VIBRATION ANALYSIS OF AN ISOLATED MASS
WITH THREE-DEGREES-OF-FREEDOM

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Derivation

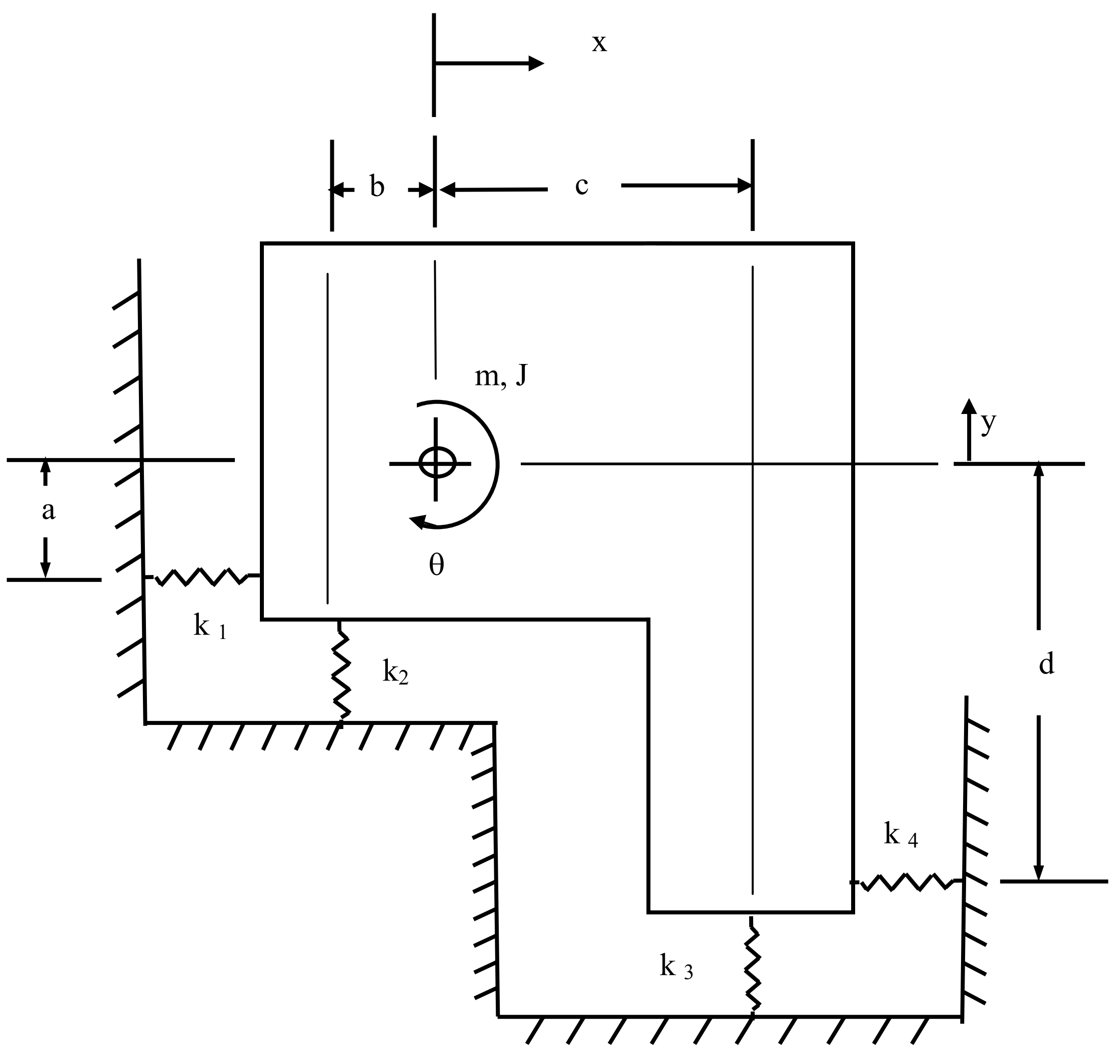

Figure 1.

The total kinetic energy is

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} J \dot{\theta}^2 \tag{1}$$

The total potential energy is

$$V = \frac{1}{2} k_1 (x - a\theta)^2 + \frac{1}{2} k_2 (y + b\theta)^2 + \frac{1}{2} k_3 (y - c\theta)^2 + \frac{1}{2} k_4 (x - d\theta)^2 \tag{2}$$

The energy is

$$E = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} k_1 (x - a\theta)^2 + \frac{1}{2} k_2 (y + b\theta)^2 + \frac{1}{2} k_3 (y - c\theta)^2 + \frac{1}{2} k_4 (x - d\theta)^2 \tag{3}$$

The energy method is based on conservation of energy.

$$\frac{d}{dt} E = 0 \tag{4}$$

$$\begin{aligned} m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + J\dot{\theta}\ddot{\theta} + k_1 (x - a\theta)\dot{x} + k_2 (y + b\theta)\dot{y} + k_3 (y - c\theta)\dot{y} + k_4 (x - d\theta)\dot{x} \\ - k_1 a(x - a\theta)\dot{\theta} + k_2 b(y + b\theta)\dot{\theta} - k_3 c(y - c\theta)\dot{\theta} - k_4 d(x - d\theta)\dot{\theta} = 0 \end{aligned} \tag{5}$$

Equation (5) can be separated into three individual equations.

$$m\dot{x}\ddot{x} + k_1 (x - a\theta)\dot{x} + k_4 (x - d\theta)\dot{x} = 0 \tag{6}$$

$$m\dot{y}\ddot{y} + k_2 (y + b\theta)\dot{y} + k_3 (y - c\theta)\dot{y} = 0 \tag{7}$$

$$J\dot{\theta}\ddot{\theta} - k_1 a(x - a\theta)\dot{\theta} + k_2 b(y + b\theta)\dot{\theta} - k_3 c(y - c\theta)\dot{\theta} - k_4 d(x - d\theta)\dot{\theta} = 0 \tag{8}$$

$$m\ddot{x} + k_1 (x - a\theta) + k_4 (x - d\theta) = 0 \tag{9}$$

$$m\ddot{y} + k_2 (y + b\theta) + k_3 (y - c\theta) = 0 \tag{10}$$

$$J\ddot{\theta} - k_1 a(x - a\theta) + k_2 b(y + b\theta) - k_3 c(y - c\theta) - k_4 d(x - d\theta) = 0 \tag{11}$$

The equations are assembled in matrix format.

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} k_1+k_4 & 0 & -k_1 a-k_4 d \\ 0 & k_2+k_3 & k_2 b-k_3 c \\ -k_1 a-k_4 d & k_2 b-k_3 c & k_1 a^2+k_2 b^2+k_3 c^2+k_4 d^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

The associated eigenvalue problem is

$$\det\left\{\begin{bmatrix} k_1+k_4 & 0 & -k_1 a-k_4 d \\ 0 & k_2+k_3 & k_2 b-k_3 c \\ -k_1 a-k_4 d & k_2 b-k_3 c & k_1 a^2+k_2 b^2+k_3 c^2+k_4 d^2 \end{bmatrix}-\omega^2\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}\right\}=0 \tag{13}$$

<u>Example 1</u>

An isolated avionics component has the following parameters. Note that the isolators are at the same position with respect to the Y-axis since a = d.

m	=	5 lbm (0.013 lbf sec^2/in)
J	=	10.83 lbm in^2 (0.028 lbf sec^2 in)
a	=	0.5 inch
b	=	2 inch
c	=	3 in
d	=	0.5 in
k1	=	160 lbf/in
k2	=	160 lbf/in
k3	=	160 lbf/in
k4	=	160 lbf/in

$$\det\left\{\begin{bmatrix} 320 & 0 & -160 \\ 0 & 320 & -160 \\ -160 & -160 & 2160 \end{bmatrix} - \omega^2 \begin{bmatrix} 0.013 & 0 & 0 \\ 0 & 0.013 & 0 \\ 0 & 0 & 0.028 \end{bmatrix}\right\} = 0 \tag{14}$$

The resulting natural frequencies are

f1 = 23.7 Hz

f2 = 25.0 Hz

f3 = 44.9 Hz

The corresponding mode shapes in column format are

Mode 1	Mode 2	Mode 3
6.06	-6.202	-1.31
6.06	-6.202	-1.31
1.26	0	5.84

The mode shape format is $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$.

Example 2

This example is the same as the previous example except that the isolators are offset with respect to the Y-axis since $a \neq d$.

m	=	5 lbm (0.013 lbf sec^2/in)
J	=	10.83 lbm in^2 (0.028 lbf sec^2 in)
a	=	0.5 inch
b	=	2 inch
c	=	3 in
d	=	1.5 in
k1	=	160 lbf/in
k2	=	160 lbf/in
k3	=	160 lbf/in
k4	=	160 lbf/in

$$\det\left\{\begin{bmatrix} 320 & 0 & -320 \\ 0 & 320 & -160 \\ -320 & -160 & 2480 \end{bmatrix} - \omega^2 \begin{bmatrix} 0.013 & 0 & 0 \\ 0 & 0.013 & 0 \\ 0 & 0 & 0.028 \end{bmatrix}\right\} = 0 \tag{15}$$

The resulting natural frequencies are

f1	=	22.2 Hz
f2	=	25.0 Hz
f3	=	48.7 Hz

The Y-axis offset of the isolators lowers the fundamental frequency from 23.7 to 22.2 Hz.

The corresponding mode shapes in column format are

Mode 1	Mode 2	Mode 3
7.57	3.92	-2.06
3.79	-7.85	-1.03
1.57	0	5.77

Again, the mode shape format is $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$.