MATRIX DEFLATION FOR REMOVING THE RIGID-BODY MODE FROM THE GENERALIZED EIGENVALUE PROBLEM

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The purpose of the tutorial is to build upon the matrix deflation method from Reference 1.

Consider the longitudinal vibration of a free-free rod. Use the finite element method to model the rod with two elements. The solution is taken from Reference 2.

The generalized eigenvalue problem can be expressed as

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \left(\frac{\lambda}{6}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1}$$

Each eigenvalue is represented by λ.

The eigenvector is $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

The eigenvector represents the modal displacement.

The eigenvalues for the sample problem are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 2.0 \end{bmatrix} \tag{2}$$

The first eigenvalue is zero since the free-free beam has a rigid-body mode.

Now remove the rigid-body mode from the eigenvalue problem.

The eigenvalue problem has the form

$$\underline{K}\,\bar{u} = \lambda\,\underline{M}\,\bar{u} \tag{3}$$

where

$\underline{K}$ is the stiffness matrix

$\underline{M}$ is the mass matrix

For a problem with three degrees-of-freedom,

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{4}$$

$$\underline{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{6}$$

Consider the rigid-body mode. The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

Any other natural mode must be orthogonal to the rigid-body mode. Thus

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \tag{8}$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ m_{21}u_1 + m_{22}u_2 + m_{23}u_3 \\ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 \end{bmatrix} = 0 \tag{9}$$

$$\begin{aligned} & m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ & + m_{21}u_1 + m_{22}u_2 + m_{23}u_3 \\ & + m_{31}u_1 + m_{32}u_2 + m_{33}u_3 = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} & (m_{11} + m_{21} + m_{31})u_1 \\ & + (m_{12} + m_{22} + m_{32})u_2 \\ & + (m_{13} + m_{23} + m_{33})u_3 = 0 \end{aligned} \tag{11}$$

$$u_3 = \frac{(m_{11} + m_{21} + m_{31})}{(m_{13} + m_{23} + m_{33})} u_1 + \frac{(m_{12} + m_{22} + m_{32})}{(m_{13} + m_{23} + m_{33})} u_2 \tag{12}$$

Equation (12) is a constraint equation.

The constraint is applied as follows

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{(m_{11}+m_{21}+m_{31})}{(m_{13}+m_{23}+m_{33})} & -\dfrac{(m_{12}+m_{22}+m_{32})}{(m_{13}+m_{23}+m_{33})} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\dfrac{(m_{11}+m_{21}+m_{31})}{(m_{13}+m_{23}+m_{33})} & -\dfrac{(m_{12}+m_{22}+m_{32})}{(m_{13}+m_{23}+m_{33})} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{13}$$

The transformation matrix is

$$\underline{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\dfrac{(m_{11}+m_{21}+m_{31})}{(m_{13}+m_{23}+m_{33})} & -\dfrac{(m_{12}+m_{22}+m_{32})}{(m_{13}+m_{23}+m_{33})} \end{bmatrix} \tag{14}$$

Transform the eigenvalue problem as follows

$$\underline{C}^T \, \underline{K} \, \underline{C} \, \bar{u} = \lambda \, \underline{C}^T \, \underline{M} \, \underline{C} \, \bar{u} \tag{15}$$

The T superscript indicates transpose. The transformed problem has the same eigenvalues as the original problem excluding the rigid body eigenvalue.

The transposed mass matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix} \tag{16}$$

The transposed stiffness matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 4 \\ -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix} \tag{17}$$

The transposed eigenvalue problem is thus

$$\begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left(\frac{\hat{\lambda}}{6} \right) \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{18}$$

The resulting eigenvalues are

$$\begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.0 \end{bmatrix} \tag{19}$$

The matrix deflation thus successfully removed the zero eigenvalue, which corresponded to the rigid-body mode.

The method is extended to beam bending in Appendix A, where two rigid-body modes must be removed from the eigenvalue problem.

<u>References</u>

1. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.

2. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Rev A, Vibrationdata, 2003.

3. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Vibrationdata, Rev D, 2004.

APPENDIX A

Matrix Deflation for Beam Bending

Consider the bending vibration for a free-free beam. Model the beam using the finite element method with one element. The generalized eigenvalue problem is

$$\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} \tag{A-1}$$

The mass and stiffness matrices in equation (A-1) are taken from Reference 3.

The eigenvalues are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.714 \\ 20 \end{bmatrix} \tag{A-2}$$

Let

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} \tag{A-3}$$

Consider the translational rigid-body mode. The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{A-4}$$

Derive the constraint equation for the translational rigid-body mode.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0 \tag{A-5}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11}u_1 + m_{12}u_2 + m_{13}u_3 + m_{14}u_4 \\ m_{21}u_1 + m_{22}u_2 + m_{23}u_3 + m_{24}u_4 \\ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 + m_{34}u_4 \\ m_{41}u_1 + m_{42}u_2 + m_{43}u_3 + m_{44}u_4 \end{bmatrix} = 0 \tag{A-6}$$

$$\begin{aligned} &m_{11}u_1 + m_{12}u_2 + m_{13}u_3 + m_{14}u_4 \\ &+ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 + m_{34}u_4 = 0 \end{aligned} \tag{A-7}$$

The resulting constraint equation is thus

$$(m_{11} + m_{31})u_1 + (m_{12} + m_{32})u_2 + (m_{13} + m_{33})u_3 + (m_{14} + m_{34})u_4 = 0 \tag{A-8}$$

$$u_4 = -\frac{(m_{11}+m_{31})}{(m_{14}+m_{34})}u_1 - \frac{(m_{12}+m_{32})}{(m_{14}+m_{34})}u_2 - \frac{(m_{13}+m_{33})}{(m_{14}+m_{34})}u_3 \tag{A-9}$$

$$u_4 = 6u_1 + u_2 + 6u_3 \tag{A-10}$$

Form the constraint matrix.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{A-11}$$

Transform the mass matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 144 & 15 & -12 \\ 15 & 2 & -3 \\ -12 & -3 & 36 \end{bmatrix} \tag{A-12}$$

Transform the stiffness matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 288 & 48 & 132 \\ 48 & 12 & 24 \\ 132 & 24 & 84 \end{bmatrix} \tag{A-13}$$

$$\begin{bmatrix} 288 & 48 & 132 \\ 48 & 12 & 24 \\ 132 & 24 & 84 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda \begin{bmatrix} 144 & 15 & -12 \\ 15 & 2 & -3 \\ -12 & -3 & 36 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{A-14}$$

The eigenvalues are

$$\lambda = \begin{bmatrix} 0 \\ 1.714 \\ 20 \end{bmatrix} \tag{A-15}$$

Thus, the translation rigid-body mode has been successfully removed from the generalized eigenvalue problem.

Consider the rotational rigid-body mode. Determine the eigenvector by setting the eigenvalue to zero in equation (A-14). The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \tag{A-16}$$

Derive the constraint equation for the rotational rigid-body mode using the matrices with the translation rigid-body mode already removed.

$$\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \tag{A-17}$$

$$\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ m_{21}u_1 + m_{22}u_2 + m_{23}u_3 \\ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 \end{bmatrix} = 0 \tag{A-18}$$

$$\begin{aligned} & m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ & -2\,m_{21}u_1 - 2\,m_{22}u_2 - 2\,m_{23}u_3 \\ & -m_{31}u_1 - m_{32}u_2 - m_{33}u_3 = 0 \end{aligned} \tag{A-19}$$

$$\begin{aligned} & \left(m_{11} - 2\,m_{21} - m_{31}\right)u_1 \\ & + \left(m_{12} - 2\,m_{22} - m_{32}\right)u_2 \\ & + \left(m_{13} - 2\,m_{23} - m_{33}\right)u_3 = 0 \end{aligned} \tag{A-20}$$

$$u_3 = -\frac{(m_{11} - 2\,m_{21} - m_{31})}{(m_{13} - 2\,m_{23} - m_{33})} u_1 - \frac{(m_{12} - 2\,m_{22} - m_{32})}{(m_{13} - 2\,m_{23} - m_{33})} u_2 \tag{A-21}$$

$$u_3 = 3\,u_1 + \frac{1}{3} u_2 \tag{A-22}$$

The resulting constraint equation is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A-23}$$

Transform the mass matrix.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} 144 & 15 & -12 \\ 15 & 2 & -3 \\ -12 & -3 & 36 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 396 & 38 \\ 38 & 4 \end{bmatrix} \tag{A-24}$$

Transform the stiffness matrix.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} 288 & 48 & 132 \\ 48 & 12 & 24 \\ 132 & 24 & 84 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.7760 & 0.2480 \\ 0.2480 & 0.0373 \end{bmatrix} \tag{A-25}$$

$$\begin{bmatrix} 1.7760 & 0.2480 \\ 0.2480 & 0.0373 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} 396 & 38 \\ 38 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A-26}$$

The resulting eigenvalues are

$$\lambda = \begin{bmatrix} 1.714 \\ 20 \end{bmatrix} \tag{A-27}$$