

THE LONGITUDINAL VIBRATION RESPONSE
OF A ROD TO AN APPLIED FORCE
Revision D

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Consider a thin, fixed-free rod subjected to an applied force at the free end.

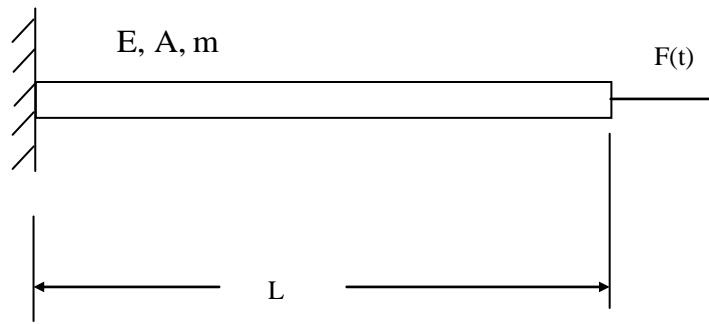


Figure 1.

The variables are

Cross-section area	A
Elastic Modulus	E
Length	L
Mass per Volume	ρ
Mass per Length	m
Displacement	$u(x,t)$
Applied Force	$F(t)$
Excitation Frequency (rad/sec)	ω
Natural Frequency (rad/sec)	ω_n
Viscous Damping Ratio	ξ

Assume uniform mass density and constant cross-section. The governing equation from Reference 1 is

$$EA \frac{\partial^2 u}{\partial x^2} - m \frac{\partial^2 u}{\partial t^2} = 0 \quad (1)$$

The boundary conditions are

$$u(0, t) = 0 \quad (2)$$

$$EA \frac{\partial}{\partial x} u(x, t) \Big|_{x=L} = F(t) \quad (3)$$

The free end thus has a time-varying boundary condition. Reference 2 gives solution methods for this equation and boundary condition set, but the methods are abstruse.

Instead, this paper will use the virtual work method from Reference 3.

The displacement due to the force F can be represented by the series

$$u(x, t) = \phi_1 \sin\left(\frac{\omega_1 x}{c}\right) + \phi_3 \sin\left(\frac{\omega_3 x}{c}\right) + \phi_5 \sin\left(\frac{\omega_5 x}{c}\right) + \dots = \sum_{i=1,3,5,\dots}^{\infty} \phi_i \sin\left(\frac{\omega_i x}{c}\right) \quad (4)$$

The ϕ_i terms are some unknown functions of time which will be determined by the principle of virtual work.

The natural frequencies for a fixed-free rod are

$$\omega_i = \frac{i \pi c}{2L}, \quad i = 1, 3, 5, \dots \quad (5)$$

The displacement mode shape function for the fixed-free rod is

$$U_i(x) = D_i \sin\left(\frac{\omega_i x}{c}\right) \quad (6)$$

The virtual longitudinal displacement δu_i in terms of the mode shapes are

$$\delta u_i = D_i \sin\left(\frac{\omega_i x}{c}\right) \quad (7)$$

The mass of an element between two adjacent cross sections of the rod is $\rho A dx$.

The work δW_I done by inertial forces on the assumed virtual displacement is

$$\delta W_I = \int_0^L (-\rho A dx) \ddot{u} \delta u_i = -\rho A \int_0^L \ddot{u} D_i \sin\left(\frac{\omega_i x}{c}\right) dx \quad (8)$$

By substitution,

$$\delta W_I = -\rho A \int_0^L \left\{ \sum_{j=1,3,5,\dots}^{\infty} \ddot{\phi}_j \sin\left(\frac{\omega_j x}{c}\right) \right\} D_i \sin\left(\frac{\omega_i x}{c}\right) dx \quad (9)$$

$$\delta W_I = -\rho A D_i \left\{ \sum_{j=1,3,5,\dots}^{\infty} \ddot{\phi}_j \int_0^L \sin\left(\frac{\omega_j x}{c}\right) \sin\left(\frac{\omega_i x}{c}\right) dx \right\} \quad (10)$$

The orthogonality of the normal mode shapes is such that

$$\int_0^L \sin\left(\frac{\omega_j x}{c}\right) \sin\left(\frac{\omega_i x}{c}\right) dx = 0, \quad \text{for } i \neq j \quad (11)$$

$$\int_0^L \sin\left(\frac{\omega_j x}{c}\right) \sin\left(\frac{\omega_i x}{c}\right) dx = \frac{L}{2}, \quad \text{for } i = j, \quad i = 1, 3, 5, \dots \quad (12)$$

$$\delta W_I = \frac{-\rho A L}{2} D_i \ddot{\phi}_i \quad (13)$$

Now calculate the virtual work δW_E produced by the elastic forces.

The force on each element is $EAu'' dx$.

$$\delta W_E = \int_0^L [EAu'' dx] \delta u_i \quad (14)$$

By substitution,

$$\delta W_E = \int_0^L \left[EA \frac{d^2}{dx^2} \left\{ \sum_{j=1,3,5,\dots}^{\infty} \phi_j \sin\left(\frac{\omega_j x}{c}\right) \right\} dx \right] D_i \sin\left(\frac{\omega_i x}{c}\right) \quad (15)$$

$$\delta W_E = - \int_0^L \left[EA \left\{ \sum_{j=1,3,5,\dots}^{\infty} \phi_j \left(\frac{\omega_j}{c}\right)^2 \sin\left(\frac{\omega_j x}{c}\right) \right\} dx \right] D_i \sin\left(\frac{\omega_i x}{c}\right) \quad (16)$$

$$\delta W_E = -EAD_i \left\{ \sum_{j=1,3,5,\dots}^{\infty} \phi_j \left(\frac{\omega_j}{c}\right)^2 \int_0^L \sin\left(\frac{\omega_j x}{c}\right) \sin\left(\frac{\omega_i x}{c}\right) dx \right\} \quad (17)$$

Apply the orthogonality relationships. The resulting virtual work due to elastic forces is

$$\delta W_E = -\frac{EA L}{2} \left(\frac{\omega_i}{c} \right)^2 D_i \phi_i \quad (18)$$

Note that the term in equation (18) could have been derived via the incremental strain energy, as shown in Reference 3.

The virtual work δW_F due to the force $F(t)$ at the free end is

$$\delta W_F = D_i \sin\left(\frac{\omega_i L}{c}\right) F(t) \quad (19)$$

The total virtual work is thus

$$\frac{-\rho AL}{2} D_i \ddot{\phi}_i - \frac{EA L}{2} \left(\frac{\omega_i}{c} \right)^2 D_i \phi_j + D_i \sin\left(\frac{\omega_i L}{c}\right) F(t) = 0 \quad (20)$$

$$\frac{\rho AL}{2} D_i \ddot{\phi}_i + \frac{EA L}{2} \left(\frac{\omega_i}{c} \right)^2 D_i \phi_j = D_i \sin\left(\frac{\omega_i L}{c}\right) F(t) \quad (21)$$

$$\frac{\rho AL}{2} \ddot{\phi}_i + \frac{EA L}{2} \left(\frac{\omega_i}{c} \right)^2 \phi_j = \sin\left(\frac{\omega_i L}{c}\right) F(t) \quad (22)$$

$$\ddot{\phi}_i + \frac{E}{\rho} \left(\frac{\omega_i}{c} \right)^2 \phi_i = \frac{2}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right) F(t) \quad (23)$$

$$c = \sqrt{E/\rho} \quad (24)$$

$$\ddot{\phi}_i + \omega_i^2 \phi_i = \frac{2}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right) F(t) \quad (25)$$

Let

$$F(t) = P \sin(\omega t) \quad (26)$$

The equation of motion with an added damping term is

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = \frac{2P}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right) \sin(\omega t) \quad (27)$$

Time Domain

Assume that the initial conditions are zero.

Let

$$F_i = \frac{2P}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right) \quad (28)$$

$$\omega_{d,i} = \omega_i \sqrt{1 - \xi_i^2} \quad (29)$$

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = F_i \sin(\omega t) \quad (30)$$

$$\begin{aligned}
\phi_i(t) = & + \frac{F_i}{(\omega^2 - \omega_i^2)^2 + (2\xi_i \omega \omega_i)^2} \left\{ 2\xi_i \omega_i \omega \cos(\omega t) - (\omega^2 - \omega_i^2) \sin(\omega t) \right\} \\
& + \frac{1}{\omega_{d,i}} \left\{ \frac{\omega F_i}{(\omega^2 - \omega_i^2)^2 + (2\xi_i \omega \omega_i)^2} \right\} \left\{ e^{-\xi_i \omega_i t} \right\} \left\{ 2\xi_i \omega_i \omega_{d,i} \cos(\omega_{d,i} t) \right\} \\
& + \frac{1}{\omega_{d,i}} \left\{ \frac{\omega F_i}{(\omega^2 - \omega_i^2)^2 + (2\xi_i \omega \omega_i)^2} \right\} \left\{ e^{-\xi_i \omega_i t} \right\} \left\{ \left[\omega^2 + \omega_i^2 \left[-1 + 2\xi_i^2 \right] \right] \sin(\omega_{d,i} t) \right\}
\end{aligned} \tag{31}$$

The displacement can then be found via

$$u(x, t) = \phi_1 \sin\left(\frac{\omega_1 x}{c}\right) + \phi_3 \sin\left(\frac{\omega_3 x}{c}\right) + \phi_5 \sin\left(\frac{\omega_5 x}{c}\right) + \dots = \sum_{i=1,3,5,\dots}^{\infty} \phi_i \sin\left(\frac{\omega_i x}{c}\right) \tag{32}$$

Steady-State Frequency Response Function

Change the forcing function to a harmonic excitation exponential term.

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = F_i \exp(j\omega t) \tag{33}$$

$$\phi_i(t) = \exp(j\omega t) \tag{34}$$

$$\dot{\phi}_i(t) = j\omega \exp(j\omega t) \tag{35}$$

$$\ddot{\phi}_i(t) = -\omega^2 \exp(j\omega t) \tag{36}$$

$$\left\{ -\omega^2 + j\omega 2\xi_i\omega_i + \omega_i^2 \right\} \phi_i \exp(j\omega t) = F_i \exp(j\omega t) \quad (37)$$

$$\left\{ \left(\omega_i^2 - \omega^2 \right) + j 2\xi_i\omega \omega_i \right\} \phi_i = F_i \quad (38)$$

The frequency response function $H_i(f)$ is thus

$$\phi_i = \frac{F_i}{(\omega_i^2 - \omega^2) + j 2\xi_i\omega \omega_i} \quad (39)$$

$$\phi_i = \frac{\frac{2P}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right)}{(\omega_i^2 - \omega^2) + j 2\xi_i\omega \omega_i} \quad (40)$$

The Fourier transform $\hat{u}(x, \omega)$ of the displacement response is

$$\hat{u}(x, \omega) = \sum_{i=1,3,5,\dots}^{\infty} \phi_i \sin\left(\frac{\omega_i x}{c}\right) \quad (41)$$

$$\hat{u}(x, \omega) = \sum_{i=1,3,5,\dots}^{\infty} \frac{\frac{2P}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right) \sin\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2\xi_i\omega \omega_i} \quad (42)$$

$$\hat{u}(x, \omega) = \left[\frac{2P}{\rho AL} \right] \sum_{i=1,3,5,\dots}^{\infty} \frac{\sin\left(\frac{\omega_i L}{c}\right) \sin\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2\xi_i\omega \omega_i} \quad (43)$$

The receptance frequency response function $H_d(x, \omega)$ is

$$H_d(x, \omega) = \frac{\hat{u}(x, \omega)}{P} \quad (44)$$

$$H_d(x, \omega) = \left[\frac{2}{\rho AL} \right]_{i=1,3,5,\dots}^{\infty} \frac{\sin\left(\frac{\omega_i L}{c}\right) \sin\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (45)$$

Receptance is (displacement/force).

The mobility frequency response function $H_v(x, f)$ is

$$H_v(x, \omega) = j\omega \left[\frac{2}{\rho AL} \right]_{i=1,3,5,\dots}^{\infty} \frac{\sin\left(\frac{\omega_i L}{c}\right) \sin\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (46)$$

Mobility is (velocity/force).

Example, Sinusoidal Excitation

Consider the longitudinal vibration of an aluminum, fixed-free, circular rod with the following properties.

Length	L	=	24 inch
Diameter	D	=	1 inch
Area	A	=	0.785 inch ²
Area Moment of Inertia	I	=	0.0491 inch ⁴
Elastic Modulus	E	=	1.0e+07 lbf/in ²
Mass Density	ρ	=	0.1 lbm/in ³
Speed of Sound in Material	c	=	1.96e+05 in/sec
Viscous Damping Ratio	ξ	=	0.05

The first three natural frequencies are

Table 1. Natural Frequencies	
i	f_i (Hz)
1	2047
3	6140
5	10,230

The following calculations are made via Matlab script: long_force_sine.m.

Let the applied sinusoidal force be $P = 100$ lbf at $f = 2047$ Hz, which coincides with the fundamental frequency.

Determine the displacement at the free end via equation (43) using the Matlab script.

$$\hat{u}(L, f) = 0.0025 \text{ in} \quad (47)$$

Determine the stress, strain and reaction force at the fixed end.

The strain equation is

$$\frac{\partial}{\partial x} \hat{u}(x, f) = \left[\frac{2P}{\rho AL} \right] \sum_{i=1,3,5,\dots}^{\infty} \left(\frac{\omega_i}{c} \right) \frac{\sin\left(\frac{\omega_i L}{c}\right) \cos\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (48)$$

The strain ε_0 at the fixed end per equation (48) and the Matlab script is

$$\varepsilon_0 = \left| \frac{\partial}{\partial x} \hat{u}(x, f) \right|_{x=0} = 0.000162 \quad (49)$$

The reaction force R at the fixed end is

$$\begin{aligned} R &= EA \varepsilon_0 \\ &= (1.0e + 07 \text{ lbf} / \text{in}^2)(0.785 \text{ in}^2)(0.000162) \\ &= 1272 \text{ lbf} \end{aligned} \quad (50)$$

The normal stress σ_0 at the fixed end is

$$\begin{aligned} \sigma_0 &= R / A \\ &= 1272 \text{ lbf} / 0.785 \text{ in}^2 \\ &= 1620 \text{ psi} \end{aligned} \quad (51)$$

Example, Random Excitation, Frequency Domain

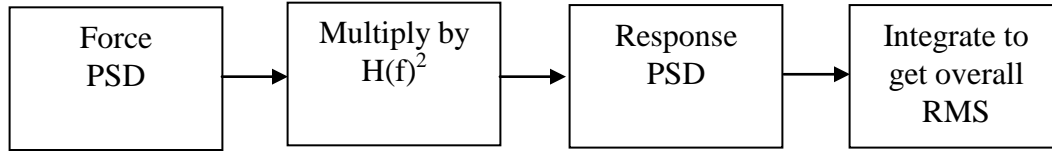
The rod is subjected to a random force at its free end as shown in the following force power spectral density.

Table 2. Force PSD	
f (Hz)	Force (lbf ² /Hz)
10	1
8000	1

The overall level is 89.4 lbf rms.

Determine the displacement of the free end and the reaction force at the fixed end.

The calculations are made using the following process.



where $H(f)$ is the frequency response function magnitude for either displacement or strain.

The $H(f)^2$ function is also known as the power transmissibility function. These functions are shown in Figure 2 and 3 for the displacement and strain, respectively.

The calculations are made using Matlab script: long_force_frf.m.

The displacement is

$$\hat{u} = 0.00045 \text{ inch RMS at } x = L \quad (52)$$

The strain ε_0 is

$$\varepsilon_0 = 3.3e-05 \text{ RMS at } x = 0 \quad (53)$$

The reaction force R at the fixed end is

$$\begin{aligned} R &= EA \varepsilon_o \\ &= (1.0e+07 \text{ lbf / in}^2)(0.785 \text{ in}^2)(3.3e-05 \text{ RMS}) \\ &= 259 \text{ lbf RMS} \end{aligned} \tag{54}$$

The normal stress σ_o at the fixed end is

$$\begin{aligned} \sigma_o &= R / A \\ &= 259 \text{ lbf RMS} / 0.785 \text{ in}^2 \\ &= 330 \text{ psi RMS} \end{aligned} \tag{55}$$

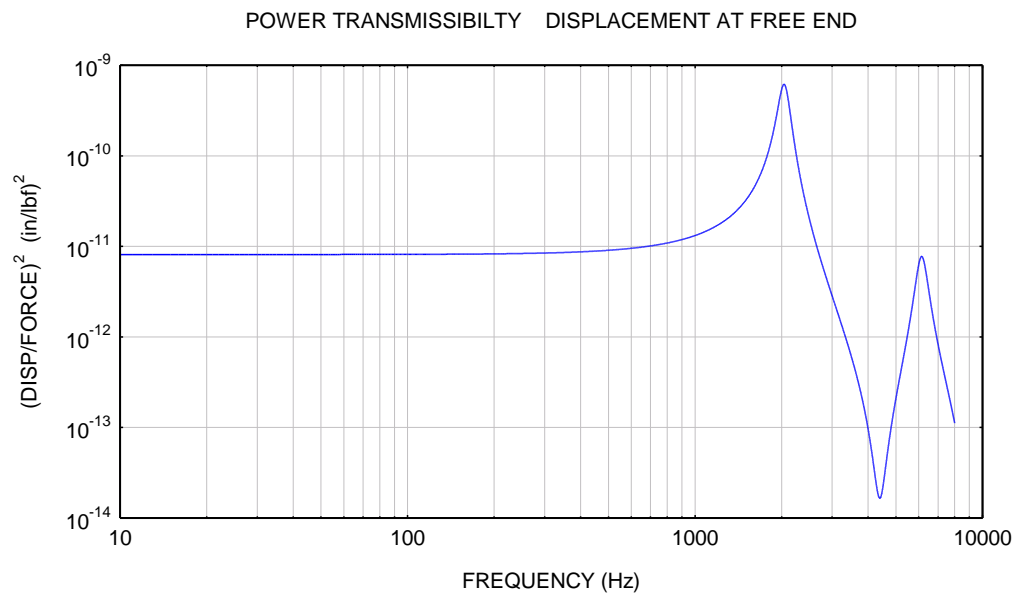


Figure 2.

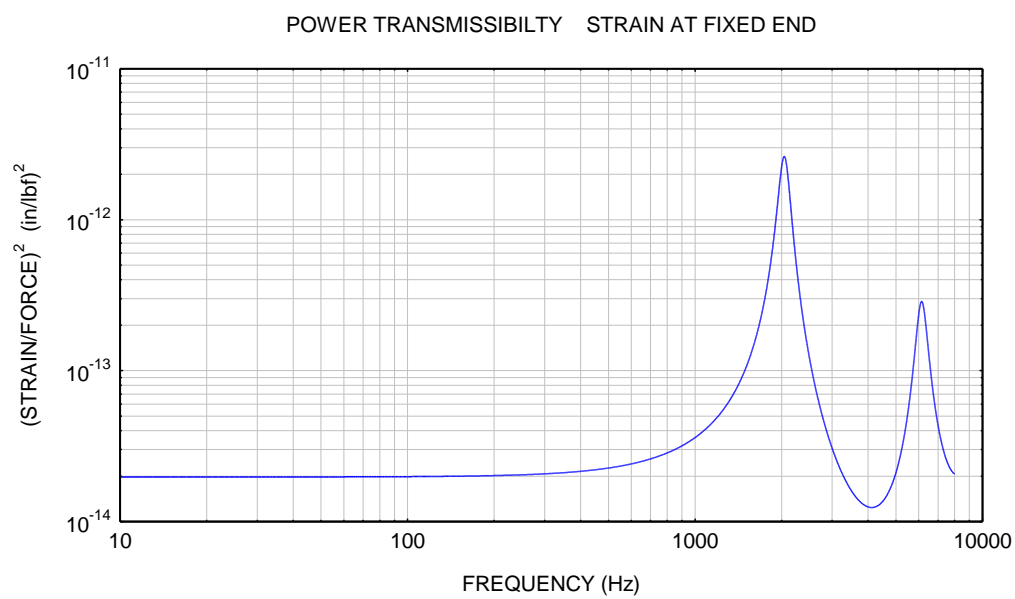
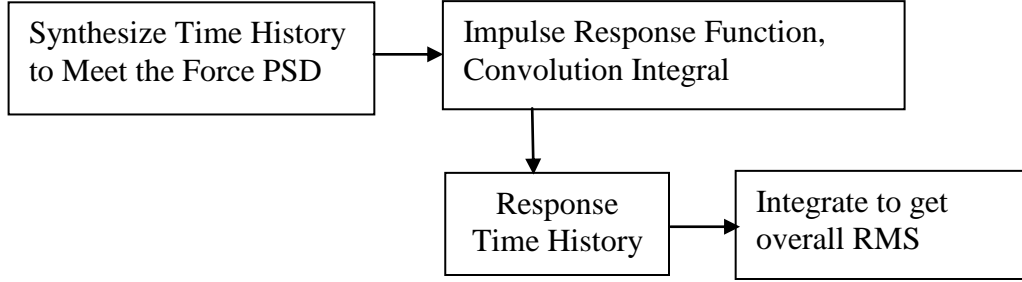


Figure 3.

Example, Random Excitation, Time Domain

The calculations are made using the following process.



The Force PSD is the same as that in Table 2.

The convolution integration is performed via a digital recursive filtering relationship for computational efficiency per Reference 5.

The modal equation is

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = \left[\frac{2}{\rho AL} \sin\left(\frac{\omega_i L}{c}\right) \right] F(t) \quad (56)$$

The displacement is

$$u(x, t) = \phi_1 \sin\left(\frac{\omega_1 x}{c}\right) + \phi_3 \sin\left(\frac{\omega_3 x}{c}\right) + \phi_5 \sin\left(\frac{\omega_5 x}{c}\right) + \dots = \sum_{i=1,3,5,\dots}^{\infty} \phi_i \sin\left(\frac{\omega_i x}{c}\right) \quad (57)$$

The time history was synthesized using the method in Reference 4 via program: psdgen.exe. The resulting time history and the corresponding PSD are shown in Figures 4 and 5, respectively.

The response calculation is performed via Matlab script: rod_arbit_f.m. The resulting displacement, velocity and strain time histories are shown in Figures 6, 7, and 8, respectively.

The overall response levels from the time history analysis agree with the corresponding levels from the steady-state, frequency-domain analysis.

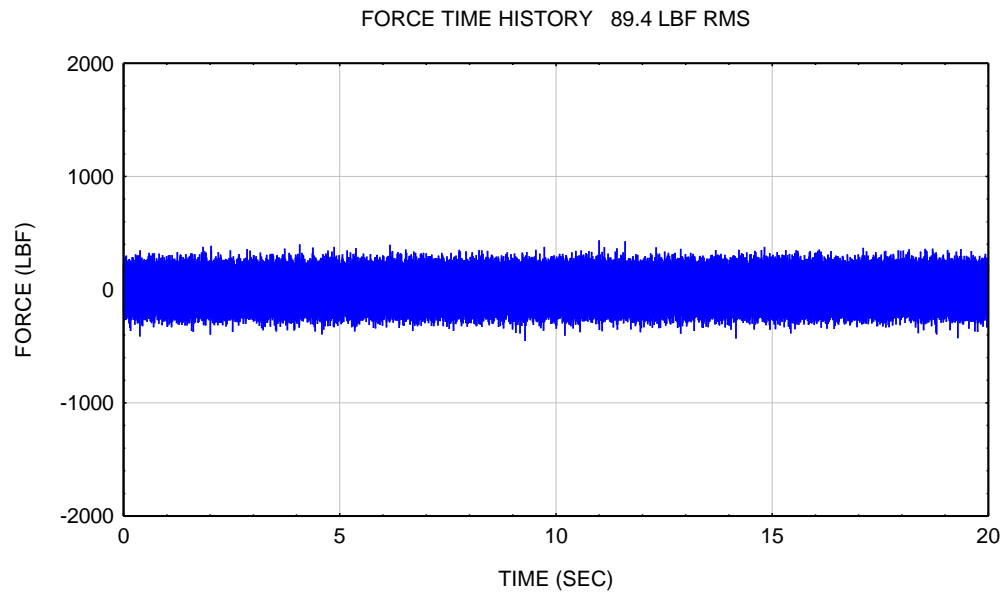


Figure 4.

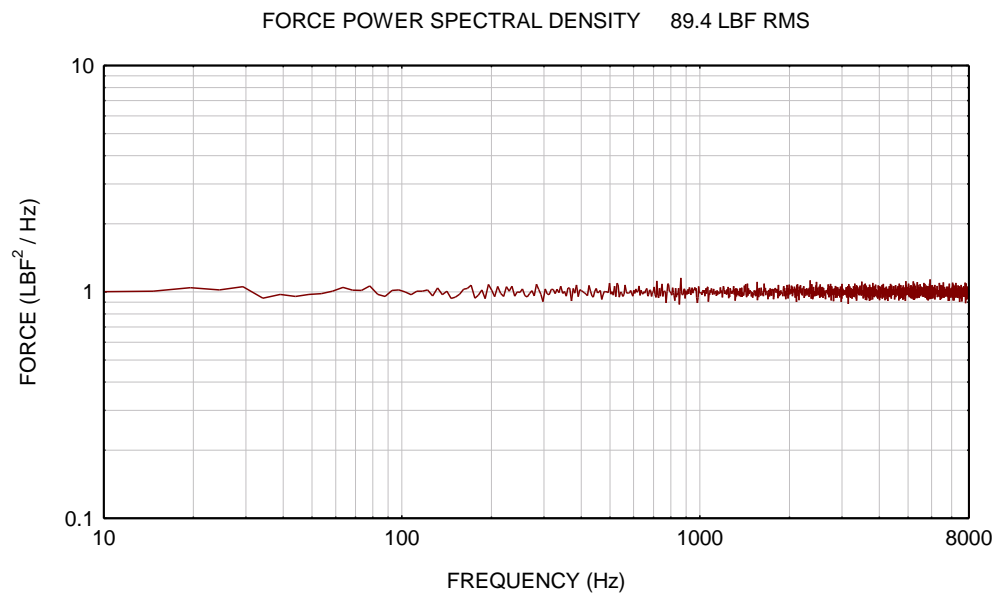


Figure 5.

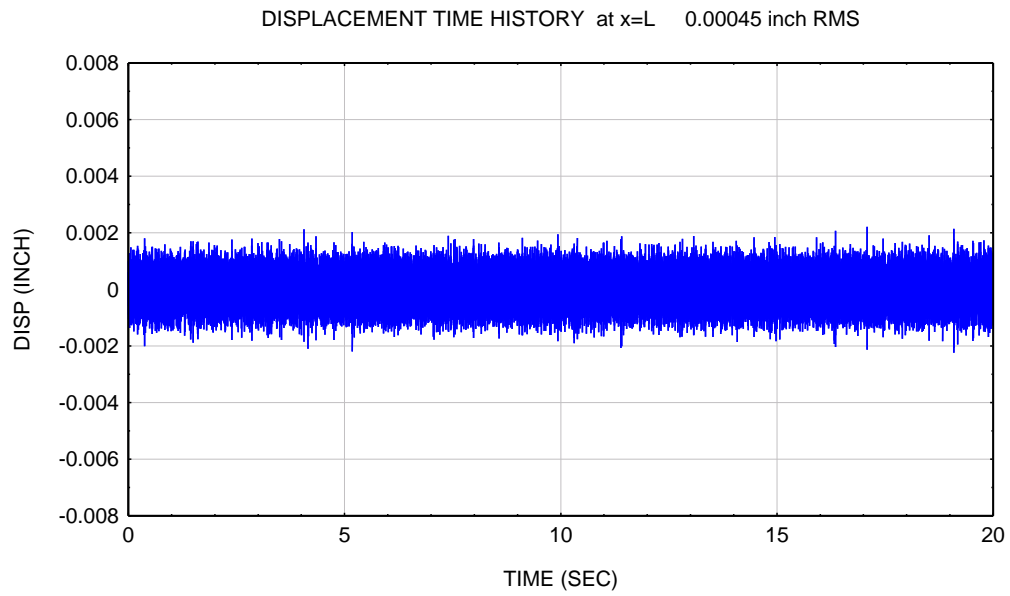


Figure 6.

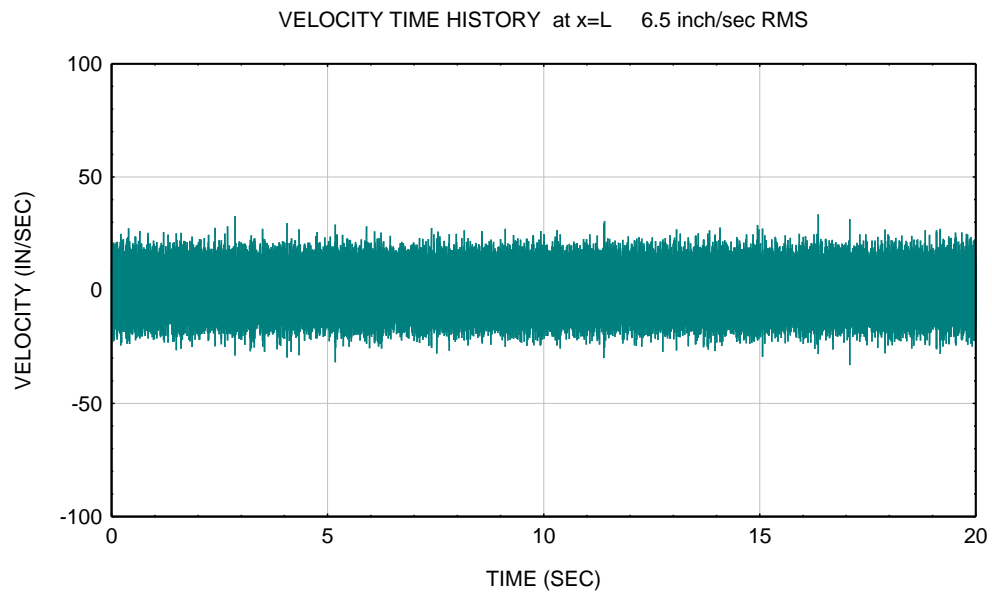


Figure 7.

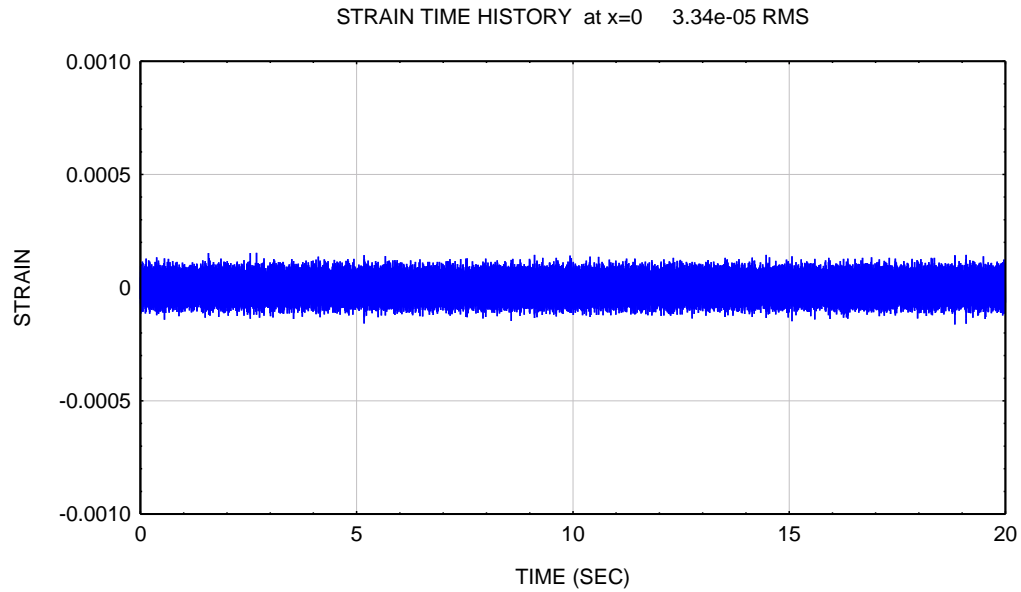


Figure 8.

Matlab Output

Natural Frequency (Hz)

2047
6140
1.023e+004

Displacement (inch) at x=L

mean = 4.77e-008 std = 0.0004534 rms = 0.0004534
max = 0.002211 min = -0.002239

Velocity (in/sec) at x=L

mean = -3.578e-005 std = 6.555 rms = 6.555
max = 33.49 min = -33.06

Strain at x=0

mean = 2.368e-009 std = 3.34e-005 rms = 3.34e-005
max = 0.0001538 min = -0.0001626

References

1. T. Irvine, Longitudinal Natural Frequencies of Rods and Response to Initial Conditions, Revision B, Vibrationdata, 2009.
2. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.
3. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
4. T. Irvine, A Method for Power Spectral Density Synthesis, Revision B, Vibrationdata, 2000.
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APPENDIX A

Stress-Velocity Relationship

The normal stress is

$$\sigma(x, \omega) = \left[\frac{2PE}{\rho AL} \right] \sum_{i=1,3,5,\dots}^{\infty} \left(\frac{\omega_i}{c} \right) \frac{\sin\left(\frac{\omega_i L}{c}\right) \cos\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (\text{A-1})$$

$$\sigma(0, \omega) = \left[\frac{2PE}{\rho AL} \right] \sum_{i=1,3,5,\dots}^{\infty} \left(\frac{\omega_i}{c} \right) \frac{\sin\left(\frac{\omega_i L}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (\text{A-2})$$

For the first mode only,

$$\sigma(0, \omega) = \left[\frac{2PE}{\rho AL} \right] \left(\frac{\omega_1}{c} \right) \frac{\sin\left(\frac{\omega_1 L}{c}\right)}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (\text{A-3})$$

The natural frequencies for a fixed-free rod are

$$\omega_i = \frac{i \pi c}{2L}, \quad i = 1, 3, 5, \dots \quad (\text{A-4})$$

$$\sigma(0, \omega) = \left[\frac{2PE}{\rho AL} \right] \left(\frac{\omega_1}{c} \right) \frac{1}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (\text{A-5})$$

The velocity is

$$\dot{u}(x, \omega) = j\omega \left[\frac{2P}{\rho AL} \right] \sum_{i=1,3,5,\dots}^{\infty} \frac{\sin\left(\frac{\omega_i L}{c}\right) \sin\left(\frac{\omega_i x}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (\text{A-6})$$

$$\dot{u}(L, \omega) = j\omega \left[\frac{2P}{\rho AL} \right] \sum_{i=1,3,5,\dots}^{\infty} \frac{\sin\left(\frac{\omega_i L}{c}\right) \sin\left(\frac{\omega_i L}{c}\right)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (\text{A-7})$$

For the first mode only,

$$\dot{u}(L, \omega) = j\omega \left[\frac{2P}{\rho AL} \right] \frac{\sin\left(\frac{\omega_1 L}{c}\right) \sin\left(\frac{\omega_1 L}{c}\right)}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (\text{A-8})$$

$$\dot{u}(L, \omega) = j\omega \left[\frac{2P}{\rho AL} \right] \frac{1}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (\text{A-9})$$

$$\sigma(0, \omega) = \frac{E}{c} \left(\frac{\omega_1}{j\omega} \right) \dot{u}(L, \omega) \quad (\text{A-10})$$

This relationship can be extended to a multi-modal response with some extra work.

The stress magnitude is

$$|\sigma(0, \omega)| = |\dot{u}(L, \omega)| \frac{E}{c} \left(\frac{\omega_1}{\omega} \right) \quad (\text{A-11})$$