MODAL DENSITY Revision C

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Introduction

Modal density is an important parameter for vibrational power flow analysis, specifically statistical energy analysis. This purpose of this tutorial is to give formulas for modal density. Pertinent variables are shown in Table 1. Modal density formulas are shown in Table 2.

Table 1. Variables

Variable	Description
ω	Frequency (radians/time)
f	Frequency (cycles/time)
λ	Wavelength
N	Number of modes with resonant frequencies below ω
ΔN	Number of modes in frequency band $\Delta\omega$.
$\Delta N / \Delta\omega$	Modal density (modes/radian)
k_L	Wavenumber of longitudinal waves
k_0	Wavenumber of sound in air
k_B	Wavenumber of bending waves
C_L	Wavespeed of longitudinal waves
C_0	Wavespeed of sound in air
C_B	Wavespeed of bending waves
l	Length
h	Thickness
S	Area
V	Volume
α	Radius
ν	$\omega\alpha / C_L$

Table 2. Modal Density Formulas		
Structure	Number of modes N with resonant frequencies below ω	Modal Density $\Delta N / \Delta\omega$ (modes/radian)
Beam, Longitudinal	$= k_L l / \pi$ $= \omega l / C_L \pi$	$= l / C_L \pi$
Beam, Bending	$= k_B l / \pi$ $= \sqrt{\omega}\, l / 1.7 \sqrt{C_L h}$	$= k_B l / 2\pi\omega$ $= l / 3.4 \sqrt{\omega C_L h}$
Plate, Bending	$= k_B{}^2 S / 4\pi$ $= \omega S / 3.6 C_L h$	$= k_B{}^2 S / 4\pi\omega$ $= S / 3.6 C_L h$
Volume, Airborne Sound	$= k_o{}^3 V / 6\pi^2$ $= \omega^3 V / 6\pi^2 C_o^3$	$= k_o{}^2 V / 2\pi^2 C_o^3$ $= \omega^2 V / 2\pi^2 C_o^3$
Ring, Radially Excited	$= 2 k_B \alpha$ $= 3.7\, \alpha \sqrt{\omega} / \sqrt{C_L h}$	$= k_B \alpha / \omega$ $= 1.9\, \alpha / \sqrt{c_L h \omega}$
Thin-walled Tube, for $\nu < 1$	$\approx 3\sqrt{3}\, l \nu^{3/2} / 2\pi h$	$\approx 2\sqrt{\omega}\; l \alpha^{3/2} / 1.6\, h C_L^{3/2}$
Thin-walled Tube, for $\nu > 1$	$\approx \sqrt{3}\; l \alpha \omega / C_L h$	$\approx \sqrt{3}\; l \alpha / C_L h$

The formulas in Table 2 are taken from Reference 1.

The modal density values in the third column can be converted to (modes/Hz) by multiplying by 2π.

Note that harmonic waves have the following relationships

$$k = \frac{\omega}{c} \tag{1}$$

$$k = \frac{2\pi}{\lambda} \tag{2}$$

Furthermore,

$$\omega = 2\pi f \tag{3}$$

The modal density for a cylinder is given in Appendix A.

The modal density for a honeycomb-sandwich flat panel is given in Appendix B.

<u>Wavespeed Values</u>

Wavespeeds for solids and gases are given in Tables 3 and 4, respectively.

Table 3. Solid Wavespeed Values

Solid	Density (kg/m^3)	Elastic Modulus (Pa)	Shear Modulus (Pa)	Poisson's Ratio	C_L (m/sec)
Aluminum	2700	$7.0 (10^{10})$	$2.4 (10^{10})$	0.33	5100 to 5200
Brass	8500	$10.4 (10^{10})$	$3.8 (10^{10})$	0.37	3200 to 3500
Copper	8900	$12.2 (10^{10})$	$4.4 (10^{10})$	0.35	3700
Steel	7700	$19.5 (10^{10})$	$8.3 (10^{10})$	0.28	5050 to 5100

Table 4. Gases at a pressure of 1 atmosphere

Gases	Molecular Mass (kg/kgmole)	Temperature (°C)	Density (kg/m^3)	Ratio of Specific Heats	C_0 (m/sec)
Air	28.97	0	1.293	1.402	332
Air	28.97	20	1.21	1.402	343
Oxygen (O_2)	32.00	0	1.43	1.40	317
Hydrogen (H_2)	2.016	0	0.09	1.41	1270
Steam	-	100	0.60	-	404.8

Note: 1 (kg/kgmole) = 1 (lbm/lbmole)

Bending Waves

The bending wave phase velocity C_B is related to the frequency f and wavelength λ by

$$C_B = f\,\lambda \tag{4}$$

The phase velocity C_B is not a constant, however. Bending waves are *dispersive.*

Consider a plate with thickness h and a longitudinal wave velocity of C_L. The longitudinal velocity is a constant for a given material. The phase velocity is

$$C_B \approx \sqrt{1.8\ C_L h f} \tag{5}$$

By substitution,

$$f\,\lambda \approx \sqrt{1.8\ C_L h f} \tag{6}$$

Dividing both sides by $\sqrt{f}$,

$$\lambda\sqrt{f} \approx \sqrt{1.8\ C_L h} \tag{7}$$

Examples (metric units)

Table 5a. Modal Density, Aluminum Panel, 5 mm Thick, CL=5100 m/sec

Area (meters^2)	$\Delta N / \Delta\omega$ (Modes per radian/sec)	$\Delta N / \Delta f$ (Modes per Hz)	$\Delta f / \Delta N$ (Hz per Mode)
0.5	0.005	0.034	29.2
1.0	0.011	0.068	14.6
1.5	0.016	0.103	9.7
2.0	0.022	0.137	7.3
2.5	0.027	0.171	5.8
3.0	0.033	0.205	4.9

Table 5b. Modal Density, Aluminum Panel, 10 mm Thick, CL = 5100 m/sec			
Area (meters^2)	ΔN / Δω (Modes per radian/sec)	ΔN / Δf (Modes per Hz)	Δf / ΔN (Hz per Mode)
0.5	0.003	0.017	58.4
1.0	0.005	0.034	29.2
1.5	0.008	0.051	19.5
2.0	0.011	0.068	14.6
2.5	0.014	0.086	11.7
3.0	0.016	0.103	9.7

Examples (English units)

Table 6a. Modal Density, Aluminum Panel, 0.25 in Thick, CL = 16732 ft/sec			
Area (ft^2)	ΔN / Δω (Modes per radian/sec)	ΔN / Δf (Modes per Hz)	Δf / ΔN (Hz per Mode)
2	0.002	0.010	99.9
4	0.003	0.020	49.9
6	0.005	0.030	33.3
8	0.006	0.040	25.0
10	0.008	0.050	20.0
12	0.010	0.060	16.6
14	0.011	0.070	14.3
16	0.013	0.080	12.5
18	0.014	0.090	11.1
20	0.016	0.100	10.0
22	0.018	0.110	9.1
24	0.019	0.120	8.3
26	0.021	0.130	7.7
28	0.022	0.140	7.1
30	0.024	0.150	6.7
32	0.026	0.160	6.2
34	0.027	0.170	5.9

Table 6b. Modal Density, Aluminum Panel, 0.50 in Thick, CL = 16732 ft/sec			
Area (ft^2)	ΔN / Δω (Modes per radian/sec)	ΔN / Δf (Modes per Hz)	Δf / ΔN (Hz per Mode)
2	0.001	0.005	199.7
4	0.002	0.010	99.9
6	0.002	0.015	66.6
8	0.003	0.020	49.9
10	0.004	0.025	39.9
12	0.005	0.030	33.3
14	0.006	0.035	28.5
16	0.006	0.040	25.0
18	0.007	0.045	22.2
20	0.008	0.050	20.0
22	0.009	0.055	18.2
24	0.010	0.060	16.6
26	0.010	0.065	15.4
28	0.011	0.070	14.3
30	0.012	0.075	13.3
32	0.013	0.080	12.5
34	0.014	0.085	11.7

Reference

1. L. Cremer and M. Heckl, Structure-Borne Sound, Springer-Verlag, New York, 1988.
2. R. Lyon, Random Noise and Vibration in Space Vehicles, SVM-1, The Shock and Vibration Information Center, United States Department of Defense, 1967.
3. Ferguson & Clarkson, The Modal Density of Honeycomb Shells, Journal of Vibration, Acoustics, Stress, and Reliability in Design, October 1986.
4. J. Wilkinson, Modal Densities of Certain Shallow Structural Elements, The Journal of the Acoustical Society of America, Volume 48, 1968.
5. T. Irvine, Natural Frequencies of a Honeycomb Sandwich Plate, Rev F, Vibrationdata, 2008.

APPENDIX A

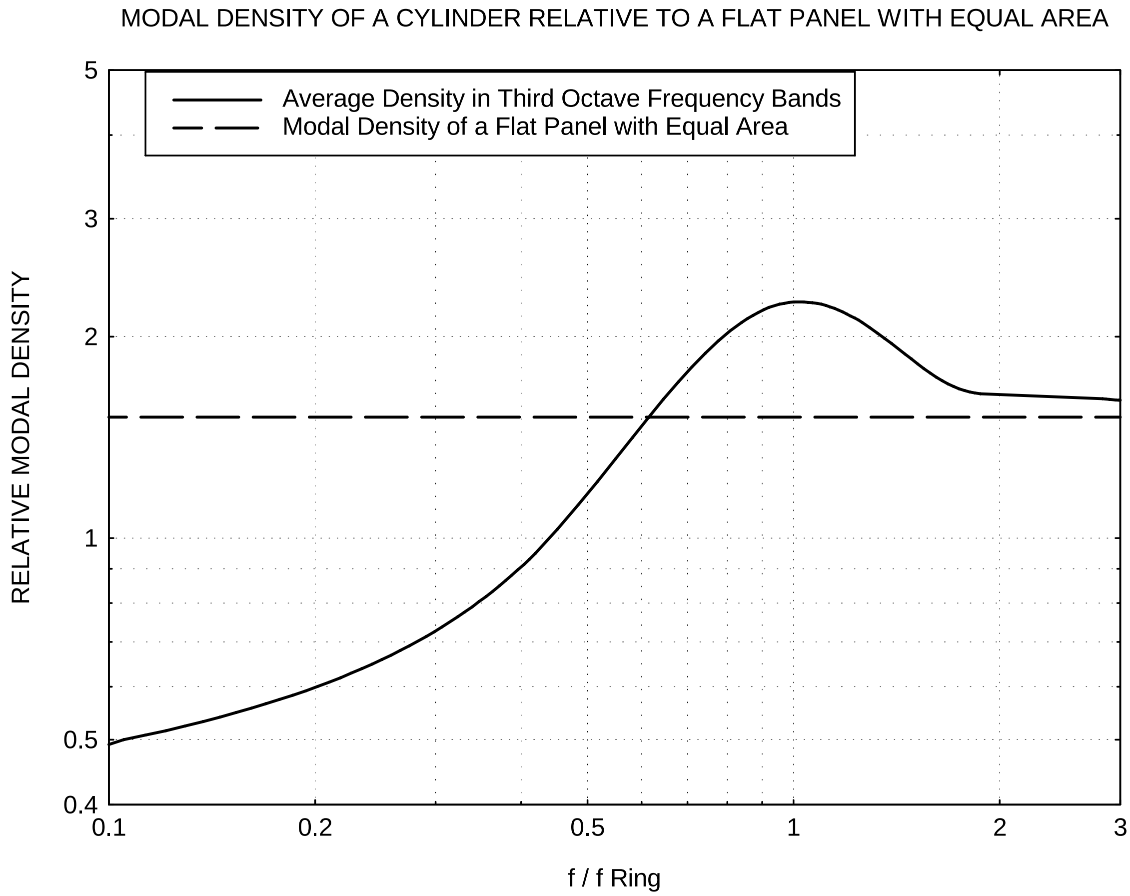
Modal Density of Cylinder

The following section is based on Reference 2.

The ring frequency f_r is the frequency at which the longitudinal wavelength in the skin material is equal to the vehicle circumference $2\pi a$.

$$f_r = \frac{C_L}{2\pi a} \qquad \text{(A-1)}$$

The modal density of a cylinder is a function of the one-third octave band center frequency and the ring frequency as shown in Figure A-1.



MODAL DENSITY OF A CYLINDER RELATIVE TO A FLAT PANEL WITH EQUAL AREA
Average Density in Third Octave Frequency Bands
Modal Density of a Flat Panel with Equal Area
RELATIVE MODAL DENSITY
5
3
2
1
0.5
0.4
0.1
0.2
0.5
1
2
3
f / f Ring

Figure A-1.

APPENDIX B

Flat Honeycomb Sandwich Panel

The variables for this section are

n(f)	=	modal density
a, b	=	length of the sides of the rectangular plate
G	=	shear modulus of core
E	=	elastic modulus of face sheets
h_1	=	half-thickness of the core
h_2	=	individual face sheet thickness
ρ_1	=	mass density of the core
ρ_2	=	mass density of the face sheets
f	=	frequency
ν	=	Poisson Ratio

The following formula is taken from References 3 and 4.

The modal density of the flat honeycomb panel is

$$n(f) = \left(\frac{a\,b}{\pi \hat{S} h_1}\right)\left(\frac{\Omega^2}{f}\right)\left\{1 + \frac{\Omega^2 + 2(1-\nu^2)\hat{S}^2}{\sqrt{\Omega^4 + 4(1-\nu^2)\hat{S}^2\Omega^2}}\right\} \quad \text{(B-1)}$$

where

$$\Omega^2 = 4\pi^2(\rho_1 h_1 + \rho_2 h_2)\frac{f^2 h_1^2}{E h_2} \quad \text{(B-2)}$$

$$\hat{S} = \frac{G h_1}{E h_2} \quad \text{(B-3)}$$

Recall the example from Reference 5, Appendix D.

Table B-1. Honeycomb Sandwich Panel Parameters	
Parameter	Value
Length	72 inch
Width	48 inch
Honeycomb Core Thickness	1.0 inch
Thickness of Each Skin	0.063 inch
Total Thickness	1.125 inch
Skin Elastic Modulus	10.0e+06 lbf/in^2
Core Elastic Modulus	40,000 lbf/in^2
Core Shear Modulus	15,385 lbf/in^2
Skin Density	0.10 lbm/in^3
Core Density	0.005 lbm/in^3
Poisson's Ratio	0.3

The panel is simply-supported on all four sides.

The natural frequencies were calculated via the finite element method.

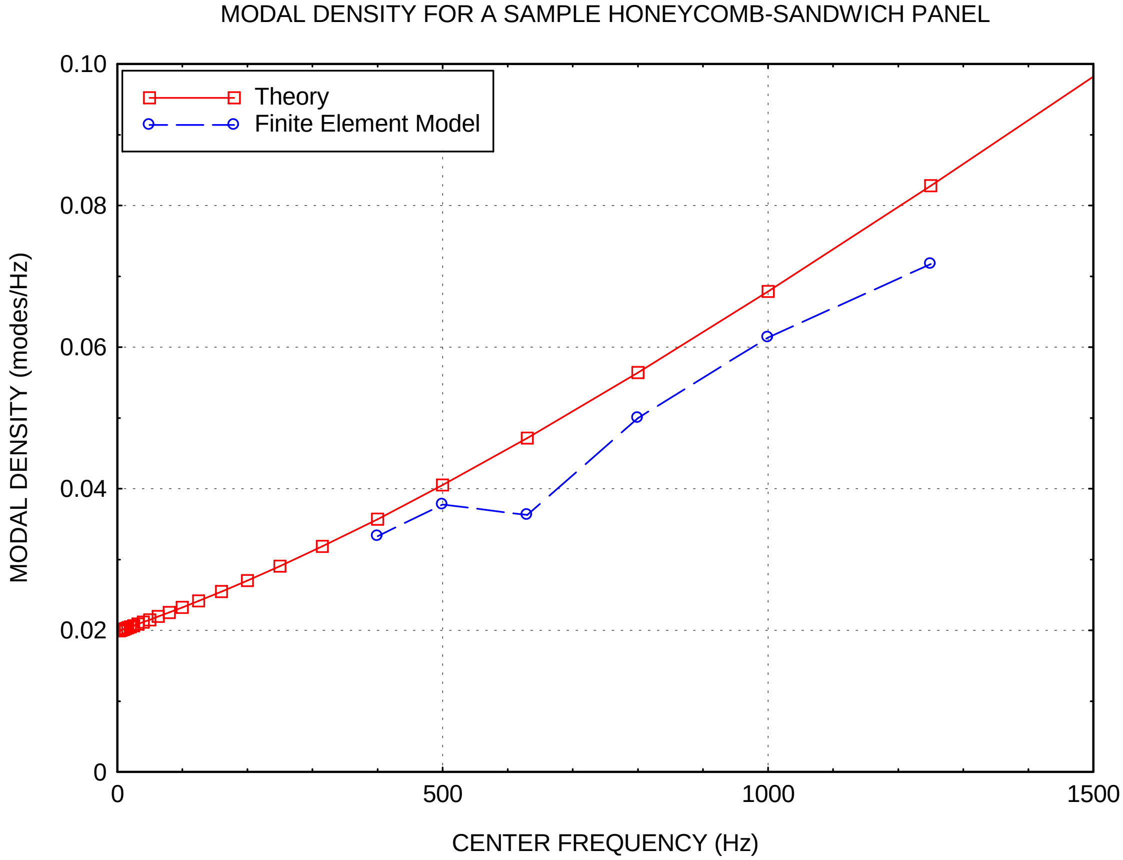


Figure B-1.

The theoretical curve was calculated using equation (B-1).

The theoretical curve predicts 50 modes from 400 to 1250 Hz, as obtained via integration.

The finite element curve yields 44 modes for this same domain.

The theoretical curve is thus 12% higher, which is reasonably good.