VIBRATION ANALYSIS OF A MASS WITH
AN ARBITRARY NUMBER OF ISOLATORS
Revision B

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Introduction

An avionics component may be mounted with isolator grommets, which act as soft springs. The goal of the isolator design is to provide attenuation of shock and vibration energy. This is achieved primarily by lowering the natural frequency of the component system. In addition, the isolators provide damping.

Consider a component with a complex geometry that is to be mounted via an arbitrary number of isolators, as shown in Figures 1 and 2. Assume that the component's hardmounted natural frequency is at least one octave greater than any of its isolation frequencies.

The objective is to derive the equations of motion for this system, accounting for six degrees-of-freedom.

Derivation

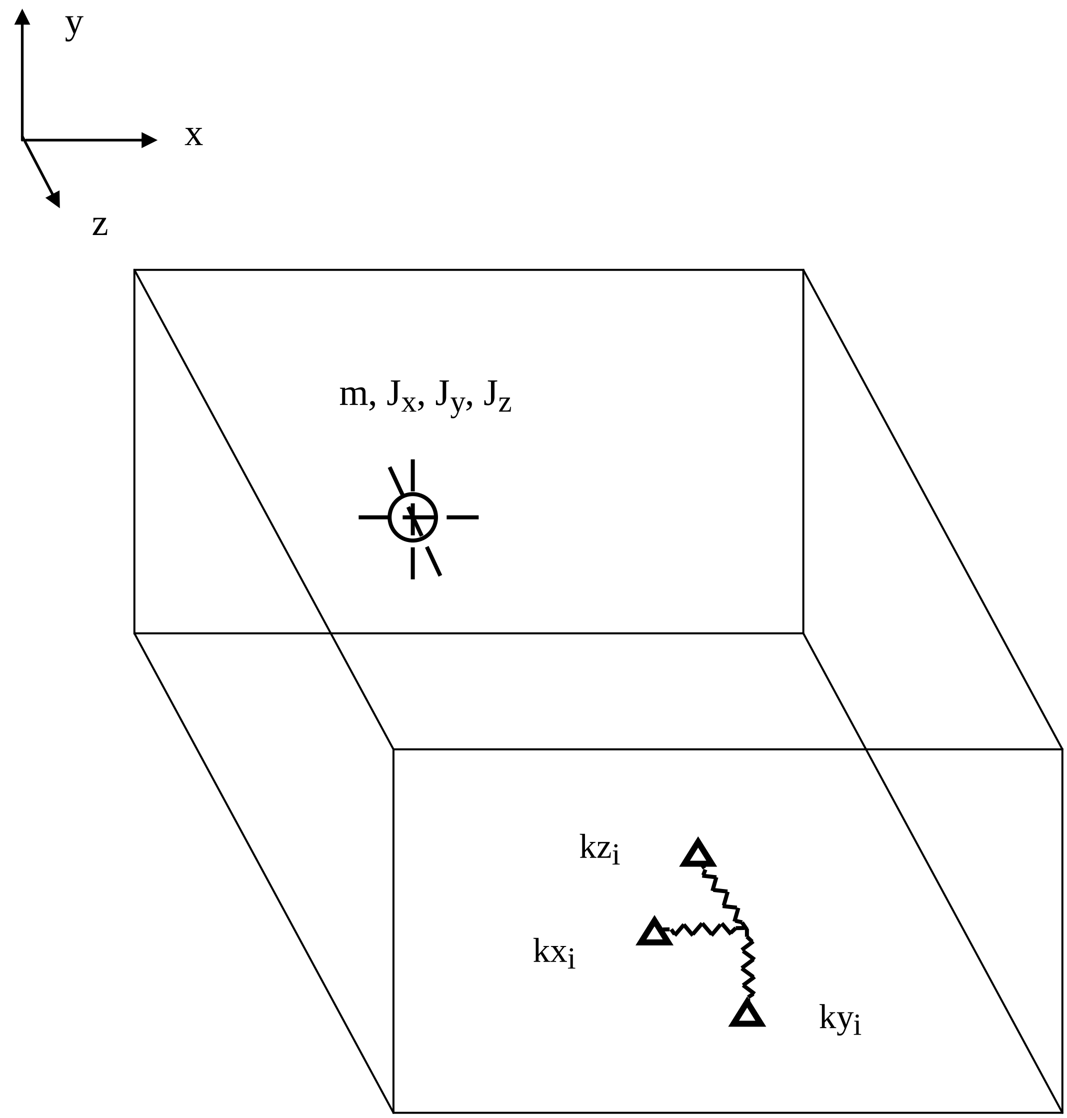

Figure 1. Isolated Avionics Component Model

The mass and inertia are represented at a point with the circle symbol. Each isolator is modeled by three orthogonal DOF springs. Only one spring set is shown for brevity. The triangles indicate fixed constraints. The origin is at the center of gravity.

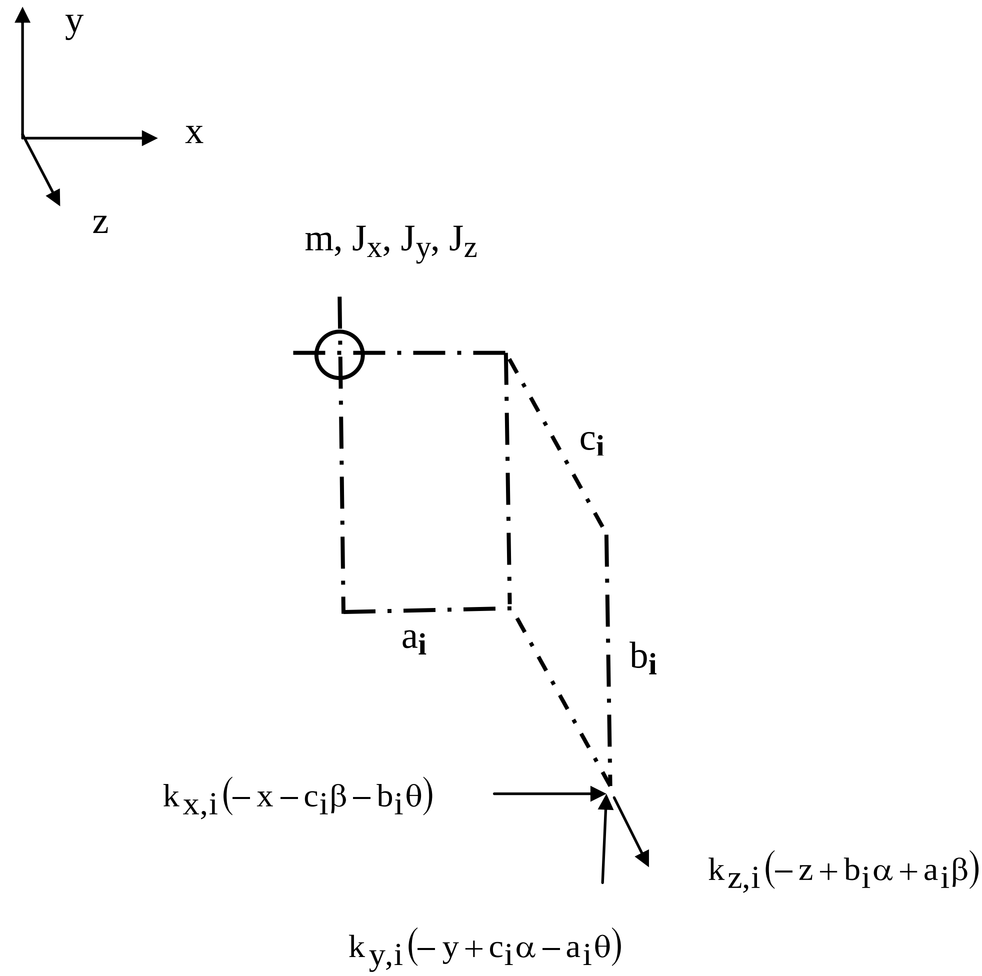

Figure 2. Isolated Avionics Component Model with Dimensions

The variables α, β, and θ represent rotations about the X, Y, and Z axes, respectively, using the right-hand rule convention.

Derive the equations of motion. Let n be the number of isolators.

$$m\ddot{x} = \sum_{i=1}^{n} k_{x,i}\left(-x - c_i\beta - b_i\theta\right) \tag{1}$$

$$m\ddot{x} + \sum_{i=1}^{n} k_{x,i}\left(x + c_i\beta + b_i\theta\right) = 0 \tag{2}$$

$$m\ddot{x} + x\sum_{i=1}^{n} k_{x,i} + \beta\sum_{i=1}^{n} k_{x,i}\, c_i + \theta\sum_{i=1}^{n} k_{x,i} b_i = 0 \tag{3}$$

$$m\ddot{y} = \sum_{i=1}^{n} k_{y,i}\left(-y + c_i\alpha - a_i\theta\right) \tag{4}$$

$$m\ddot{y} + \sum_{i=1}^{n} k_{y,i}\left(y - c_i\alpha + a_i\theta\right) = 0 \tag{5}$$

$$m\ddot{y} + y\sum_{i=1}^{n} k_{y,i} - \alpha\sum_{i=1}^{n} k_{y,i}\, c_i + \theta\sum_{i=1}^{n} k_{y,i}\, a_i = 0 \tag{6}$$

$$m\ddot{z} = \sum_{i=1}^{n} k_{z,i}\left(-z + b_i\alpha + a_i\beta\right) \tag{7}$$

$$m\ddot{z} + \sum_{i=1}^{n} k_{z,i}\left(z - b_i\alpha - a_i\beta\right) = 0 \tag{8}$$

$$m\ddot{z} + z\sum_{i=1}^{n} k_{z,i} - \alpha\sum_{i=1}^{n} k_{z,i} b_i - \beta\sum_{i=1}^{n} k_{z,i} a_i = 0 \tag{9}$$

$$J_x\,\ddot{\alpha} = -\sum_{i=1}^{n} k_{y,i}\left(-y + c_i\alpha - a_i\theta\right)c_i - \sum_{i=1}^{n} k_{z,i}\left(-z + b_i\alpha + a_i\beta\right)b_i \tag{10}$$

$$J_x\,\ddot{\alpha} + \sum_{i=1}^{n} k_{y,i}\left(-y + c_i\alpha - a_i\theta\right)c_i + \sum_{i=1}^{n} k_{z,i}\left(-z + b_i\alpha + a_i\beta\right)b_i = 0 \tag{11}$$

$$\begin{aligned} & J_x\,\ddot{\alpha} - y\sum_{i=1}^{n} k_{y,i}\,c_i - z\sum_{i=1}^{n} k_{z,i}b_i \\ & + \alpha\sum_{i=1}^{n}\left[k_{y,i}c_i^{\,2} + k_{z,i}b_i^{\,2}\right] + \beta\sum_{i=1}^{n}\left[k_{z,i}a_i b_i\right] - \theta\sum_{i=1}^{n}\left[k_{y,i}\,a_i c_i\right] = 0 \end{aligned} \tag{12}$$

$$J_y\,\ddot{\beta} + \sum_{i=1}^{n} k_{x,i}\left(-x - c_i\beta - b_i\theta\right)c_i - \sum_{i=1}^{n} k_{z,i}\left(-z + b_i\alpha + a_i\beta\right)a_i \tag{13}$$

$$\begin{aligned} & J_y\,\ddot{\beta} + x\sum_{i=1}^{n} k_{x,i}\,c_i - z\sum_{i=1}^{n} k_{z,i}a_i \\ & + \alpha\sum_{i=1}^{n} k_{z,i}\,a_i b_i + \beta\sum_{i=1}^{n}\left[k_{x,i}c_i^{\,2} + k_{z,i}a_i^{\,2}\right] + \theta\sum_{i=1}^{n} k_{x,i}\,b_i c_i = 0 \end{aligned} \tag{14}$$

$$J_z\,\ddot{\theta} = \sum_{i=1}^{n} k_{x,i}\left(-x - c_i\beta - b_i\theta\right)b_i + \sum_{i=1}^{n} k_{y,i}\left(-y + c_i\alpha - a_i\theta\right)a_i \tag{15}$$

$$J_z\,\ddot{\theta} + \sum_{i=1}^{n} k_{x,i}\left(x + c_i\beta + b_i\theta\right)b_i + \sum_{i=1}^{n} k_{y,i}\left(y - c_i\alpha + a_i\theta\right)a_i = 0 \tag{16}$$

$$J_z\,\ddot{\theta} + x\sum_{i=1}^{n} k_{x,i} b_i + y\sum_{i=1}^{n} k_{y,i} a_i$$

$$-\alpha\sum_{i=1}^{n} k_{y,i}\, a_i c_i + \beta\sum_{i=1}^{n} k_{x,i} b_i c_i + \theta\sum_{i=1}^{n}\left[k_{y,i} a_i{}^2 + k_{x,i} b_i{}^2\right] = 0$$

(17)

Typically

$$k_{x,i} = k_x \quad \text{for all i} \tag{18}$$

$$k_{y,i} = k_y \quad \text{for all i} \tag{19}$$

$$k_{z,i} = k_z \quad \text{for all i} \tag{20}$$

By substitution, the equations of motion simplify to

$$m\ddot{x} + x\,n\,k_x + \beta k_x \sum_{i=1}^{n} c_i + \theta k_x \sum_{i=1}^{n} b_i = 0 \tag{21}$$

$$m\ddot{y} + y\,n\,k_y - \alpha k_y \sum_{i=1}^{n} c_i + \theta k_y \sum_{i=1}^{n} a_i = 0 \tag{22}$$

$$m\ddot{z} + z\,n\,k_z - \alpha k_z \sum_{i=1}^{n} b_i - \beta k_z \sum_{i=1}^{n} a_i = 0 \tag{23}$$

$$J_x\,\ddot{\alpha} - y k_y \sum_{i=1}^{n} c_i - z k_z \sum_{i=1}^{n} b_i$$

$$+\alpha\sum_{i=1}^{n}\left[k_y c_i{}^2 + k_z b_i{}^2\right] + \beta k_z \sum_{i=1}^{n}\left[a_i b_i\right] - \theta k_y \sum_{i=1}^{n}\left[a_i c_i\right] = 0$$

(24)

$$J_y \ddot{\beta} + x\, k_x \sum_{i=1}^{n} c_i - z\, k_z \sum_{i=1}^{n} a_i$$

$$+ \alpha\, k_z \sum_{i=1}^{n} a_i b_i + \beta \sum_{i=1}^{n} \left[k_x c_i^2 + k_z a_i^2 \right] + \theta\, k_x \sum_{i=1}^{n} b_i c_i = 0$$

(25)

$$J_z \ddot{\theta} + x\, k_x \sum_{i=1}^{n} b_i + y\, k_y \sum_{i=1}^{n} a_i$$

$$- \alpha\, k_y \sum_{i=1}^{n} a_i c_i + \beta\, k_x \sum_{i=1}^{n} b_i c_i + \theta \sum_{i=1}^{n} \left[k_y a_i^2 + k_x b_i^2 \right] = 0$$

(26)

The equations can be arranged in matrix format.

$$\underline{M} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\theta} \end{bmatrix} + \underline{K} \begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(27)

The mass and stiffness matrices are shown in upper triangular form due to symmetry.

$$\underline{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 \\ & & m & 0 & 0 & 0 \\ & & & J_x & 0 & 0 \\ & & & & J_y & 0 \\ & & & & & J_z \end{bmatrix}$$

(28)

$$\underline{K} = \begin{bmatrix} nk_x & 0 & 0 & 0 & k_x \sum_{i=1}^{n} c_i & k_x \sum_{i=1}^{n} b_i \\ & nk_y & 0 & -k_y \sum_{i=1}^{n} c_i & 0 & k_y \sum_{i=1}^{n} a_i \\ & & nk_z & -k_z \sum_{i=1}^{n} b_i & k_z \sum_{i=1}^{n} a_i & 0 \\ & & & \sum_{i=1}^{n} \left[k_y c_i{}^2 + k_z b_i{}^2 \right] & k_z \sum_{i=1}^{n} \left[a_i b_i \right] & -k_y \sum_{i=1}^{n} \left[a_i c_i \right] \\ & & & & \sum_{i=1}^{n} \left[k_x c_i{}^2 + k_z a_i{}^2 \right] & k_x \sum_{i=1}^{n} b_i c_i \\ & & & & & \sum_{i=1}^{n} \left[k_y a_i{}^2 + k_x b_i{}^2 \right] \end{bmatrix} \tag{29}$$

Base Excitation

Consider three separate base excitation cases, following the convention in Reference 1.

The mass and stiffness matrices in equations (28) and (29) apply in each of the cases.

Let u, x, and w be the base displacement in the X, Y and Z-axes, respectively.

X-axis Excitation

Let

$$r_1 = x - u \tag{30}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{r}_1\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}r_1\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}-m\ddot{u}\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{31}$$

Y-axis Excitation

Let

$$r_2 = y - v \tag{32}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{r}_2\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}x\\ r_2\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ -m\ddot{v}\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{33}$$

Z-axis Excitation

Let

$$r_3 = z - w \tag{34}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{r}_3\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}x\\ y\\ r_3\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ 0\\ -m\ddot{w}\\ 0\\ 0\\ 0\end{bmatrix} \tag{35}$$

The acceleration transmissibility functions of equations (31), (33) and (35) can then be determined via Reference 2.

The modal transient response to a base input time history can be calculated via References 3 and 4.

<u>References</u>

1. T. Irvine, Vibration Analysis of an Isolated Mass with Six Degrees of Freedom, Revision F, Vibrationdata, 2011.

2. T. Irvine, Frequency Response Function Analysis of a Multi-degree-of-freedom System with Enforced Motion, Vibrationdata, 2011.

3. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Subjected to Base Excitation, Revision B, Vibrationdata, 2004.

4. T. Irvine, Shock Response of Multi-degree-of-freedom Systems, Revision F, Vibrationdata, 2010.

APPENDIX A

Example 1

A mass is mounted to a surface with six isolators. The system has the following properties.

M	=	35 lbm
Jx	=	804 lbm in^2
Jy	=	1213 lbm in^2
Jz	=	1035 lbm in^2
kx	=	1200 lbf/in
ky	=	1600 lbf/in
kz	=	1200 lbf/in

The six springs mount to the box at the following distances (inches) from the box C.G. The pattern is shown in Figure A-1.

Isolator	X	Y	Z
1	5.0	-3.91	-5.3
2	5.0	-3.91	5.3
3	7.1	-3.91	0
4	-5.0	-3.91	5.3
5	-5.0	-3.91	-5.3
6	-7.1	-3.91	0

Assume uniform modal damping of 10%, which is equivalent to Q=5.

The results are calculated via Matlab script: arbitrary_isolators.m

A partial output listing is given on the next page. The results agree with a separate finite element model which is omitted for brevity.

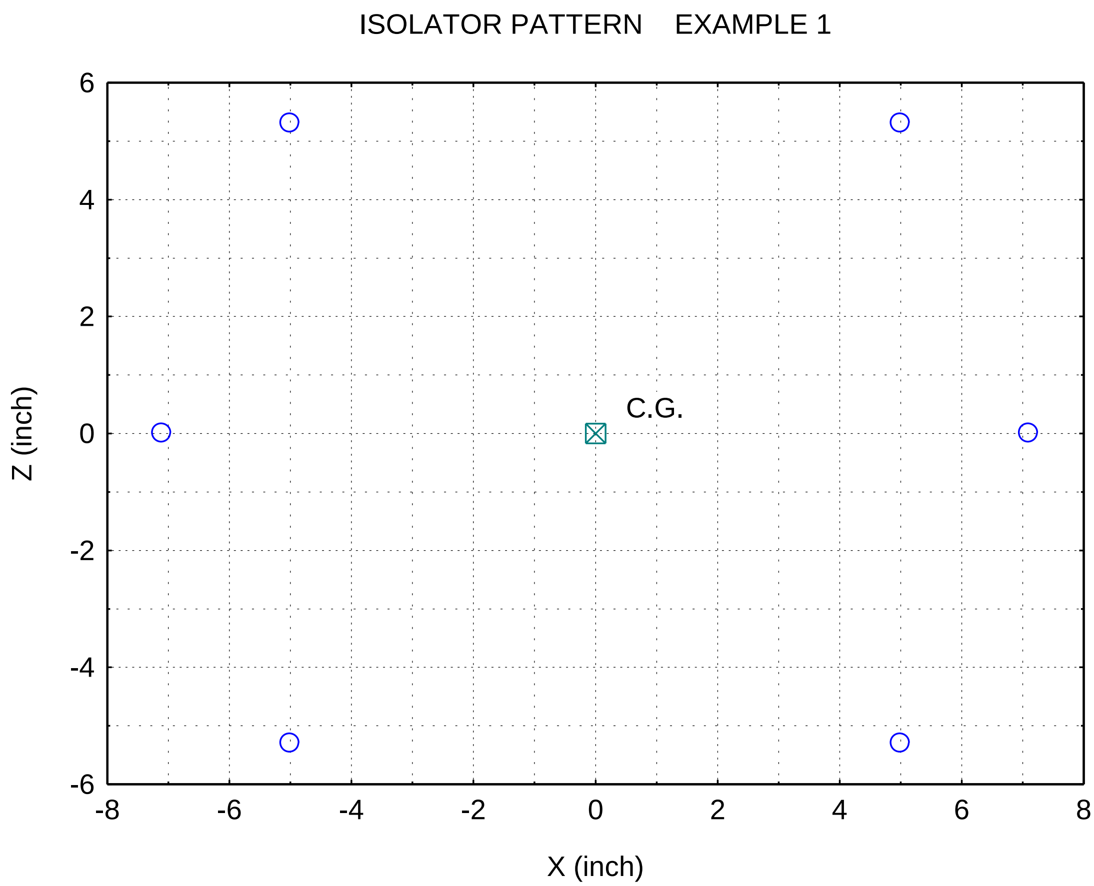

Figure A-1.

The circles represent the isolator mounting points. The view is looking down along the Y-axis to the XZ plane.

```
The mass matrix is

m =

    0.0907         0         0         0         0         0
         0    0.0907         0         0         0         0
         0         0    0.0907         0         0         0
         0         0         0    2.0829         0         0
         0         0         0         0    3.1425         0
         0         0         0         0         0    2.6813


 The stiffness matrix is

k =

  1.0e+005 *

    0.0720         0         0         0         0    0.2815
         0    0.0960         0         0         0    0.0000
         0         0    0.0720   -0.2815    0.0000         0
         0         0   -0.2815    2.8985    0.0000         0
         0         0    0.0000    0.0000    3.7582         0
    0.2815    0.0000         0         0         0    4.3139


 Eigenvalues

lambda =

  1.0e+005 *

    0.3794    0.5001    1.0587    1.1959    1.8062    1.9028


  Natural Frequencies =
 1.        31 Hz
 2.     35.59 Hz
 3.     51.79 Hz
 4.     55.04 Hz
 5.     67.64 Hz
 6.     69.43 Hz
```

Modes Shapes (rows represent modes)

x	y	z	alpha	beta	theta
0	0	2.8	0.374	0	0
-2.95	0	0	0	0	0.28
0	-3.32	0	0	0	0
0	0	0	0	-0.564	0
0	0	-1.79	0.584	0	0
-1.52	0	0	0	0	-0.543

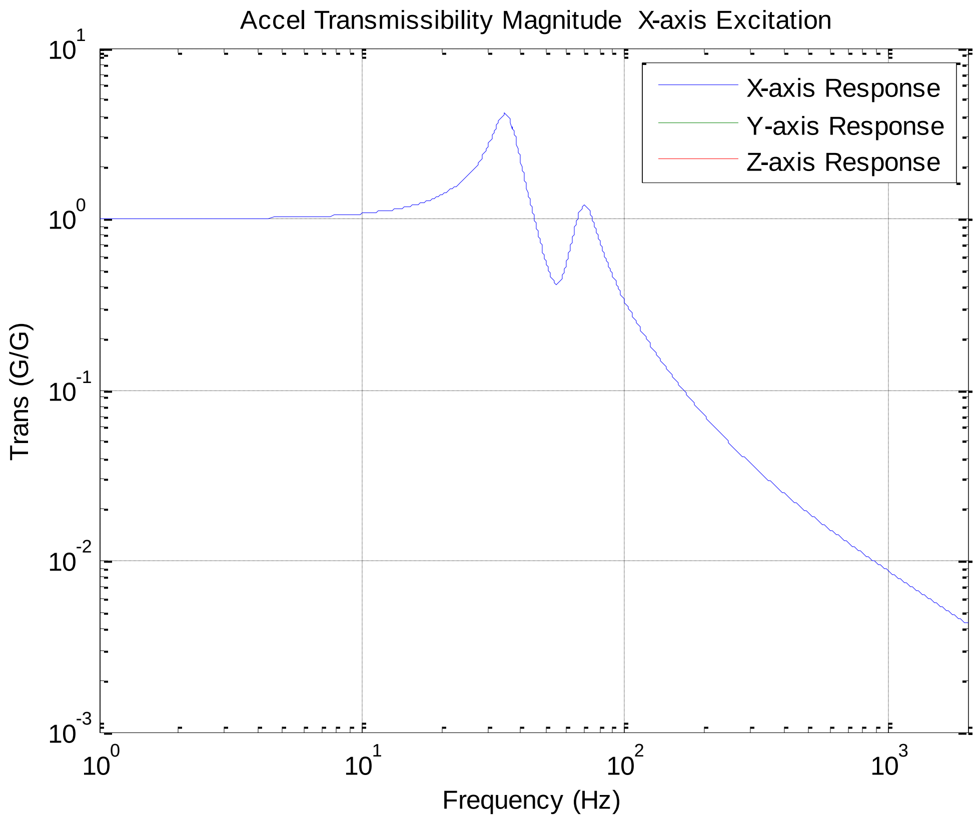

Figure A-2.

Both the Y and Z-axes responses were below the lower amplitude plot limit.

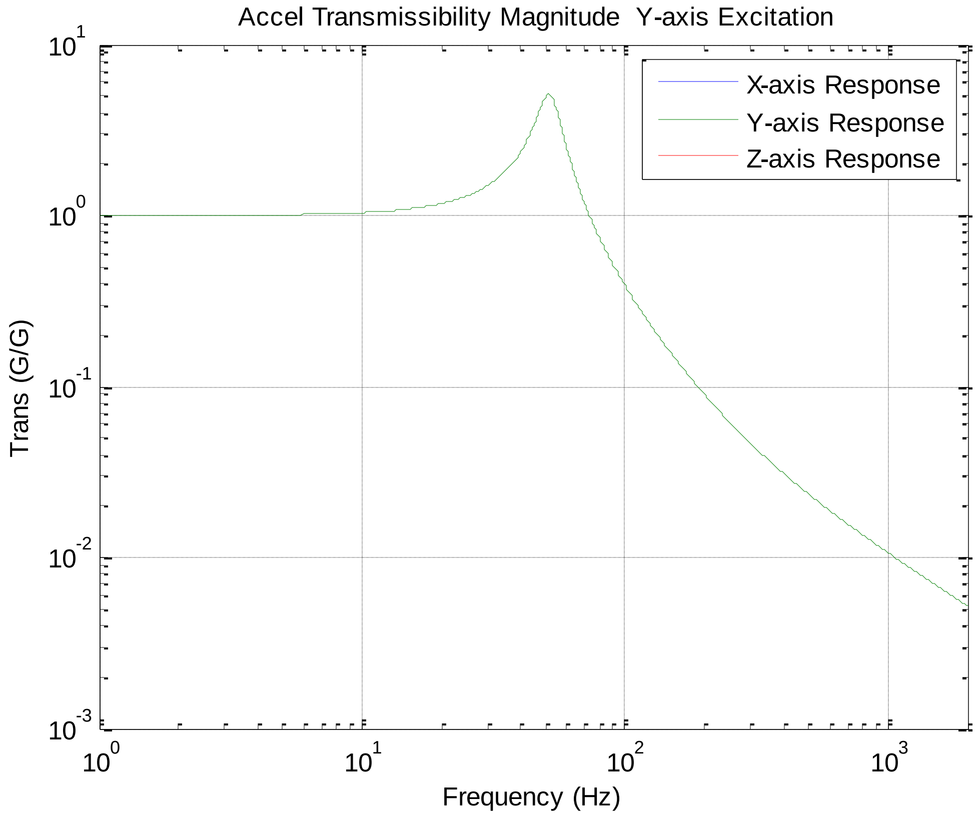

Figure A-3.

Both the X and Z-axes responses were below the lower amplitude plot limit.

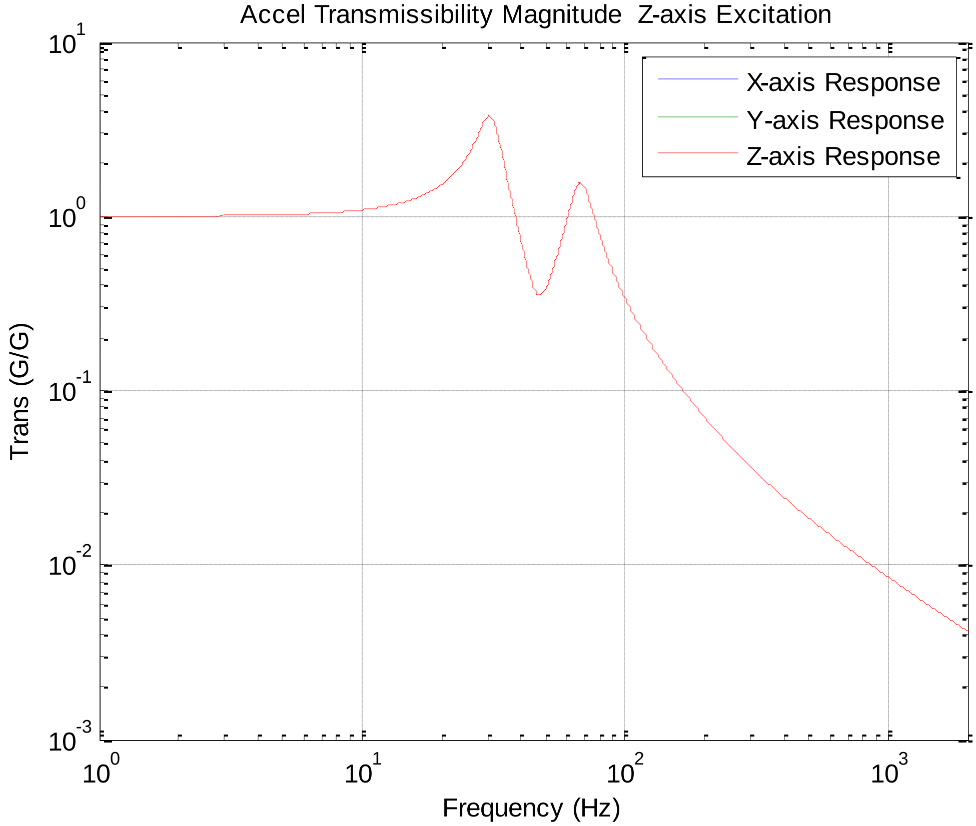

Figure A-4.

Both the X and Y-axes responses were below the lower amplitude plot limit.

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APPENDIX B

Example 2

A mass is mounted to a surface with six isolators. The system has the following properties.

M	=	9.0 lbm
Jx	=	90.17 lbm in^2
Jy	=	105.2 lbm in^2
Jz	=	42.08 lbm in^2
kx	=	500 lbf/in
ky	=	500 lbf/in
kz	=	500 lbf/in

The six springs mount to the box at the following distances (inches) from the box C.G. The pattern is shown in Figure B-1.

Isolator	X	Y	Z
1	-3	-2	-3
2	-3	-2	1
3	-3	-2	5
4	3	-2	-3
5	3	-2	1
6	3	-2	5

The results are calculated via Matlab script: arbitrary_isolators.m

A partial output listing is given on the next page. The results agree with the example in Reference 1.

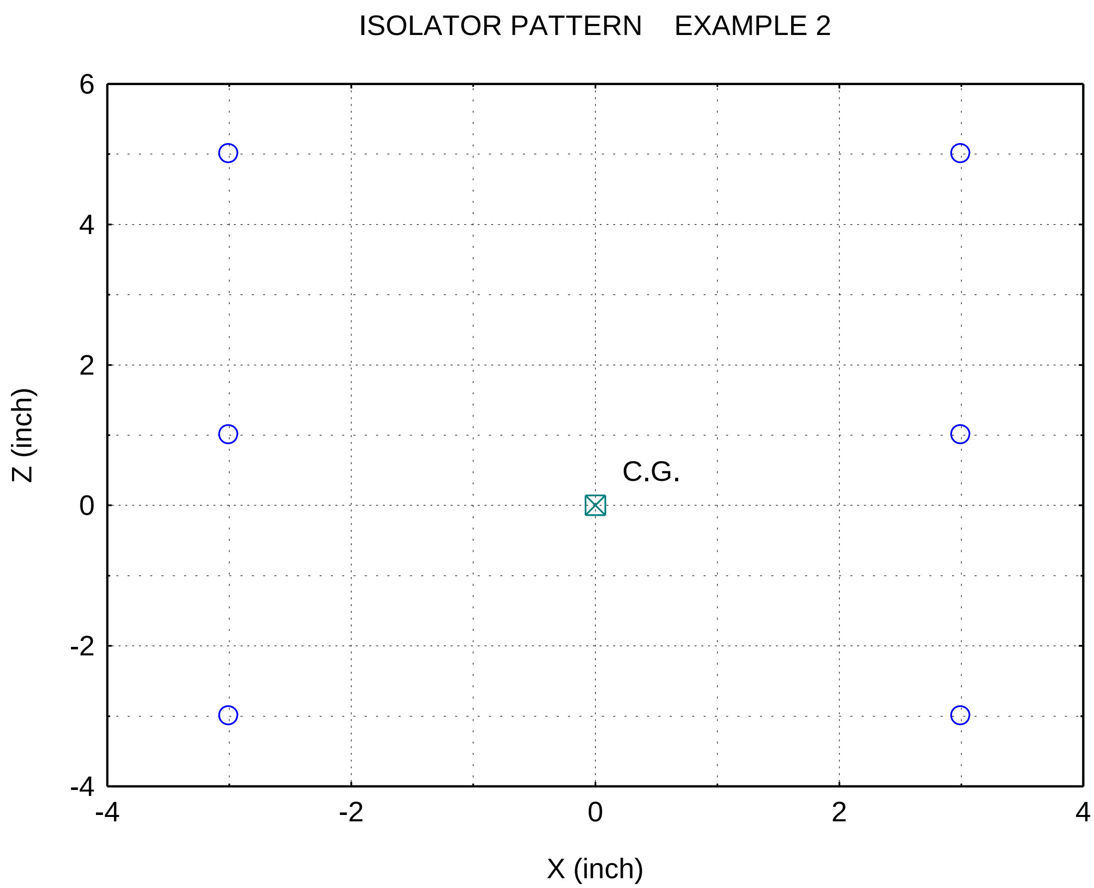

Figure B-1.

The circles represent the isolator mounting points. The view is looking down along the Y-axis to the XZ plane.

```
The mass matrix is

m =

 0.0233       0       0       0       0       0
      0  0.0233       0       0       0       0
      0       0  0.0233       0       0       0
      0       0       0  0.2336       0       0
      0       0       0       0  0.2725       0
      0       0       0       0       0   0.109

 The stiffness matrix is

k =

   3000       0       0       0    3000    6000
      0    3000       0   -3000       0       0
      0       0    3000   -6000       0       0
      0   -3000   -6000   47000       0       0
   3000       0       0       0   62000    6000
   6000       0       0       0    6000   39000

Eigenvalues

lambda =

  1.0e+005 *

    0.6707    0.7501    1.2867    2.1907    2.6280    4.1983

  Natural Frequencies =
 1.      41.22 Hz
 2.      43.59 Hz
 3.      57.09 Hz
 4.      74.49 Hz
 5.      81.59 Hz
 6.      103.1 Hz
```

Modes Shapes (rows represent modes)

	x	y	z	alpha	beta	theta
1	0	2.42	4.85	1.16	0	0
2	6.01	0	0	0	-0.273	-1.12
3	0	-5.86	2.93	0	0	0
4	-0.254	0	0	0	-1.84	0.831
5	0	-1.64	-3.29	1.71	0	0
6	2.58	0	0	0	0.455	2.69