ROD RESPONSE TO LONGITUDINAL BASE EXCITATION,
STEADY-STATE AND TRANSIENT
Revision B

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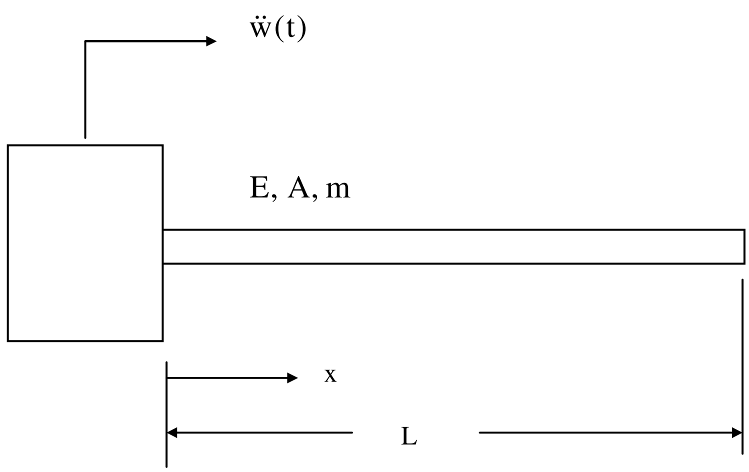
Consider a thin rod subjected to base excitation.



The variables are

Cross-section area	A
Elastic Modulus	E
Length	L
Mass per Volume	ρ
Mass per Length	m
Relative Displacement	u(x,t)
Base Acceleration	$\ddot{w}(t)$
Base Excitation Frequency (rad/sec)	ω
Natural Frequency (rad/sec)	ω_n
Viscous Damping Ratio	ξ

Consider a rod with uniform mass density and constant cross-section. The rod is driven by base excitation. The governing equation is

$$EA\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^2 u}{\partial t^2} = m\frac{\partial^2 w}{\partial t^2} \tag{1}$$

The term on the right-hand-side is the inertial force per unit length.

Assume separation of variables.

$$u(x,t) = \sum_{n=1}^{m} U_n(x) T_n(t) \tag{2}$$

By substitution,

$$EA\frac{\partial^2}{\partial x^2}\left\{\sum_{n=1}^{m} U_n(x) T_n(t)\right\} - m\frac{\partial^2}{\partial t^2}\left\{\sum_{n=1}^{m} U_n(x) T_n(t)\right\} = m\frac{d^2 w}{dt^2} \tag{3}$$

$$EA\left\{\sum_{n=1}^{m}\left[\frac{\partial^2}{\partial x^2} U_n(x)\right] T_n(t)\right\} - m\left\{\sum_{n=1}^{m} U_n(x)\left[\frac{\partial^2}{\partial t^2} T_n(t)\right]\right\} = m\frac{d^2 w}{dt^2} \tag{4}$$

Change to ordinary derivatives.

$$EA\left\{\sum_{n=1}^{m}\left[\frac{d^2}{dx^2} U_n(x)\right] T_n(t)\right\} - m\left\{\sum_{n=1}^{m} U_n(x)\left[\frac{d^2}{dt^2} T_n(t)\right]\right\} = m\frac{d^2 w}{dt^2} \tag{5}$$

Note from Reference 1 that

$$\frac{d^2}{dx^2}U_n(x) = \frac{-\omega_n{}^2}{c^2}U_n(x) \tag{6}$$

By substitution

$$-EA\left\{\sum_{n=1}^{m}\frac{\omega_n{}^2}{c^2}U_n(x)T_n(t)\right\} - m\left\{\sum_{n=1}^{m}U_n(x)\left[\frac{d^2}{dt^2}T_n(t)\right]\right\} = m\frac{d^2w}{dt^2} \tag{7}$$

$$-EA\frac{\omega_n{}^2}{c^2}\left\{\sum_{n=1}^{m}U_n(x)T_n(t)\right\} - m\left\{\sum_{n=1}^{m}U_n(x)\left[\frac{d^2}{dt^2}T_n(t)\right]\right\} = m\frac{d^2w}{dt^2} \tag{8}$$

Multiply each term by $U_p(x)$.

$$-EA\frac{\omega_n{}^2}{c^2}\left\{\sum_{n=1}^{m}U_n(x)U_p(x)T_n(t)\right\} - m\left\{\sum_{n=1}^{m}U_n(x)U_p(x)\left[\frac{d^2}{dt^2}T_n(t)\right]\right\}$$
$$= m\,U_p(x)\frac{d^2w}{dt^2} \tag{9}$$

Integrate with respect to length.

$$\int_0^L\left\{-EA\frac{\omega_n{}^2}{c^2}\left\{\sum_{n=1}^{m}U_n(x)U_p(x)T_n(t)\right\}\right\}dx$$
$$\int_0^L\left\{-m\left\{\sum_{n=1}^{m}U_n(x)U_p(x)\left[\frac{d^2}{dt^2}T_n(t)\right]\right\}\right\}dx$$
$$= \int_0^L\left\{-m\,U_p(x)\frac{d^2w}{dt^2}\right\}dx$$

(10)

$$\left\{-EA\frac{\omega_n^{2}}{c^{2}}\ T_n(t)\right\}\left\{\sum_{n=1}^{m}\int_0^L U_n(x)U_p(x)dx\right\}-m\left[\frac{d^2}{dt^2}T_n(t)\right]\left\{\sum_{n=1}^{m}\int_0^L U_n(x)U_p(x)dx\right\}$$
$$=m\frac{d^2w}{dt^2}\int_0^L\{U_p(x)\}dx$$

(11)

The eigenvectors are orthogonal such that

$$m\int_0^L U_n(x)U_p(x)dx=0 \qquad \text{for} \quad n\neq p \tag{12}$$

$$m\int_0^L U_n(x)U_p(x)dx=1 \qquad \text{for} \quad n=p \tag{13}$$

Thus,

$$\left\{-EA\frac{\omega_n^{2}}{c^{2}}\ T_n(t)\right\}-m\left[\frac{d^2}{dt^2}T_n(t)\right]=m\frac{d^2w}{dt^2}\int_0^L\{U_n(x)\}dx \tag{14}$$

$$-\frac{d^2}{dt^2}T_n(t)-\left[\frac{EA}{m}\right]\frac{\omega_n^{2}}{c^{2}}\ T_n(t)=\frac{d^2w}{dt^2}\int_0^L\{U_n(x)\}dx \tag{15}$$

$$-\frac{d^2}{dt^2}T_n(t)-\omega_n^{2}\ T_n(t)=\frac{d^2w}{dt^2}\int_0^L\{U_n(x)\}dx \tag{16}$$

$$\frac{d^2}{dt^2} T_n(t) + \omega_n{}^2 \; T_n(t) = -\frac{d^2 w}{dt^2} \int_0^L \{ U_n(x) \} dx \tag{17}$$

Define a participation factor.

$$\Gamma_n = \int_0^L \{ U_n(x) \} dx \tag{18}$$

By substitution,

$$\frac{d^2}{dt^2} T_n(t) + \omega_n{}^2 \; T_n(t) = -\Gamma_n \frac{d^2 w}{dt^2} \tag{19}$$

Add a modal damping term.

$$\frac{d^2}{dt^2} T_n(t) + 2\xi_n \omega_n \frac{d}{dt} T_n(t) + \omega_n{}^2 \; T_n(t) = -\Gamma_n \frac{d^2 w}{dt^2} \tag{20}$$

Change notation.

$$\ddot{T}_n(t) + 2\xi_n \omega_n \dot{T}_n(t) + \omega_n{}^2 \; T_n(t) = -\Gamma_n \; \ddot{w}(t) \tag{21}$$

Take the Fourier transform of each side

$$\int_{-\infty}^{\infty} \left\{ \ddot{T}_n(t) + 2\xi_n \omega_n \dot{T}_n(t) + \omega_n{}^2 \; T_n(t) \right\} \exp(j\omega t) dt = -\Gamma_n \int_{-\infty}^{\infty} \ddot{w}(t) \exp(j\omega t) dt \tag{22}$$

$$\int_{-\infty}^{\infty} \ddot{T}_n(t)\exp(j\omega t)dt + 2\xi_n\omega_n \int_{-\infty}^{\infty} \dot{T}_n(t)\exp(j\omega t)dt + \omega_n{}^2 \int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt$$
$$= -\Gamma_n \int_{-\infty}^{\infty} \ddot{w}(t)\exp(j\omega t)dt \tag{23}$$

Note that

$$\int_{-\infty}^{\infty} \dot{T}_n(t)\exp(j\omega t)dt = j\omega\int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt \tag{24}$$

$$\int_{-\infty}^{\infty} \ddot{T}_n(t)\exp(j\omega t)dt = -\omega^2\int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt \tag{25}$$

$$-\omega^2\int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt + j2\xi_n\omega_n\omega\int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt + \omega_n{}^2\int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt$$
$$= -\Gamma_n \int_{-\infty}^{\infty} \ddot{w}(t)\exp(j\omega t)dt \tag{26}$$

$$\left\{\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right\}\int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt = -\Gamma_n \int_{-\infty}^{\infty} \ddot{w}(t)\exp(j\omega t)dt \tag{27}$$

Let

$$\hat{T}_n(\omega) = \int_{-\infty}^{\infty} T_n(t)\exp(j\omega t)dt \tag{28}$$

$$\ddot{W}(\omega) = \int_{-\infty}^{\infty} \ddot{w}(t)\exp(j\omega t)dt \tag{29}$$

By substitution,

$$\left\{\left(\omega_n{}^2-\omega^2\right)+j2\xi_n\omega_n\omega\right\}\hat{T}_n(\omega)=-\Gamma_n\ \ddot{W}(\omega) \tag{30}$$

$$\hat{T}_n(\omega)=\frac{-\Gamma_n\ \ddot{W}(\omega)}{\left\{\left(\omega_n{}^2-\omega^2\right)+j2\xi_n\omega_n\omega\right\}} \tag{31}$$

Recall

$$u(x,t)=\sum_{n=1}^{m}U_n(x)T_n(t) \tag{32}$$

The Fourier transform of the relative displacement is

$$\hat{U}(x,\omega)=\sum_{n=1}^{m}U_n(x)\hat{T}_n(\omega) \tag{33}$$

$$\hat{U}(x,\omega)=\sum_{n=1}^{m}U_n(x)\frac{-\Gamma_n\ \ddot{W}(\omega)}{\left\{\left(\omega_n{}^2-\omega^2\right)+j2\xi_n\omega_n\omega\right\}} \tag{34}$$

$$\hat{U}(x,\omega)=\ddot{W}(\omega)\sum_{n=1}^{m}U_n(x)\frac{-\Gamma_n}{\left\{\left(\omega_n{}^2-\omega^2\right)+j2\xi_n\omega_n\omega\right\}} \tag{35}$$

The ω term is given a subscript n because there are multiple roots. The natural frequencies for a fixed-free rod are

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi\frac{c}{L}, \quad n = 1,2,3,... \tag{36}$$

The displacement function for the fixed-free rod is

$$U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{37}$$

$$U_n(x) = d_n \sin\left(\frac{(2n-1)\pi\ x}{2L}\right) \tag{38}$$

Need to normalize with respect to mass

$$m\int_0^L U_n(x)U_p(x)dx = 1 \qquad \text{for} \quad n = p \tag{39}$$

$$m\int_0^L d_n{}^2 \sin^2\left(\frac{\omega_n x}{c}\right)dx = 1 \tag{40}$$

$$m\,d_n{}^2\int_0^L \left[\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\omega_n x}{c}\right)\right]dx = 1 \tag{41}$$

Recall

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi\frac{c}{L} \tag{42}$$

$$m\, d_n{}^2 \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos\left((2n-1)\frac{\pi x}{L}\right)\right] dx = 1 \tag{43}$$

$$\frac{1}{2} m\, d_n{}^2 \left\{ x - \left[\frac{L}{(2n-1)\pi}\right] \sin\left((2n-1)\frac{\pi x}{L}\right)\right\}\Bigg|_0^L = 1 \tag{44}$$

$$\frac{1}{2} m\, d_n{}^2\, L = 1 \tag{45}$$

$$d_n{}^2 = \frac{2}{mL} \tag{46}$$

$$d_n = \sqrt{\frac{2}{mL}} \tag{47}$$

The participation factor is

$$\Gamma_n = m \int_0^L U_n(x) dx \tag{48}$$

$$\Gamma_n = m \int_0^L \left\{ d_n \sin\left(\frac{\omega_n x}{c}\right)\right\} dx \tag{49}$$

$$\Gamma_n = m\ d_n \int_0^L \left\{ \sin\left(\frac{\omega_n x}{c}\right)\right\} dx \tag{50}$$

$$\Gamma_n = -\left(\frac{m\, c\, d_n}{\omega_n}\right) \left\{ \cos\left(\frac{\omega_n x}{c}\right)\Bigg|_0^L \right\} \tag{51}$$

$$\Gamma_n = -\left(\frac{m c d_n}{\omega_n}\right)\left\{\cos\left(\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right)\Big|_0^L\right\} \tag{52}$$

$$\Gamma_n = \left(\frac{m c d_n}{\omega_n}\right) \tag{53}$$

$$d_n = \sqrt{\frac{2}{mL}} \tag{54}$$

$$\Gamma_n = \left(\frac{m c}{\omega_n}\sqrt{\frac{2}{mL}}\right) \tag{55}$$

$$\Gamma_n = \left(\frac{c}{\omega_n}\sqrt{\frac{2m}{L}}\right) \tag{56}$$

Again,

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi\frac{c}{L},\ n = 1,2,3,... \tag{57}$$

By substitution,

$$\Gamma_n = \left(\frac{c}{\left(\left(\frac{2n-1}{2}\right)\pi\frac{c}{L}\right)}\sqrt{\frac{2m}{L}}\right) \tag{58}$$

$$\Gamma_n = \left(\frac{2L}{((2n-1)\pi)}\sqrt{\frac{2m}{L}}\right) \tag{59}$$

$$\Gamma_n = \left(\frac{2}{((2n-1)\pi)} \sqrt{2mL} \right) \tag{60}$$

As a summary,

Mode	Natural Frequency ω_n	d_n	Γ_n
1	$0.5\pi\ c/L$	$\sqrt{2/(mL)}$	$\frac{2}{\pi}\sqrt{2mL}$
2	$1.5\pi\ c/L$	$\sqrt{2/(mL)}$	$\frac{2}{3\pi}\sqrt{2mL}$
3	$2.5\pi\ c/L$	$\sqrt{2/(mL)}$	$\frac{2}{5\pi}\sqrt{2mL}$

The relative displacement is found using the method in Reference 2.

$$U(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{p} \left[\frac{-\Gamma_n \hat{U}_n(x)}{(\omega_n{}^2 - \omega^2) + j(2\xi\omega\omega_n)} \right] \tag{61}$$

where p is the total number of modes included in the analysis.

Let

$$H_{rn}(x,\omega) = \frac{-\Gamma_n \hat{U}_n(x)}{(\omega_n{}^2 - \omega^2) + j(2\xi\omega\omega_n)} \tag{62}$$

Thus

$$U(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{p} \left[H_{rn}(x,\omega) \right] \tag{63}$$

Again,

$$\hat{U}_n(x) = \sqrt{\frac{2}{mL}} \sin\left(\frac{\omega_n x}{c}\right) \tag{64}$$

The strain is

$$\frac{\partial}{\partial x} U(x, \omega) = \ddot{W}(\omega) \sum_{n=1}^{p} \left[\frac{-\Gamma_n \frac{d}{dx} \hat{U}_n(x)}{(\omega_n{}^2 - \omega^2) + j(2\xi\,\omega\omega_n)} \right] \tag{65}$$

where

$$\frac{d}{dx} \hat{U}_n(x) = \frac{\omega_n}{c} \sqrt{\frac{2}{mL}} \cos\left(\frac{\omega_n x}{c}\right) \tag{66}$$

The response absolute acceleration is

$$\ddot{U}(x, \omega) = \left\{ 1 + \sum_{n=1}^{p} \left[\frac{\omega^2 \Gamma_n \hat{U}_n(x)}{(\omega_n{}^2 - \omega^2) + j(2\xi\,\omega\omega_n)} \right] \right\} \ddot{W}(\omega) \tag{67}$$

Let

$$H_{an}(x, \omega) = \left[\frac{\omega^2 \Gamma_n \hat{U}_n(x)}{(\omega_n{}^2 - \omega^2) + j(2\xi\,\omega\omega_n)} \right] \tag{68}$$

Thus

$$\ddot{U}(x, \omega) = \left\{ 1 + \sum_{n=1}^{p} H_{an}(x, \omega) \right\} \ddot{W}(\omega) \tag{69}$$

The transient response is derived in Appendix A.

References

1. T. Irvine, Longitudinal Natural Frequencies of Rods and Response to Initial Conditions, Revision B, Vibrationdata, 2009.
2. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Revision A, Vibrationdata, 2009.

APPENDIX A

Transient Response

Recall the modal transfer function for the relative displacement.

$$H_{rn}(x,\omega) = \frac{-\Gamma_n \hat{U}_n(x)}{(\omega_n{}^2 - \omega^2) + j(2\xi\omega\omega_n)} \tag{A-1}$$

The roots of the denominator are

$$\omega_d \pm j\xi\omega_n \tag{A-2}$$

where

$$\omega_d = \omega_n\sqrt{1-\xi^2} \tag{A-3}$$

The partial fraction expansion is

$$\frac{-1}{(\omega_n{}^2 - \omega^2) + j(2\xi\omega\omega_n)} = \left\{\frac{1}{2\omega_d}\right\}\left\{\frac{1}{\omega + \omega_d + j\xi\omega_n} + \frac{-1}{\omega - \omega_d + j\xi\omega_n}\right\} \tag{A-4}$$

The impulse response function is

$$\hat{h}_{rn}(t) = \frac{1}{\omega_d}\left[\exp\{-\xi\omega_n t\}\right]\left[\sin\omega_d t\right] \quad , \quad t \geq 0 \tag{A-5}$$

The relative displacement response to an arbitrary acceleration $\ddot{w}(t)$ is

$$U(x,t) = -\sum_{n=1}^{p}\left\{\Gamma_n \hat{U}_n(x)\int_0^t \hat{h}_{rn}(t-\tau)\ddot{w}(\tau)d\tau\right\} \qquad \text{(A-6)}$$

For digital data,

$$U(x,t_i) = -\sum_{n=1}^{p}\left\{\Gamma_n \hat{U}_n(x)\sum_{i=1}^{q}\hat{h}_{rn}(t_i-\tau_i)\ddot{w}(\tau_i)\Delta\tau\right\} \qquad \text{(A-7)}$$

where q is the number of sample needed for the time to advance to t_i .

APPENDIX B

Indirect Verification of the Impulse Response Function

$$H_{rn}(x,\omega) = -\Gamma_n \hat{U}_n(x) \int_0^{\infty} \hat{h}_{rn}(t) \exp(-j\omega t) dt \tag{B-1}$$

$$H_{rn}(x,\omega) = -\Gamma_n \hat{U}_n(x) \int_0^{\infty} \frac{1}{\omega_d} [\exp(-\xi\omega_n t)][\sin \omega_d t] \exp(-j\omega t) dt \tag{B-2}$$

$$H_{rn}(x,\omega) = \frac{-\Gamma_n \hat{U}_n(x)}{\omega_d} \int_0^{\infty} [\exp(-\xi\omega_n t)][\sin \omega_d t] \exp(-j\omega t) dt \tag{B-3}$$

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d} \int_0^{\infty} [\exp(-\xi\omega_n t)][\exp(j\omega_d t) - \exp(-j\omega_d t)] \exp(-j\omega t) dt \tag{B-4}$$

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d} \int_0^{\infty} [\exp(-\xi\omega_n t)][\exp j(\omega_d - \omega)t - \exp j(\omega_d + \omega)t] dt \tag{B-5}$$

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d} \int_0^{\infty} [\exp[-\xi\omega_n + j(\omega_d - \omega)]t - \exp[-\xi\omega_n + j(\omega_d + \omega)]t] dt \tag{B-6}$$

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d} \int_0^{\infty} \left[\exp\left[-\xi\omega_n + j(\omega_d - \omega)\right]t - \exp\left[-\xi\omega_n + j(\omega_d + \omega)\right]t\right]dt$$

(B-7)

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d} \int_0^{\infty} \left[\exp\left[-\xi\omega_n + j(\omega_d - \omega)\right]t\right]dt$$

$$+\frac{\Gamma_n \hat{U}_n(x)}{j2\omega_d} \int_0^{\infty} \left[\exp\left[-\xi\omega_n + j(\omega_d + \omega)\right]t\right]dt$$

(B-8)

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d - \omega)\right]} \exp\left[-\xi\omega_n + j(\omega_d - \omega)\right]t \Big|_0^{\infty}$$

$$+\frac{\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d + \omega)\right]} \exp\left[-\xi\omega_n + j(\omega_d + \omega)\right]t \Big|_0^{\infty}$$

(B-9)

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d - \omega)\right]} \exp\left[-\xi\omega_n + j(\omega_d - \omega)\right]t\,\Big|_0^\infty$$

$$+\frac{\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d + \omega)\right]} \exp\left[-\xi\omega_n + j(\omega_d + \omega)\right]t\,\Big|_0^\infty$$

(B-10)

$$H_{rn}(x,\omega) =$$

$$\frac{-\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d - \omega)\right]} \exp\left[-\xi\omega_n t\right]\exp\left[j(\omega_d - \omega)\right]t\,\Big|_0^\infty$$

$$+\frac{\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d + \omega)\right]} \exp\left[-\xi\omega_n t\right]\exp\left[j(\omega_d + \omega)\right]t\,\Big|_0^\infty$$

(B-11)

$$H_{rn}(x,\omega) =$$

$$\frac{\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d - \omega)\right]} - \frac{\Gamma_n \hat{U}_n(x)}{j2\omega_d\left[-\xi\omega_n + j(\omega_d + \omega)\right]}$$

(B-12)

$$H_{rn}(x,\omega) =$$

$$\frac{\Gamma_n \hat{U}_n(x)}{2\omega_d\left[-j\xi\omega_n - (\omega_d - \omega)\right]} - \frac{\Gamma_n \hat{U}_n(x)}{2\omega_d\left[-j\xi\omega_n - (\omega_d + \omega)\right]}$$

(B-13)

$H_{rn}(x,\omega) =$

$$\frac{\Gamma_n \hat{U}_n(x)}{2\omega_d\left[-(\omega_d-\omega)-j\xi\omega_n\right]} - \frac{\Gamma_n \hat{U}_n(x)}{2\omega_d\left[-(\omega_d+\omega)-j\xi\omega_n\right]}$$

(B-14)

$H_{rn}(x,\omega) =$

$$\frac{-\Gamma_n \hat{U}_n(x)}{2\omega_d\left[(\omega_d-\omega)j\xi\omega_n\right]} + \frac{\Gamma_n \hat{U}_n(x)}{2\omega_d\left[(\omega_d+\omega)+j\xi\omega_n\right]}$$

(B-15)

$H_{rn}(x,\omega) =$

$$\frac{\Gamma_n \hat{U}_n(x)}{2\omega_d\left[(\omega_d+\omega)+j\xi\omega_n\right]} + \frac{-\Gamma_n \hat{U}_n(x)}{2\omega_d\left[(\omega_d-\omega)+j\xi\omega_n\right]}$$

(B-16)

$$H_{rn}(x,\omega) = \left\{\frac{\Gamma_n \hat{U}_n(x)}{2\omega_d}\right\}\left\{\frac{1}{(\omega_d+\omega)+j\xi\omega_n} + \frac{-1}{(\omega_d-\omega)+j\xi\omega_n}\right\}$$

(B-17)

Recall

$$\frac{-\Gamma_n \hat{U}_n(x)}{(\omega_n{}^2-\omega^2)+j(2\xi\omega\omega_n)} = \left\{\frac{\Gamma_n \hat{U}_n(x)}{2\omega_d}\right\}\left\{\frac{1}{\omega+\omega_d+j\xi\omega_n} + \frac{-1}{\omega-\omega_d+j\xi\omega_n}\right\}$$

(B-18)

Thus, the impulse response function is verified.