POLYNOMIAL INTERPOLATION FOR FINITE ELEMENT ANALYSIS

By Tom Irvine
Email: tomirvine@aol.com

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Linear Interpolation

Consider a line which passes through the points: (x_1, y_1) and (x_2, y_2).

Determine an interpolation equation for the domain: $x_1 < x < x_2$

$$f(x) = y_1\left[\frac{x_2 - x}{x_2 - x_1}\right] + y_2\left[\frac{x - x_1}{x_2 - x_1}\right] \tag{1}$$

Transform the interpolation equation to local coordinates.

Let

$$\xi = x - x_1 \tag{2}$$

$$x = \xi + x_1 \tag{3}$$

$$L = x_2 - x_1 \tag{4}$$

$$x_2 = x_1 + L \tag{5}$$

$$x_2 - x = x_1 + L - x \tag{6}$$

$$x_2 - x = x_1 + L - \xi - x_1 \tag{7}$$

By substitution,

$$f(\xi) = y_1\left[\frac{L - \xi}{L}\right] + y_2\left[\frac{\xi}{L}\right] \tag{8}$$

$$f(\xi) = y_1\left[1 - \frac{\xi}{L}\right] + y_2\left[\frac{\xi}{L}\right] \tag{9}$$

Quadratic Interpolation

Consider a curve which passes through the points: (x_1, y_1) , (x_2, y_2) & (x_3, y_3).

Determine an interpolation equation for the domain: $x_1 < x < x_3$

$$f(x) = y_1\left[\frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}\right] + y_2\left[\frac{(x_3 - x)(x - x_1)}{(x_3 - x_2)(x_2 - x_1)}\right] + y_3\left[\frac{(x - x_2)(x - x_1)}{(x_3 - x_2)(x_3 - x_1)}\right] \quad (10)$$

Transform the interpolation equation to local coordinates.

Let

$$\xi = x - x_1 \quad (11)$$

$$x = \xi + x_1 \quad (12)$$

$$L_{12} = x_2 - x_1 \quad (13)$$

$$L_{13} = x_3 - x_1 \quad (14)$$

$$L_{23} = x_3 - x_2 \quad (15)$$

$$f(\xi) = y_1\left[\frac{(x_2 - \xi - x_1)(x_3 - \xi - x_1)}{L_{12} L_{13}}\right] + y_2\left[\frac{(x_3 - \xi - x_1)\xi}{L_{23} L_{12}}\right] + y_3\left[\frac{(\xi + x_1 - x_2)\xi}{L_{23} L_{13}}\right] \quad (16)$$

$$f(\xi) = y_1\left[\frac{(L_{12} - \xi)(L_{13} - \xi)}{L_{12} L_{13}}\right] + y_2\left[\frac{(L_{13} - \xi)\xi}{L_{23} L_{12}}\right] + y_3\left[\frac{(\xi - L_{12})\xi}{L_{23} L_{13}}\right] \quad (17)$$

$$f(\xi) = y_1\left[\left(1-\frac{\xi}{L_{12}}\right)\left(1-\frac{\xi}{L_{13}}\right)\right] + y_2\left[\frac{L_{13}}{L_{23}L_{12}}\right]\left[\frac{(L_{13}-\xi)\xi}{L_{13}}\right] + y_3\left[\frac{L_{12}}{L_{23}L_{13}}\right]\left[\frac{(\xi-L_{12})\xi}{L_{12}}\right] \tag{18}$$

$$f(\xi) = y_1\left[\left(1-\frac{\xi}{L_{12}}\right)\left(1-\frac{\xi}{L_{13}}\right)\right] + y_2\left[\frac{L_{13}}{L_{23}L_{12}}\right]\left[\left(1-\frac{\xi}{L_{13}}\right)\xi\right] + y_3\left[\frac{L_{12}}{L_{23}L_{13}}\right]\left[\left(\frac{\xi}{L_{12}}-1\right)\xi\right] \tag{19}$$

Now consider the special case where

$$L_{13} = L \tag{20}$$

$$L_{12} = L_{23} = L/2 \tag{21}$$

By substitution,

$$f(\xi) = y_1\left[\left(1-\frac{2\xi}{L}\right)\left(1-\frac{\xi}{L}\right)\right] + y_2[4]\left[\left(1-\frac{\xi}{L}\right)\left(\frac{\xi}{L}\right)\right] + y_3\left[\left(\frac{2\xi}{L}-1\right)\left(\frac{\xi}{L}\right)\right] \tag{22}$$

Cubic Interpolation

Consider a curve which passes through the points: (x_1, y_1) , (x_2, y_2) , (x_3, y_3) & (x_4, y_4)

Determine an interpolation equation for the domain: $x_1 < x < x_4$

Fix here

$$f(x) = y_1\left[\frac{(x_2 - x)(x_3 - x)(x_4 - x)}{(x_2 - x_1)(x_3 - x_1)(x_4 - x_1)}\right] + y_2\left[\frac{(x_4 - x)(x_3 - x)(x - x_1)}{(x_3 - x_2)(x_2 - x_1)(x_4 - x_2)}\right]$$

$$+ y_3\left[\frac{(x - x_2)(x - x_1)(x_4 - x)}{(x_3 - x_2)(x_3 - x_1)(x_4 - x_3)}\right] + y_4\left[\frac{(x - x_3)(x - x_2)(x - x_1)}{(x_4 - x_3)(x_4 - x_2)(x_4 - x_1)}\right] \tag{23}$$

Let

$$\xi = x - x_1 \tag{24}$$

$$x = \xi + x_1 \tag{25}$$

$$L_{12} = x_2 - x_1 \tag{26}$$

$$L_{13} = x_3 - x_1 \tag{27}$$

$$L_{14} = x_4 - x_1 \tag{28}$$

$$L_{23} = x_3 - x_2 \tag{29}$$

$$L_{24} = x_4 - x_2 \tag{30}$$

$$L_{34} = x_4 - x_3 \tag{31}$$

$$f(\xi) = y_1\left[\frac{(x_2-\xi-x_1)(x_3-\xi-x_1)(x_4-\xi-x_1)}{L_{12}L_{13}L_{14}}\right] + y_2\left[\frac{(x_4-\xi-x_1)(x_3-\xi-x_1)\xi}{L_{23}L_{12}L_{24}}\right]$$

$$+ y_3\left[\frac{(\xi+x_1-x_2)(x_4-\xi-x_1)\xi}{L_{23}L_{13}L_{34}}\right] + y_4\left[\frac{(\xi+x_1-x_3)(\xi+x_1-x_2)\xi}{L_{34}L_{24}L_{14}}\right] \tag{32}$$

$$f(\xi) = y_1\left[\frac{(L_{12}-\xi)(L_{13}-\xi)(L_{14}-\xi)}{L_{12}L_{13}L_{14}}\right] + y_2\left[\frac{(L_{14}-\xi)(L_{13}-\xi)\xi}{L_{23}L_{12}L_{24}}\right]$$

$$+ y_3\left[\frac{(\xi-L_{12})(L_{14}-\xi)\xi}{L_{23}L_{13}L_{34}}\right] + y_4\left[\frac{(\xi+L_{13})(\xi-L_{12})\xi}{L_{34}L_{24}L_{14}}\right] \tag{33}$$

$$f(\xi) = y_1\left[\left(1-\frac{\xi}{L_{12}}\right)\left(1-\frac{\xi}{L_{13}}\right)\left(1-\frac{\xi}{L_{14}}\right)\right]$$

$$+ y_2\left[\frac{L_{14}L_{13}}{L_{23}L_{12}L_{24}}\right]\left[\frac{(L_{14}-\xi)(L_{13}-\xi)\xi}{L_{14}L_{13}}\right]$$

$$+ y_3\left[\frac{L_{12}L_{14}}{L_{23}L_{13}L_{34}}\right]\left[\frac{(\xi-L_{12})(L_{14}-\xi)\xi}{L_{12}L_{14}}\right]$$

$$+ y_4\left[\frac{L_{13}L_{12}}{L_{34}L_{24}L_{14}}\right]\left[\frac{(\xi+L_{13})(\xi-L_{12})\xi}{L_{13}L_{12}}\right] \tag{34}$$

$$f(\xi) = y_1\left[\left(1-\frac{\xi}{L_{12}}\right)\left(1-\frac{\xi}{L_{13}}\right)\left(1-\frac{\xi}{L_{14}}\right)\right]$$

$$+ y_2\left[\frac{L_{14}L_{13}}{L_{23}L_{12}L_{24}}\right]\left[\left(1-\frac{\xi}{L_{13}}\right)\left(1-\frac{\xi}{L_{14}}\right)\xi\right]$$

$$+ y_3\left[\frac{L_{12}L_{14}}{L_{23}L_{13}L_{34}}\right]\left[\left(\frac{\xi}{L_{12}}-1\right)\left(1-\frac{\xi}{L_{14}}\right)\xi\right]$$

$$+ y_4\left[\frac{L_{13}L_{12}}{L_{34}L_{24}L_{14}}\right]\left[\left(\frac{\xi}{L_{12}}-1\right)\left(\frac{\xi}{L_{13}}-1\right)\xi\right] \tag{35}$$

Now consider the special case where

$$L_{14} = L \tag{36}$$

$$L_{13} = L_{24} = 2L/3 \tag{37}$$

$$L_{12} = L_{23} = L_{34} = L/3 \tag{38}$$

By substitution,

$$f(\xi) = y_1\left[\left(1-\frac{3\xi}{L}\right)\left(1-\frac{2\xi}{L}\right)\left(1-\frac{\xi}{L}\right)\right] \quad + \quad y_2\,[9]\left[\left(1-\frac{3\xi}{2L}\right)\left(1-\frac{\xi}{L}\right)\left(\frac{\xi}{L}\right)\right]$$

$$+ y_3\left[\frac{9}{2}\right]\left[\left(\frac{3\xi}{L}-1\right)\left(1-\frac{\xi}{L}\right)\left(\frac{\xi}{L}\right)\right] \quad + \quad y_4\left[\left(\frac{3\xi}{L}-1\right)\left(\frac{3\xi}{2L}-1\right)\left(\frac{\xi}{L}\right)\right] \tag{39}$$