NATURAL FREQUENCIES OF AN INFINITE CYLINDER WITH PLANE STRAIN
Revision C

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Variables

u	is the axial displacement
v	is the tangential displacement
w	is the radial displacement
E	is the modulus of elasticity
R	is the radius
ρ	is the mass/volume
ν	is the Poisson ratio
c	is the speed of sound in the material
t	is time
h	is the wall thickness
ω	is the excitation frequency
ω_n	is the natural frequency (radian/sec)
n	is an index, n=0,1,2,…
k	is a nondimensional thickness factor
Ω	is a nondimensional frequency factor
B	Eigenvector scale factor
C	Eigenvector scale factor

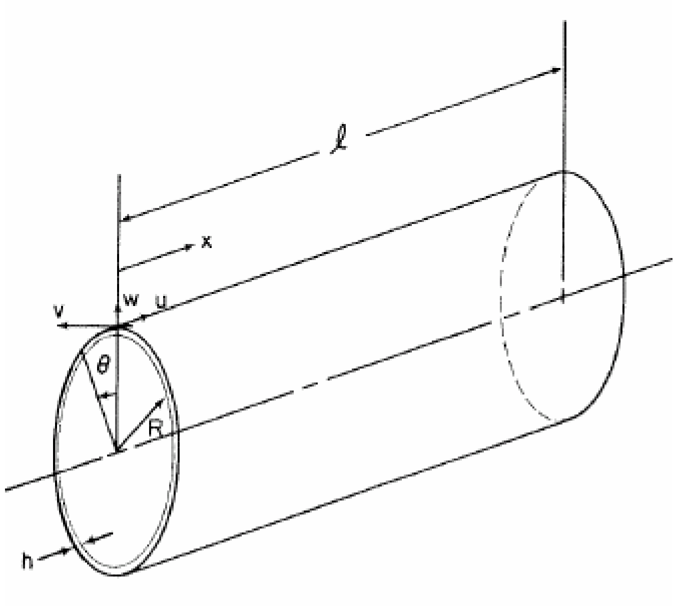

Figure 1. Cylinder Diagram

The three translation variables are

$$u = 0 \tag{1}$$

$$v = v(\theta) \tag{2}$$

$$w = w(\theta) \tag{3}$$

There are two coupled equations of motion, taken from Reference 1.

$$\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial w}{\partial \theta} = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 v}{\partial t^2} \tag{4}$$

$$\frac{\partial v}{\partial \theta} + \left[1+k\left(1+\frac{\partial^2}{\partial \theta^2}\right)^2\right]w = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{5}$$

where

$$k = \frac{h^2}{12R^2} \tag{6}$$

Equations (4) and (5) together model both the bending and membrane response.

$$\frac{\partial v}{\partial \theta} + \left[1 + k\left(\frac{\partial^4}{\partial \theta^4} + 2\frac{\partial^2}{\partial \theta^2} + 1 \right) \right] w = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{7}$$

$$\frac{\partial v}{\partial \theta} + \left[k\left(\frac{\partial^4 w}{\partial \theta^4} + 2\frac{\partial^2 w}{\partial \theta^2} + w \right) + w \right] = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{8}$$

$$\frac{\partial v}{\partial \theta} + \left[k\frac{\partial^4 w}{\partial \theta^4} + 2k\frac{\partial^2 w}{\partial \theta^2} + w(k+1) \right] = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{9}$$

Assume that the tangential displacement is

$$v = B\sin(n\theta)\cos(\omega t) \tag{10}$$

$$\frac{\partial v}{\partial \theta} = nB\cos(n\theta)\cos(\omega t) \tag{11}$$

$$\frac{\partial^2 v}{\partial \theta^2} = -n^2 B\sin(n\theta)\cos(\omega t) \tag{12}$$

$$\frac{\partial v}{\partial t} = -\omega B \sin(n\theta)\sin(\omega t) \tag{13}$$

$$\frac{\partial^2 v}{\partial t^2} = -\omega^2 B \sin(n\theta)\cos(\omega t) \tag{14}$$

Assume that the radial displacement is

$$w = C\cos(n\theta)\cos(\omega t) \tag{15}$$

$$\frac{\partial}{\partial \theta} w = -n C \sin(n\theta)\cos(\omega t) \tag{16}$$

$$\frac{\partial^2}{\partial \theta^2} w = -n^2 C \cos(n\theta)\cos(\omega t) \tag{17}$$

$$\frac{\partial^3}{\partial \theta^3} w = n^3 C \sin(n\theta)\cos(\omega t) \tag{18}$$

$$\frac{\partial^4}{\partial \theta^4} w = n^4 C \cos(n\theta)\cos(\omega t) \tag{19}$$

$$\frac{\partial}{\partial t} w = -\omega C \cos(n\theta)\sin(\omega t) \tag{20}$$

$$\frac{\partial^2}{\partial t^2} w = -\omega^2 C \cos(n\theta)\cos(\omega t) \tag{21}$$

Substitute the assumed solutions into equation (4).

$$-n^2 B\sin(n\theta)\cos(\omega t) \;\; - nC\sin(n\theta)\cos(\omega t) \;\; = -\frac{\rho\left(1-\nu^2\right)R^2}{E}\omega^2 B\sin(n\theta)\cos(\omega t) \tag{22}$$

$$-n^2 B \;\; - nC \;\; = -\frac{\rho\left(1-\nu^2\right)R^2}{E}\omega^2 B \tag{23}$$

$$\left[\frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E} - n^2\right]B \;\; - nC \;\; = 0 \tag{24}$$

Let

$$\Omega^2 \;\; = \;\; \left[\frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}\right] \tag{25}$$

$$\left[\Omega^2 - n^2\right]B \;\; - nC \;\; = 0 \tag{26}$$

$$\left[n^2 - \Omega^2\right]B \;\; + nC \;\; = 0 \tag{27}$$

Substitute the assumed solutions into equation (5).

$$n\,B\cos(n\theta)\cos(\omega t) + n^4 k\,C\cos(n\theta)\cos(\omega t)$$

$$-\,2n^2 k\,C\cos(n\theta)\cos(\omega t) + (k+1)C\cos(n\theta)\cos(\omega t)$$

$$= \frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}C\cos(n\theta)\cos(\omega t)$$

(28)

$$n\,B + n^4 k\,C - 2n^2 k\,C + (k+1)C = \frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}C \tag{29}$$

$$n\,B + \left[n^4 k - 2n^2 k + (k+1) - \frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}\right]C = 0 \tag{30}$$

$$n\,B + \left[n^4 k - 2n^2 k + k + 1 - \Omega^2\right]C = 0 \tag{31}$$

$$\begin{bmatrix} \left(n^2-\Omega^2\right) & n \\ n & n^4 k - 2n^2 k + k + 1 - \Omega^2 \end{bmatrix}\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(32)

$$\begin{bmatrix} \left(n^2 - \Omega^2\right) & n \\ n & 1 + k\left(1 - n^2\right)^2 - \Omega^2 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{33}$$

$$\left(1 + k\left(1 - n^2\right)^2 - \Omega^2\right)\left(n^2 - \Omega^2\right) - n^2 = 0 \tag{34}$$

For n = 0,

$$\left(1 + k - \Omega^2\right)\left(-\Omega^2\right) = 0 \tag{35}$$

$$\Omega^2 = 0 \quad \text{or} \quad \Omega^2 = 1 + k \tag{36}$$

For n = 1,

$$\left(1 - \Omega^2\right)\left(1 - \Omega^2\right) - 1 = 0 \tag{37}$$

$$\Omega^4 - 2\Omega^2 + 1 - 1 = 0 \tag{38}$$

$$\Omega^4 - 2\Omega^2 = 0 \tag{39}$$

$$\Omega^2\left(\Omega^2 - 2\right) = 0 \tag{40}$$

$$\Omega^2 = 0 \quad \text{or} \quad \Omega^2 = 2 \tag{41}$$

For $n \geq 1$,

$$\left(1+k\left(1-n^2\right)^2-\Omega^2\right)\left(n^2-\Omega^2\right)-n^2=0 \tag{42}$$

$$\Omega^4-\left(1+n^2+k\left(1-n^2\right)^2\right)\Omega^2+n^2\left(1+k\left(1-n^2\right)^2\right)-n^2=0 \tag{43}$$

$$\Omega^4-\left(1+n^2+k\left(1-n^2\right)^2\right)\Omega^2+k\,n^2\left(1-n^2\right)^2=0 \tag{44}$$

$$\Omega^2=\frac{1}{2}\left[\left(1+n^2+k\left(1-n^2\right)^2\right)\pm\sqrt{\left(1+n^2+k\left(1-n^2\right)^2\right)^2-4k\,n^2\left(1-n^2\right)^2}\right] \tag{45}$$

Mode Shapes

The root $\Omega^2=0$ for $n = 0$ corresponds to rigid-body torsional rotation of the shell, per Reference 1.

The root $\Omega^2>0$ for $n = 0$ has the following mode shape with pure radial motion

$$V(\theta)=0 \quad \text{(tangential)} \tag{46}$$

$$W(\theta)=C \quad \text{(radial)} \tag{47}$$

C is an arbitrary scale factor.

The mode shapes for n≥1 can be represented as

$$V(\theta) = B \sin(n\theta) \qquad \text{(tangential)} \qquad (48)$$

$$W(\theta) = \frac{-\left(n^2 - \Omega^2\right)}{n} B \cos(n\theta) \qquad \text{(radial)} \qquad (49)$$

B is an arbitrary scale factor.

Comparison with Ring Mode

Again, the first extension mode is

$$\Omega^2 = 1 + k \qquad (50)$$

$$\Omega = \sqrt{1+k} \qquad (51)$$

$$\Omega = \sqrt{1 + \frac{h^2}{12R^2}} \qquad (52)$$

$$\omega R \sqrt{\frac{\rho\left(1-\nu^2\right)}{E}} = \sqrt{1 + \frac{h^2}{12R^2}} \qquad (53)$$

$$\omega = \frac{1}{R} \sqrt{\frac{E}{\rho\left(1-\nu^2\right)}} \sqrt{1 + \frac{h^2}{12R^2}} \qquad (54)$$

$$f_{cyl} = \frac{1}{\pi d}\sqrt{\frac{E}{\rho\left(1-\nu^2\right)}}\sqrt{1+\frac{h^2}{12R^2}} \tag{55}$$

In comparison, the ring frequency is

$$f_r = \frac{1}{\pi d}\sqrt{\frac{E}{\rho}} \tag{56}$$

The infinite cylinder's first extensional mode is about 6% higher than the ring frequency due to the Poisson effect. This assumes a practical cylinder where the radius is much larger than the thickness.

References

1. Leissa, Vibration of Shells, NASA SP-288, Washington, D.C., 1973. (See section 2.2).

2. T. Irvine, Ring Vibration Modes, Revision A, Vibrationdata, 2004.

APPENDIX A

Example

Consider an infinitely long cylinder with the following properties:

Radius	19 inch
Skin Thickness	0.080 inch
Skin Material	Titanium
Speed of Sound	194,650 in/sec
Mass Density ρ	0.16 lbm/in^3
Elastic Modulus E	1.57e+07 lbf/in^2
Poisson Ratio	0.33

Note that: 386.04 lbm = 1 lbf sec^2/in

The equivalent mass density is ρ = 0.00041446 lbf sec^2/in^4.

Note that

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{1.57e+07 \text{ lbf/in\^{}2}}{0.00041446 \text{ lbf sec\^{}2/in\^{}4}}} = 194{,}650 \text{ in/sec} \tag{A-1}$$

Thus,

$$\frac{E}{\rho} = 3.79E{+}10 \text{ (in/sec)\^{}2} \tag{A-2}$$

Furthermore,

$$k = \frac{h^2}{12R^2} = \frac{(0.080 \text{ inch})^2}{12(19 \text{ inch})^2} = 1.48E-06 \tag{A-3}$$

The natural frequencies are calculated using equations (25), (36) and (45). The natural frequencies are shown in the table on the next page. Diagrams of selected mode shapes are given after the table.

The complete set of frequencies was calculated via a computer program. There are two frequencies per n value.

n	Lower Frequency (Hz)	Upper Frequency (Hz)
0	0	1727
1	0	2442
2	5.63	3862
3	15.9	5461
4	30.6	7121
5	49.4	8806
6	72.5	10505
7	99.7	12212
8	131	13923
9	167	15638
10	207	17356
11	251	19075
12	299	20796
13	352	22517
14	408	24239
15	469	25962
16	534	27686
17	604	29409
18	677	31134
19	755	32858
20	836	34583

The dashed line is the undeformed cylinder cross-section in each of the following figures. The solid line is the mode shape.

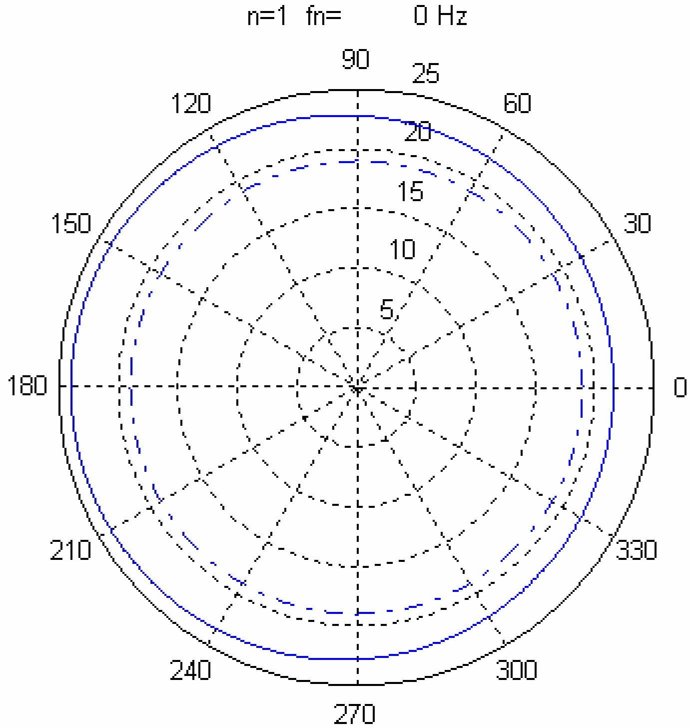

Figure A-1. n=1, Freq = 0 Hz, Rigid-body Motion

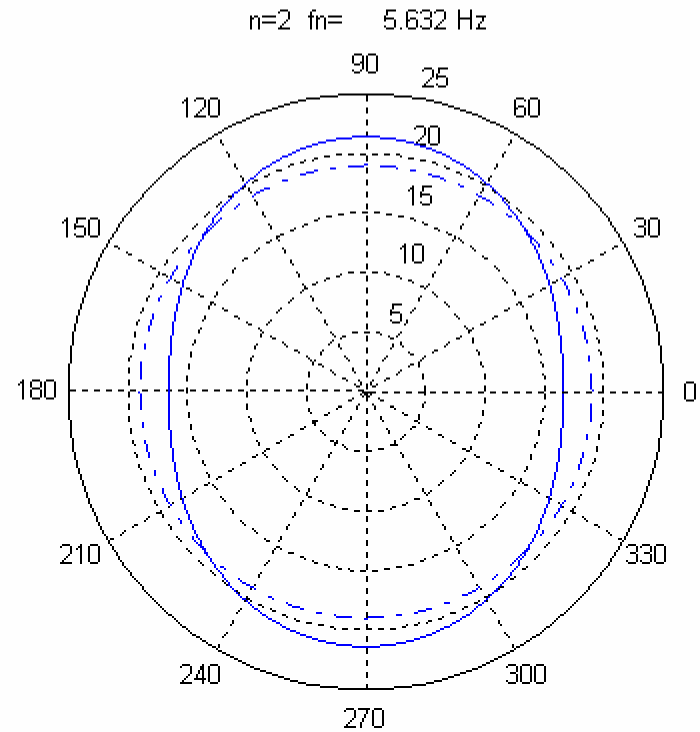

Figure A-2. n=2, Freq = 5.63 Hz, Flexural Mode

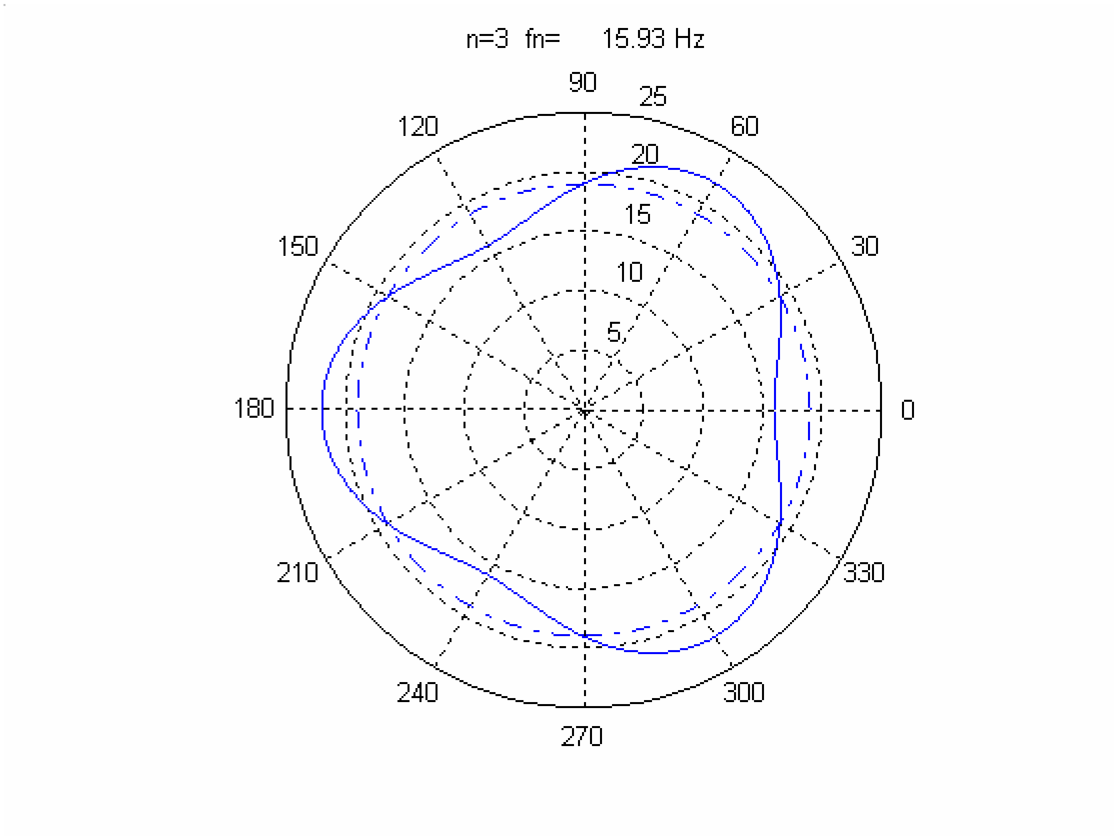

Figure A-3. n=3, Freq = 15.9 Hz, Flexural Mode

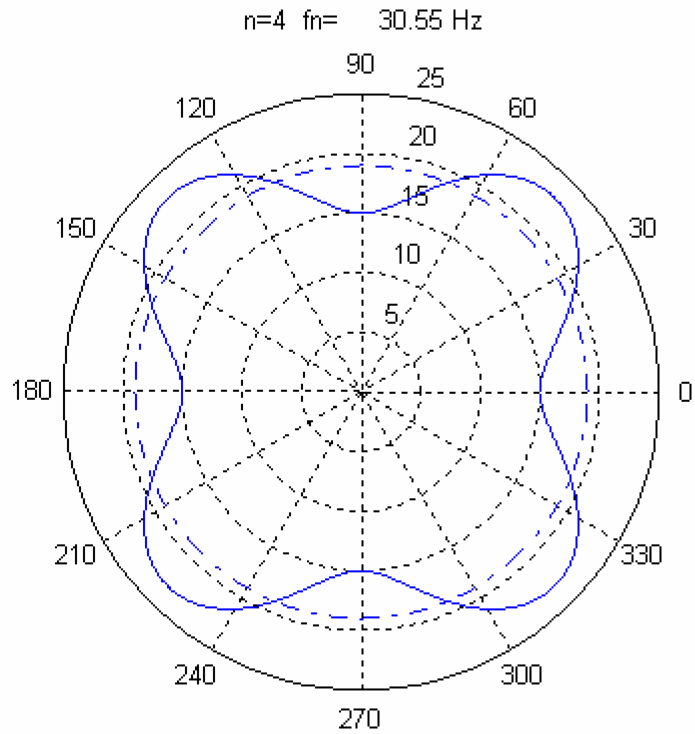

Figure A-4. n=4, Freq = 30.6 Hz, Flexural Mode

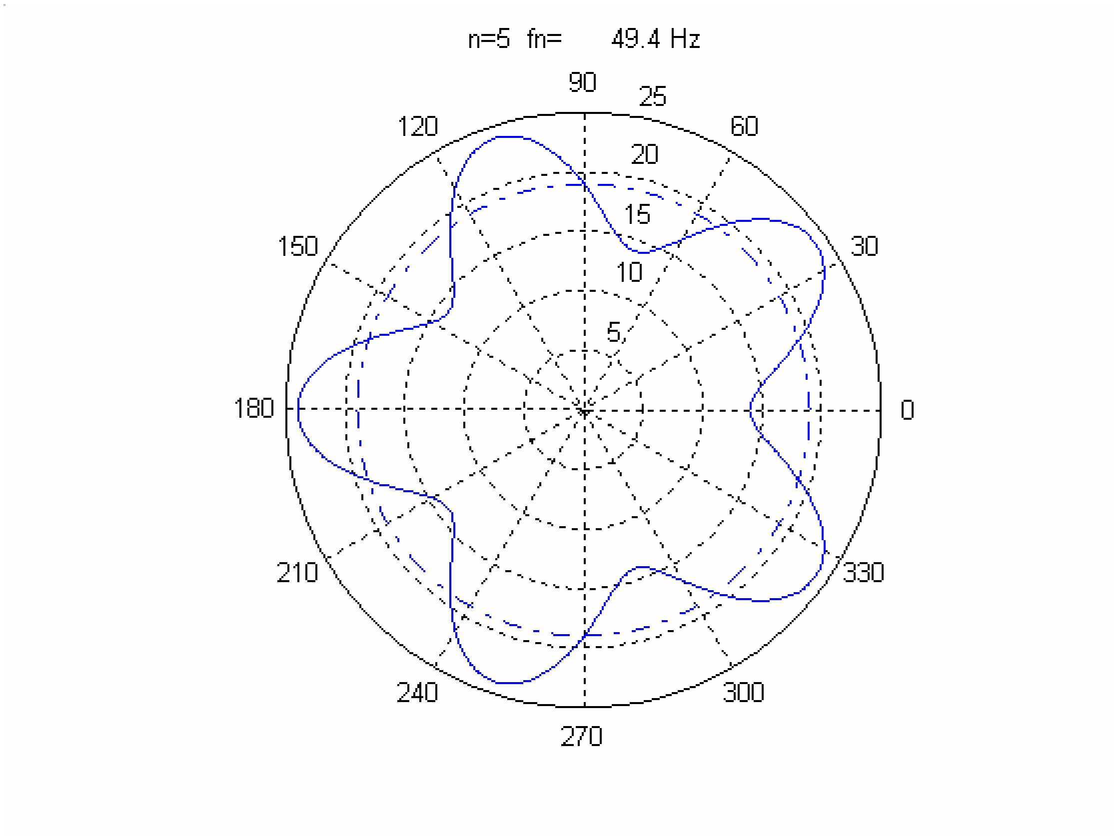

Figure A-5. n=5, Freq = 30.6 Hz, Flexural Mode

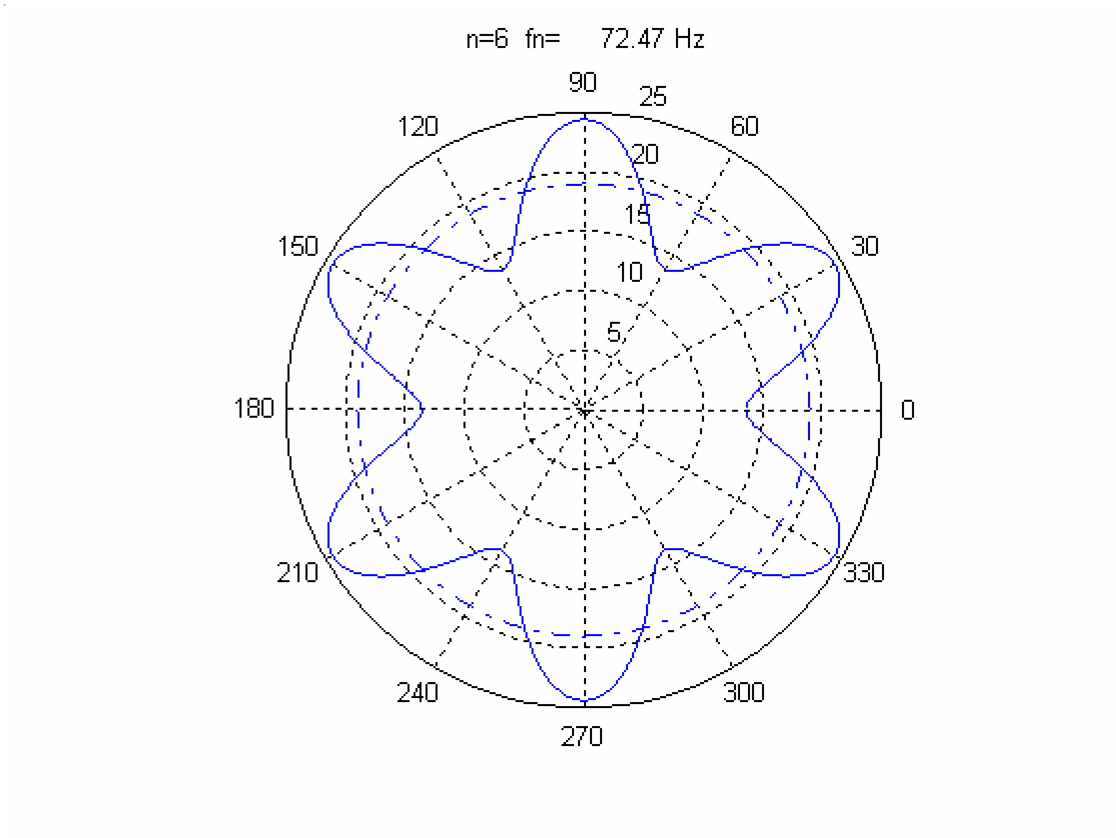

Figure A-6. n=6, Freq = 49.4 Hz, Flexural Mode

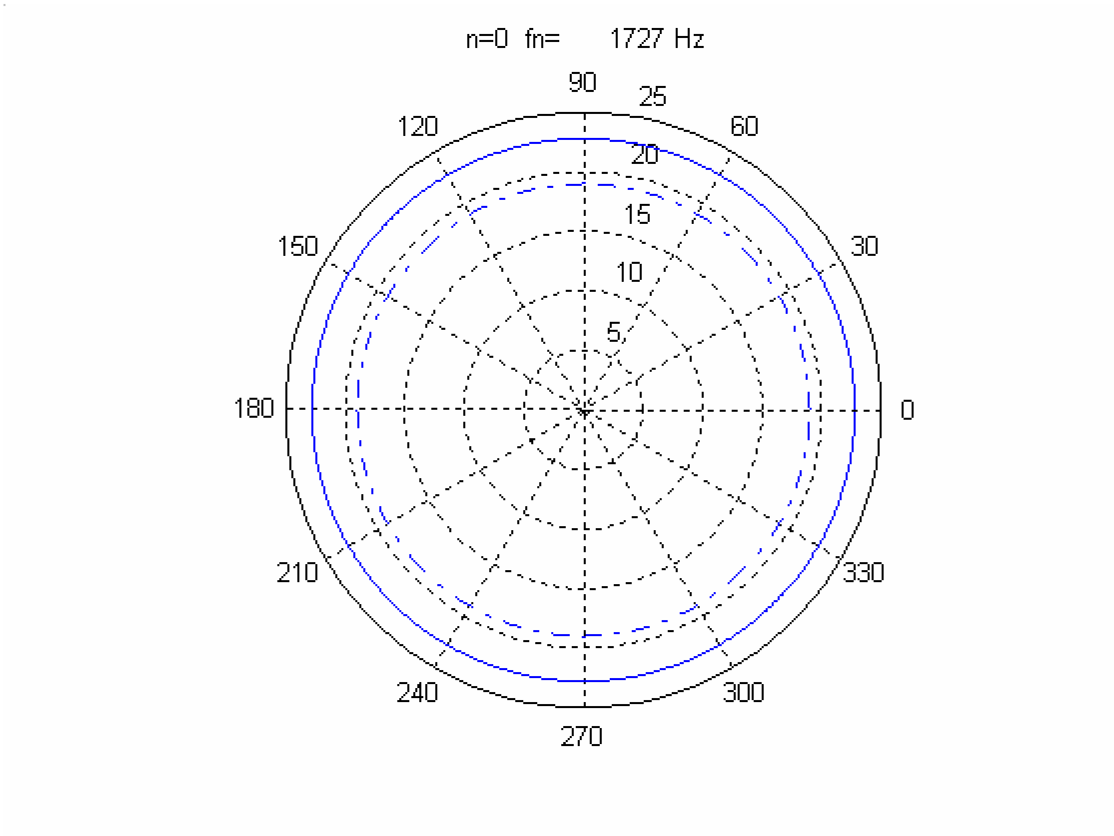

Figure A-7. n=0, Freq = 1727 Hz, Extension Mode

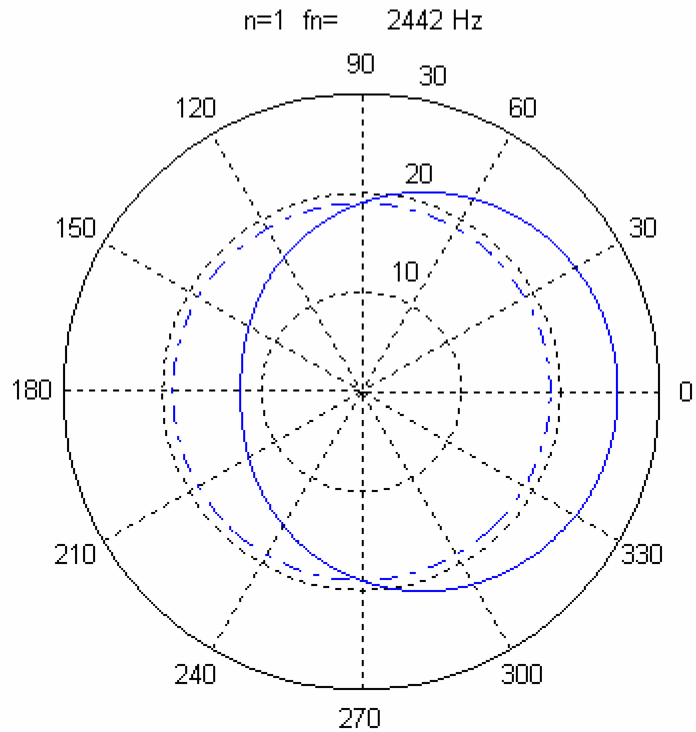

Figure A-8. n=1, Freq = 2442 Hz, Extension Mode