DYNAMIC ANALYSIS OF A MASS-DAMPER SYSTEM

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November 11, 2005

Introduction

Consider a mass-damper system.

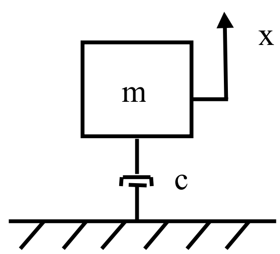

Figure 1.

The variables are

m = mass
c = viscous damping coefficient
x = absolute displacement of the mass

Four cases are analyzed. The results are given in the appendices.

Appendix	Topic
A	Homogeneous, Initial Value Problem
B	Sinusoidal Base Excitation
C	Applied Sinusoidal Force
D	Applied Constant Force

APPENDIX A

<u>Homogeneous, Initial Value Problem</u>

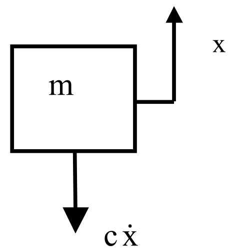

Figure A-1. Free-body Diagram

Consider the initial value problem. The equation of motion is

$$\ddot{x} + (c/m)\dot{x} = 0 \tag{A-1}$$

The Laplace transform method can be used as follows

$$L\{\ddot{x} + (c/m)\,\dot{x}\} = L\{0\} \tag{A-2}$$

$$L\{\ddot{x}\} + (c/m)\; L\{\dot{x}\} = L\{0\} \tag{A-3}$$

$$s^2 X(s) - sx(0) - \dot{x}(0) + (c/m)X(s) - (c/m)x(0) = 0 \tag{A-4}$$

$$[s^2 + (c/m)s]\, X(s) + [-s - (c/m)]x(0) - \dot{x}(0) = 0 \tag{A-5}$$

$$[s^2 + (c/m)]X(s) = [s + (c/m)]\, x(0) + \dot{x}(0) \tag{A-6}$$

$$X(s) = \frac{\{[s + (c/m)]x(0) + \dot{x}(0)\}}{\{s^2 + (c/m)s\}} \tag{A-7}$$

$$X(s)=\left\{\frac{1}{s}\right\}\left\{\frac{[s+(c/m)]x(0)+\dot{x}(0)}{s+(c/m)}\right\} \quad \text{(A-8)}$$

$$X(s) = \left\{\frac{x(0)}{s+(c/m)}\right\}+\left\{\frac{1}{s}\right\}\left\{\frac{(c/m)x(0)+\dot{x}(0)}{s+(c/m)}\right\} \quad \text{(A-9)}$$

Take the inverse Laplace transform. The absolute displacement is

$$x(t) = x(0)\exp[-(c/m)t] + \{x(0)+(m/c)\dot{x}(0)\}\{1-\exp[-(c/m)t]\} \quad \text{(A-10)}$$

$$x(t) = x(0) + \{(m/c)\dot{x}(0)\}\{1-\exp[-(c/m)t]\} \quad \text{(A-11)}$$

The absolute velocity is

$$\dot{x}(t) = \dot{x}(0)\exp[-(c/m)t] \quad \text{(A-12)}$$

The absolute acceleration is

$$\ddot{x}(t) = -(c/m)\,\dot{x}(0)\exp[-(c/m)t] \quad \text{(A-13)}$$

APPENDIX B

Sinusoidal Base Excitation

A mass-damper system subjected to base excitation is shown in Figure 1. The free-body diagram is shown in Figure B-1.

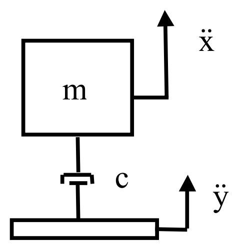

Figure B-1. Single-degree-of-freedom System

The variables are

m = mass
c = viscous damping coefficient
x = absolute displacement of the mass
y = base input displacement

The double-dot notation indicates acceleration

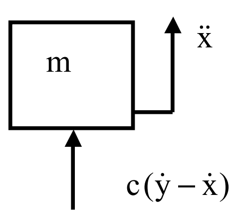

Figure B-2. Free-body Diagram

A summation of forces yields the following governing differential equation of motion:

$$m\ddot{x} + c\dot{x} = c\dot{y} \quad \text{(B-1)}$$

A relative displacement can be defined as z = x-y. The following equation is obtained by substituting this expression into equation (B-2):

$$m\ddot{z} + c\dot{z} = - m\ddot{y} \tag{B-2}$$

Substitution of these terms into equation (B-2) yields an equation of motion for the relative response

$$\ddot{z} + (c/m)\dot{z} = - \ddot{y}(t) \tag{B-3}$$

$$\ddot{z} + (c/m)\dot{z} = - \ddot{y}(t) \tag{B-4}$$

$$\ddot{z} + (c/m)\dot{z} = - A\sin\omega t \tag{B-5}$$

The Laplace transform method can be used as follows

$$L\{\ddot{z} + (c/m)\dot{z}\} = L\{- A\sin\omega t\} \tag{B-6}$$

$$L\{\ddot{z}\} + (c/m)\, L\{\dot{z}\} = -A\, L\{\sin\omega t\} \tag{B-7}$$

$$s^2\hat{Z}(s) - sz(0) - \dot{z}(0) + (c/m)\hat{Z}(s) - (c/m)z(0) = \frac{-A\omega}{s^2+\omega^2} \tag{B-8}$$

$$[s^2 + (c/m)s]\hat{Z}(s) + [-s - (c/m)]z(0) - \dot{z}(0) = \frac{-A\omega}{s^2+\omega^2} \tag{B-9}$$

$$[s^2 + (c/m)s]\hat{Z}(s) = \frac{-A\omega}{s^2+\omega^2} + [\,s(c/m)]z(0) + \dot{z}(0) \tag{B-10}$$

$$\hat{Z}(s) = \left\{\frac{-A\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+(c/m)s}\right\} + \frac{[s+(c/m)]z(0)+\dot{z}(0)}{s^2+(c/m)s} \tag{B-11}$$

$$\hat{Z}(s) = \left\{\frac{1}{s}\right\}\left\{\frac{-A\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{s+(c/m)}\right\} + \left\{\frac{1}{s}\right\}\left\{\frac{[s+(c/m)]z(0)+\dot{z}(0)}{s+(c/m)}\right\} \tag{B-12}$$

Use the partial fraction expansion from Appendix E.

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right] + \left[\frac{-1}{\alpha^2\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\beta s+\alpha^2}{s^2+\alpha^2}\right] + \left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

$\alpha = \varpi$

$\beta = c/m$

(B-13)

$$\hat{Z}(s) =$$

$$-A\omega\left\{\left[\frac{m}{\omega^2 c}\right]\left[\frac{1}{s}\right] + \left[\frac{-1}{\omega^2\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{(c/m)s+\omega^2}{s^2+\omega^2}\right] + \left[\frac{-m}{c\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{1}{s+(c/m)}\right]\right\}$$

$$+\left\{\frac{z(0)}{s+(c/m)}\right\} + \left\{\frac{1}{s}\right\}\left\{\frac{(c/m)z(0)+\dot{z}(0)}{s+(c/m)}\right\}$$

(B-14)

$$\hat{Z}(s) =$$

$$-\frac{A}{\omega^2}\left\{\left[\frac{m}{c}\right]\left[\frac{1}{s}\right] + \left[\frac{-\omega}{\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{(c/m)s+\omega^2}{s^2+\omega^2}\right] + \left[\frac{-m\omega^3}{c\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{1}{s+(c/m)}\right]\right\}$$

$$+\left\{\frac{z(0)}{s+(c/m)}\right\} + \left\{\frac{1}{s}\right\}\left\{\frac{(c/m)z(0)+\dot{z}(0)}{s+(c/m)}\right\}$$

(B-15)

Take the inverse Laplace transform. The relative displacement is

$$z(t) = \left[\frac{A/\omega}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\cos(\omega t) + \omega\sin(\omega t)\right] + \left[\frac{A\,m\,\omega}{c\left(\omega^2 + (c/m)^2\right)}\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

$$-\frac{Am}{\omega^2 c} + z(0) + \{(m/c)\dot{z}(0)\}\{1 - \exp[-(c/m)t]\} \quad \text{for } t > 0$$

(B-16)

The relative velocity is

$$\dot{z}(t) = \left[\frac{A}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{-c}{m}\right)\sin(\omega t) + \omega\cos(\omega t)\right] + \left[\frac{-A\omega}{\left(\omega^2 + (c/m)^2\right)}\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

$$+\dot{z}(0)\exp[-(c/m)t] \quad \text{for } t > 0$$

(B-17)

$$\dot{z}(t) =$$

$$\left[\frac{A}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{-c}{m}\right)\sin(\omega t) + \omega\cos(\omega t)\right] + \left[\frac{-A\omega}{\left(\omega^2 + (c/m)^2\right)} + \dot{z}(0)\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

(B-18)

The relative acceleration is

$\ddot{z}(t) =$

$$\left[\frac{-A\omega}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\cos(\omega t) + \omega\sin(\omega t)\right] + \left[\frac{c}{m}\right]\left[\frac{A\omega}{\left(\omega^2 + (c/m)^2\right)} + \dot{z}(0)\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

(B-19)

The absolute acceleration is

$\ddot{x}(t) =$

$$\left[\frac{-A\omega}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\cos(\omega t) + \omega\sin(\omega t)\right] + \left[\frac{c}{m}\right]\left[\frac{A\omega}{\left(\omega^2 + (c/m)^2\right)} + \dot{z}(0)\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

$$+ A\sin\omega t$$

(B-20)

APPENDIX C

Applied Sinusoidal Force

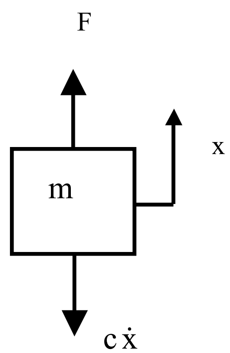

Figure C-2. Free-body Diagram

The equation of motion is

$$m\ddot{x} + c\dot{x} = F(t) \qquad \text{(C-1)}$$

$$m\ddot{x} + c\dot{x} = F\sin\omega t \qquad \text{(C-2)}$$

$$\ddot{x} + (c/m)\dot{x} = (F/m)\sin\omega t \qquad \text{(C-3)}$$

The Laplace transform method can be used as follows

$$L\{\ddot{x} + (c/m)\dot{x}\} = L\{(F/m)\sin\omega t\} \qquad \text{(C-4)}$$

$$L\{\ddot{x}\} + (c/m)\,L\{\dot{x}\} = (F/m)L\{\sin\omega t\} \qquad \text{(C-5)}$$

$$s^2X(s) - sx(0) - \dot{x}(0) + (c/m)X(s) - (c/m)x(0) = \frac{(F/m)\omega}{s^2+\omega^2} \qquad \text{(C-6)}$$

$$[s^2+(c/m)s]X(s)+[-s-(c/m)]x(0)-\dot{x}(0)=\frac{(F/m)\omega}{s^2+\omega^2} \quad \text{(C-7)}$$

$$[s^2+(c/m)s]X(s)=\frac{(F/m)\omega}{s^2+\omega^2}+[s(c/m)]x(0)+\dot{x}(0) \quad \text{(C-8)}$$

$$X(s)=\left\{\frac{(F/m)\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+(c/m)s}\right\}+\frac{[s+(c/m)]x(0)+\dot{x}(0)}{s^2+(c/m)s} \quad \text{(C-9)}$$

$$X(s)=\left\{\frac{1}{s}\right\}\left\{\frac{(F/m)\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{s+(c/m)}\right\}+\left\{\frac{1}{s}\right\}\left\{\frac{[s+(c/m)]x(0)+\dot{x}(0)}{s+(c/m)}\right\} \quad \text{(C-10)}$$

Use the partial fraction expansion from Appendix E.

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\}=\left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right]+\left[\frac{-1}{\alpha^2\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\beta s+\alpha^2}{s^2+\alpha^2}\right]+\left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

$\alpha=\varpi$

$\beta=c/m$

(C-11)

$X(s)=$

$$\left\{\frac{F\omega}{m}\right\}\left\{\left[\frac{m}{\omega^2 c}\right]\left[\frac{1}{s}\right]+\left[\frac{-1}{\omega^2\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{(c/m)s+\omega^2}{s^2+\omega^2}\right]+\left[\frac{-1}{c\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{1}{s+(c/m)}\right]\right\}$$

$$+\left\{\frac{x(0)}{s+(c/m)}\right\}+\left\{\frac{1}{s}\right\}\left\{\frac{(c/m)x(0)+\dot{x}(0)}{s+(c/m)}\right\}$$

(C-12)

$X(s) =$

$$\frac{F}{m\omega^2}\left\{\left[\frac{m}{c}\right]\left[\frac{1}{s}\right]+\left[\frac{-\omega}{\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{(c/m)s+\omega^2}{s^2+\omega^2}\right]+\left[\frac{-\omega^3}{c\left(\omega^2+(c/m)^2\right)}\right]\left[\frac{1}{s+(c/m)}\right]\right\}$$

$$+\left\{\frac{z(0)}{s+(c/m)}\right\}+\left\{\frac{1}{s}\right\}\left\{\frac{(c/m)x(0)+\dot{x}(0)}{s+(c/m)}\right\}$$

(C-13)

Take the inverse Laplace transform. The absolute displacement is

$$x(t)=\left[\frac{-F/(m\omega)}{\left(\omega^2+(c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\cos(\omega t)+\omega\sin(\omega t)\right]+\left[\frac{-F\,\omega/m}{c\left(\omega^2+(c/m)^2\right)}\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

$$+\frac{F}{\omega^2 c}+x(0)+\{(m/c)\dot{x}(0)\}\{1-\exp[-(c/m)t]\} \quad \text{for } t>0$$

(C-14)

The absolute velocity is

$$\dot{x}(t)=\left[\frac{F/m}{\left(\omega^2+(c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\sin(\omega t)+\omega\cos(\omega t)\right]+\left[\frac{F\omega/m}{\left(\omega^2+(c/m)^2\right)}\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

$$+\dot{x}(0)\exp[-(c/m)t] \quad \text{for } t>0$$

(C-15)

$\dot{x}(t) =$

$$\left[\frac{F/m}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\sin(\omega t) + \omega\cos(\omega t)\right] + \left[\frac{F\omega/m}{\left(\omega^2 + (c/m)^2\right)} + \dot{x}(0)\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

(C-16)

The absolute acceleration is

$\ddot{x}(t) =$

$$\left[\frac{F\omega/m}{\left(\omega^2 + (c/m)^2\right)}\right]\left[\left(\frac{c}{m}\right)\cos(\omega t) + \omega\sin(\omega t)\right] + \left[\frac{c}{m}\right]\left[\frac{-F\omega/m}{\left(\omega^2 + (c/m)^2\right)} + \dot{x}(0)\right]\exp\left[\left(-\frac{c}{m}\right)t\right]$$

(C-17)

APPENDIX D

Applied Constant Force

The equation of motion is

$$m\ddot{x} + c\dot{x} = F \tag{D-1}$$

$$\ddot{x} + (c/m)\dot{x} = (F/m) \tag{D-2}$$

The Laplace transform method can be used as follows

$$L\{\ddot{x} + (c/m)\dot{x}\} = L\{(F/m)\} \tag{D-3}$$

$$L\{\ddot{x}\} + (c/m)\, L\{\dot{x}\} = (F/m) L\{u(t)\} \tag{D-4}$$

where u(t) is the unit step function.

$$s^2 X(s) - sx(0) - \dot{x}(0) + (c/m)X(s) - (c/m)x(0) = \frac{(F/m)}{s} \tag{D-5}$$

$$[s^2 + (c/m)]X(s) = [s + (c/m)]\, x(0) + \dot{x}(0) + \frac{(F/m)}{s} \tag{D-6}$$

$$X(s) = \frac{\{[s + (c/m)]x(0) + \dot{x}(0)\}}{\{s^2 + (c/m)s\}} + \frac{(F/m)}{s\{s^2 + (c/m)s\}} \tag{D-7}$$

$$X(s)=\left\{\frac{1}{s}\right\}\left\{\frac{[s+(c/m)]x(0)+\dot{x}(0)}{s+(c/m)}\right\}+\frac{(F/m)}{s^{2}\{s+(c/m)\}} \tag{D-8}$$

$$X(s)=\left\{\frac{x(0)}{s+(c/m)}\right\}+\left\{\frac{1}{s}\right\}\left\{\frac{(c/m)x(0)+\dot{x}(0)}{s+(c/m)}\right\}+\frac{(F/m)}{s^{2}\{s+(c/m)\}} \tag{D-9}$$

Take the inverse Laplace transform. The absolute displacement is

$$x(t)=x(0)\exp[-(c/m)t]+\{x(0)+(m/c)\dot{x}(0)\}\{1-\exp[-(c/m)t]\}$$
$$+\left\{\frac{F}{c}\right\}\left\{t-\left[\frac{m}{c}\right][1-\exp[-(c/m)t]]\right\} \tag{D-10}$$

$$x(t)=x(0)+\{(m/c)\dot{x}(0)\}\{1-\exp[-(c/m)t]\}$$
$$+\left\{\frac{F}{c}\right\}\left\{t-\left[\frac{m}{c}\right][1-\exp[-(c/m)t]]\right\} \tag{D-11}$$

The absolute velocity is

$$\dot{x}(t)=\dot{x}(0)\exp[-(c/m)t]+\left\{\frac{F}{c}\right\}\{1-[\exp[-(c/m)t]]\} \tag{D-12}$$

The absolute acceleration is

$$\ddot{x}(t)=-\left(\frac{c}{m}\right)\dot{x}(0)\exp[-(c/m)t]+\left(\frac{F}{m}\right)\exp[-(c/m)t] \tag{D-13}$$

APPENDIX E

Partial Fraction Expansion

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\}=\frac{a}{s}+\frac{bs+c}{s^2+\alpha^2}+\frac{d}{s+\beta} \tag{E-1}$$

$$1=\frac{a}{s}\{s\}\left\{s^2+\alpha^2\right\}\{s+\beta\}+\left\{\frac{bs+c}{s^2+\alpha^2}\right\}\{s\}\left\{s^2+\alpha^2\right\}\{s+\beta\}+\left\{\frac{d}{s+\beta}\right\}\{s\}\left\{s^2+\alpha^2\right\}\{s+\beta\} \tag{E-2}$$

$$1=a\left\{s^2+\alpha^2\right\}\{s+\beta\}+\{bs+c\}\{s\}\{s+\beta\}+\{ds\}\left\{s^2+\alpha^2\right\} \tag{E-3}$$

$$1=a\left\{s^2+\alpha^2\right\}\{s+\beta\}+\{bs+c\}\left\{s^2+\beta s\right\}+\left\{ds^3+ds\alpha^2\right\} \tag{E-4}$$

$$1=\left[as^3+a\beta s^2+a\alpha^2 s+a\alpha^2\beta\right]+\left[bs^3+(c+b\beta)s^2+c\beta s\right]+\left[ds^3+ds\alpha^2\right] \tag{E-5}$$

$$1=s^3[a+b+d]+s^2[a\beta+(c+b\beta)]+s\left[a\alpha^2+c\beta+d\alpha^2\right]+\left[a\alpha^2\beta\right] \tag{E-6}$$

$$a=\frac{1}{\alpha^2\beta} \tag{E-7}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ \alpha^2 & 0 & \beta & \alpha^2\\ \beta & \beta & 1 & 0\\ 1 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}1/\alpha^2\beta\\ 0\\ 0\\ 0\end{bmatrix} \tag{E-8}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \beta & \alpha^2 \\ 0 & \beta & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ -1/\beta \\ -1/\alpha^2 \\ -1/\alpha^2\beta \end{bmatrix} \tag{E-9}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ 0 & 0 & \beta & \alpha^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ -1/\alpha^2\beta \\ -1/\alpha^2 \\ -1/\beta \end{bmatrix} \tag{E-10}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & \beta & \alpha^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ -1/\alpha^2\beta \\ 0 \\ -1/\beta \end{bmatrix} \tag{E-11}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ -1/\alpha^2\beta \\ 0 \\ -1/\beta \end{bmatrix} \tag{E-12}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ -1/\alpha^2\beta \\ 0 \\ -1/\beta\left(\alpha^2+\beta^2\right) \end{bmatrix} \tag{E-13}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ \left[-1/\alpha^2\beta\right] + \left[1/\beta\left(\alpha^2+\beta^2\right)\right] \\ -1/\left(\alpha^2+\beta^2\right) \\ -1/\beta\left(\alpha^2+\beta^2\right) \end{bmatrix} \tag{E-14}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha^2 & 0 & \beta & \alpha^2 \\ \beta & \beta & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\alpha^2\beta \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{E-15}$$

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right] + \left[\frac{\left[\left[-1/\alpha^2\beta\right]+\left[1/\beta\left(\alpha^2+\beta^2\right)\right]\right]s - 1/\left(\alpha^2+\beta^2\right)}{s^2+\alpha^2}\right]$$

$$+\left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right] \tag{E-16}$$

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right]+\left[\frac{1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\left[\left[-\left(\alpha^2+\beta^2\right)/\alpha^2\right]+1\right]s-\beta}{s^2+\alpha^2}\right]$$

$$+\left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

(E-17)

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right]+\left[\frac{1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\left[\left[\alpha^2-\left(\alpha^2+\beta^2\right)\right]/\alpha^2\right]s-\beta}{s^2+\alpha^2}\right]$$

$$+\left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

(E-18)

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right]+\left[\frac{1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\left[-\beta^2/\alpha^2\right]s-\beta}{s^2+\alpha^2}\right]$$

$$+\left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

(E-19)

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right] + \left[\frac{1}{\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\left[-\beta/\alpha^2\right]s-1}{s^2+\alpha^2}\right] + \left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

(E-20)

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s+\beta}\right\} = \left[\frac{1}{\alpha^2\beta}\right]\left[\frac{1}{s}\right] + \left[\frac{-1}{\alpha^2\left(\alpha^2+\beta^2\right)}\right]\left[\frac{\beta s+\alpha^2}{s^2+\alpha^2}\right] + \left[\frac{-1}{\beta\left(\alpha^2+\beta^2\right)}\right]\left[\frac{1}{s+\beta}\right]$$

(E-21)