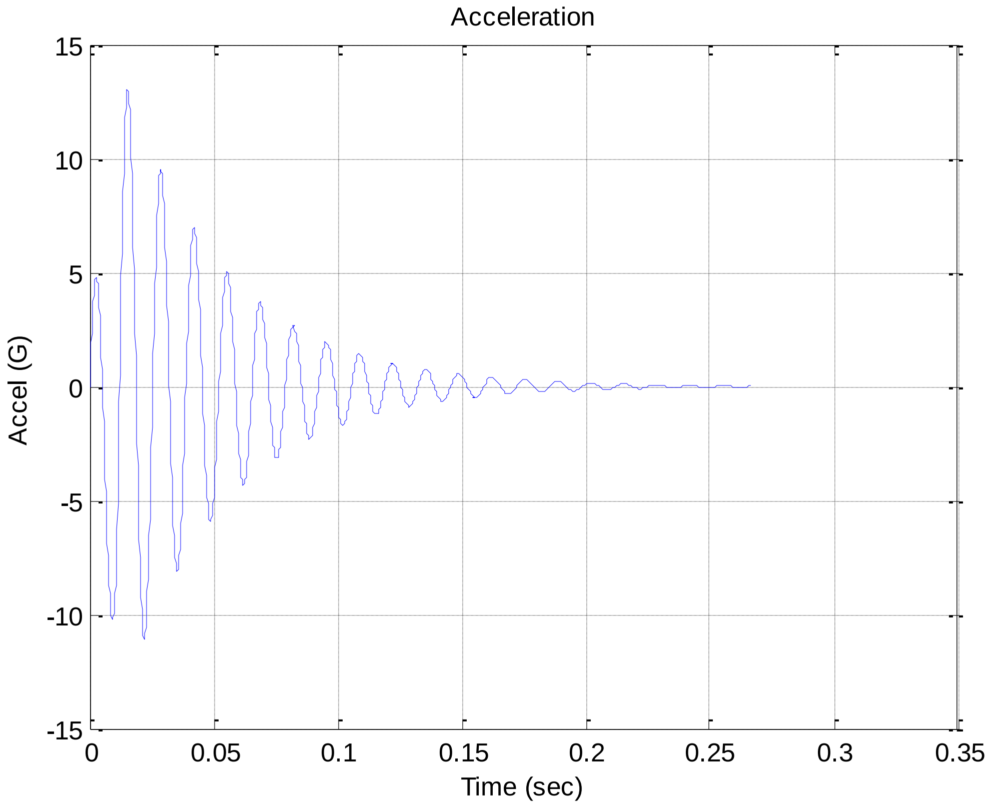
Acceleration
15
10
5
0
-5
-10
-15
Accel (G)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Time (sec)

Figure A-3.