THE STEADY-STATE FREQUENCY RESPONSE FUNCTION OF A MULTI-DEGREE-OF-FREEDOM SYSTEM TO HARMONIC BASE EXCITATION
Revision E

By Tom Irvine
Email: tom@vibrationdata.com

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Introduction

The Frequency Response Function (FRF) method is demonstrated by an example. Consider the system in Figure 1.

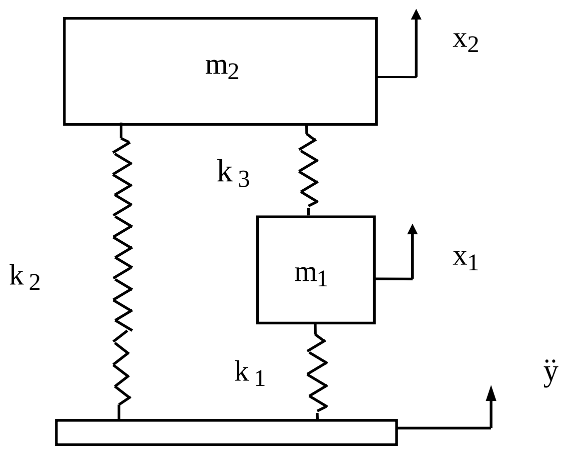

Figure 1.

The system also has damping, but it is modeled as modal damping.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

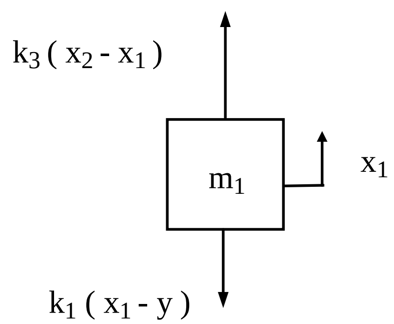

Figure 2.

Determine the equation of motion for mass 1.

$$\Sigma\, F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = k_3(x_2 - x_1) - k_1(x_1 - y) \tag{2}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_3(x_2 - x_1) = k_1 y \tag{3}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_3(x_1 - x_2) = k_1 y \tag{4}$$

$$m_1 \ddot{x}_1 + (k_1 + k_3)x_1 - k_3 x_2 = k_1 y \tag{5}$$

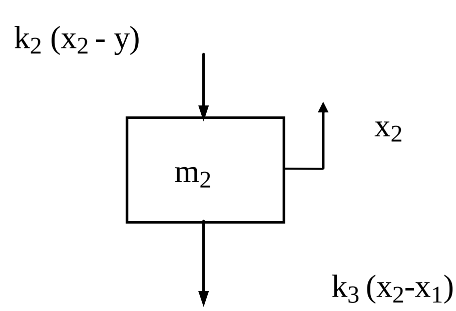

Figure 3.

Derive the equation of motion for mass 2.

$$\Sigma F = m_2 \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = -k_3(x_2 - x_1) - k_2(x_2 - y) \tag{7}$$

$$m_2 \ddot{x}_2 + k_2 x_2 + k_3(x_2 - x_1) = k_2 y \tag{8}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_3 x_1 = k_2 y \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{10}$$

Define a relative displacement z such that

$$x_1 = z_1 + y \tag{11}$$

$$x_2 = z_2 + y \tag{12}$$

Substitute equations (11) and (12) into (10).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 + \ddot{y} \\ \ddot{z}_2 + \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 + y \\ z_2 + y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1\ddot{y} \\ m_2\ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1\ddot{y} \\ m_2\ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{16}$$

Decoupling

Equation (16) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\,\ddot{\bar{z}} + K\,\bar{z} = \bar{F} \tag{17}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{18}$$

$$K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{19}$$

$$\bar{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{20}$$

$$\bar{F} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{21}$$

Consider the homogeneous form of equation (17).

$$M\,\ddot{\bar{z}} + K\,\bar{z} = \bar{0} \tag{22}$$

Seek a solution of the form

$$\bar{z} = \bar{q}\exp\left(j\omega t\right) \tag{23}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{z}} = j\omega\,\bar{q}\exp(j\omega t) \tag{24}$$

$$\ddot{\bar{z}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{25}$$

Substitute equations (23) through (25) into equation (22).

$$-\omega^2 M\,\bar{q}\exp(j\omega t) + K\bar{q}\exp(j\omega t) = \bar{0} \tag{26}$$

$$\left\{-\omega^2 M\,\bar{q} + K\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{27}$$

$$-\omega_n{}^2 M\,\bar{q} + K\bar{q} = \bar{0} \tag{28}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{29}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{30}$$

Equation (30) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{31}$$

$$\det\left\{\begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} - \omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{32}$$

$$\det\begin{bmatrix} (k_1 + k_3) - \omega^2 m_1 & -k_3 \\ -k_3 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix} = 0 \tag{33}$$

$$\left[(k_1 + k_3) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_3{}^2 = 0 \tag{34}$$

$$\omega^4 m_1 m_2 - \omega^2\left[m_1(k_2 + k_3) + m_2(k_1 + k_3)\right] - k_3{}^2 = 0 \tag{35}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1{}^2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \tag{36}$$

$$\omega_2{}^2 = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{37}$$

where

$$a = m_1 m_2 \tag{38}$$

$$b = -\left[m_1\left(k_2+k_3\right)+m_2\left(k_1+k_3\right)\right] \tag{39}$$

$$c = -k_3{}^2 \tag{40}$$

The eigenvectors are found via the following equations.

$$\left\{K-\omega_1{}^2 M\right\}\bar{q}_1 = \bar{0} \tag{41}$$

$$\left\{K-\omega_2{}^2 M\right\}\bar{q}_2 = \bar{0} \tag{42}$$

where

$$\bar{q}_1 = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \tag{34}$$

$$\bar{q}_2 = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \tag{44}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\ \bar{q}_1 \quad | \quad \bar{q}_2\ ] \tag{45}$$

$$Q = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \tag{46}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{47}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{48}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix} \tag{49a}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \tag{49b}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 5 and 6.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{z} = \hat{Q} \; \bar{\eta} \tag{50a}$$

$$z_1 = \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{50b}$$

$$z_2 = \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{50c}$$

Recall

$$x_1 = z_1 + y \tag{51a}$$

$$x_2 = z_2 + y \tag{51b}$$

The displacement terms are

$$x_1 = y + \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{52a}$$

$$x_2 = y + \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{52b}$$

The velocity terms are

$$\dot{x}_1 = \dot{y} + \hat{q}_{11}\,\dot{\eta}_1 + \hat{q}_{12}\,\dot{\eta}_2 \tag{53a}$$

$$\dot{x}_2 = \dot{y} + \hat{q}_{21}\,\dot{\eta}_1 + \hat{q}_{22}\,\dot{\eta}_2 \tag{53b}$$

The acceleration terms are

$$\ddot{x}_1 = \ddot{y} + \hat{q}_{11}\,\ddot{\eta}_1 + \hat{q}_{12}\,\ddot{\eta}_2 \tag{54a}$$

$$\ddot{x}_2 = \ddot{y} + \hat{q}_{21}\,\ddot{\eta}_1 + \hat{q}_{22}\,\ddot{\eta}_2 \tag{54b}$$

Substitute equation (50a) into the equation of motion, equation (17).

$$M\hat{Q}\,\overline{\ddot{\eta}} + K\hat{Q}\,\overline{\eta} = \overline{F} \tag{55}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\overline{\ddot{\eta}} + \hat{Q}^T K\hat{Q}\,\overline{\eta} = \hat{Q}^T\overline{F} \tag{56}$$

The orthogonality relationships yield

$$I\,\overline{\ddot{\eta}} + \Omega\,\overline{\eta} = \hat{Q}^T\overline{F} \tag{57}$$

For the sample problem, equation (57) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix}\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{58}$$

Note that the two equations are decoupled in terms of the modal coordinate.

Now assume modal damping by adding an uncoupled damping matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1 \omega_1 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix}$$

(59)

Define a participation factor.

$$\Gamma_j = \sum_{i=1}^{n} \hat{q}_{ij} m_i \tag{60}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1 \omega_1 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 \\ -\Gamma_2 \end{bmatrix} \ddot{y} \tag{61}$$

Equation (59) yields two equations

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 + \omega_1^2 \eta_1 = -\Gamma_1 \ \ddot{y} \tag{62}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 + \omega_2^2 \ \eta_2 = -\Gamma_2 \ \ddot{y} \tag{63}$$

The generic form is

$$\ddot{\eta}_i + 2\xi_i \ \omega_i + \omega_i^2 \ \eta_i = -\Gamma_i \ \ddot{y} \tag{64}$$

Now assume a harmonic base input.

$$\ddot{y} = A \exp(j\omega t) \tag{65}$$

Assume a harmonic modal displacement.

$$\eta_i = \psi_i \ \exp(j\omega t) \tag{66}$$

$$\dot{\eta}_i = j\,\omega_i\,\psi_i\;\exp(j\omega t) \tag{67}$$

$$\ddot{\eta}_i = -\omega_i^{\,2}\,\psi_i\;\exp(j\omega t) \tag{68}$$

By substitution,

$$\left\{-\omega^2 \;+ j\,2\,\xi_i\,\omega_i\,\omega \;+\omega_i^{\,2}\right\}\psi_i\,\exp(j\omega t) = -\Gamma_i\;A\exp(j\omega t) \tag{69}$$

$$\left\{\left[\omega_i^{\,2} - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}\psi_i\,\exp(j\omega t) = -\,\Gamma_i\;A\exp(j\omega t) \tag{70}$$

$$\eta_i = \psi_i\,\exp(j\omega t) = \frac{-\Gamma_i}{\left\{\left[\omega_i^{\,2} - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}}\;A\exp(j\omega t) \tag{71}$$

The modal velocity is

$$\dot{\eta}_i = \frac{-\,j\omega\,\Gamma_i}{\left\{\left[\omega_i^{\,2} - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}}\;A\exp(j\omega t) \tag{72}$$

The modal acceleration is

$$\ddot{\eta}_i = \frac{\omega^2\;\Gamma_i}{\left\{\left[\omega_i^{\,2} - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}}\;A\exp(j\omega t) \tag{73}$$

Recall

$$\ddot{x}_1 = \ddot{y} + \hat{q}_{11}\,\ddot{\eta}_1 + \hat{q}_{12}\,\ddot{\eta}_2 \tag{76}$$

$$\ddot{x}_2 = \ddot{y} + \hat{q}_{21}\,\ddot{\eta}_1 + \hat{q}_{22}\,\ddot{\eta}_2 \tag{77}$$

The generic form is

$$\ddot{x}_i = \ddot{y} + \sum_{j=1}^{n} \hat{q}_{ij}\,\ddot{\eta}_j \tag{78}$$

$$\ddot{x}_i(t) = \left\{ 1 + \omega^2 \sum_{k=1}^{n} \frac{\hat{q}_{ik}\,\Gamma_k}{\left\{ \left[\omega_k{}^2 - \omega^2 \right] + j\,2\,\xi_k\,\omega_k\,\omega \right\}} \right\} A \exp\left(j\omega t\right) \tag{79}$$

The Fourier transform equation is

$$\hat{X}_i(\omega) = \int_{-\infty}^{\infty} \ddot{x}_i(t) \exp\left[-\,j\omega t\right] dt \tag{80}$$

Take the Fourier transform of each side of equation (80).

$$\hat{X}_i(\omega) / A = \left\{ 1 + \omega^2 \sum_{k=1}^{n} \frac{\hat{q}_{ik}\,\Gamma_k}{\left\{ \left[\omega_k{}^2 - \omega^2 \right] + j\,2\,\xi_k\,\omega_k\,\omega \right\}} \right\} \tag{81}$$

Furthermore, the relative displacement is found as follows

$$z_1 = x_1 - y = \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{82}$$

$$z_2 = x_2 - y = \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{83}$$

The generic form is

$$z_i = x_i - y = \sum_{j=1}^{n} \hat{q}_{ij}\,\ddot{\eta}_j \tag{84}$$

$$z_i(t) = \left\{ \sum_{k=1}^{n} \frac{-\hat{q}_{ik}\,\Gamma_k}{\left\{[\omega_k{}^2 - \omega^2] + j\,2\,\xi_k\,\omega_k\,\omega\right\}} \right\} A \exp(j\omega t) \tag{85}$$

Take the Fourier transform of each side of equation (85). The relative displacement transfer functions are

$$\hat{Z}_i(\omega)/A = \sum_{k=1}^{n} \frac{-\hat{q}_{ik}\,\Gamma_k}{\left\{[\omega_k{}^2 - \omega^2] + j\,2\,\xi_k\,\omega_k\,\omega\right\}} \tag{86}$$

References

1. T. Irvine, An Introduction to the Shock Response Spectrum Revision P, Vibrationdata, 2002.

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5. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.

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APPENDIX A

EXAMPLE 1

Normal Modes Analysis

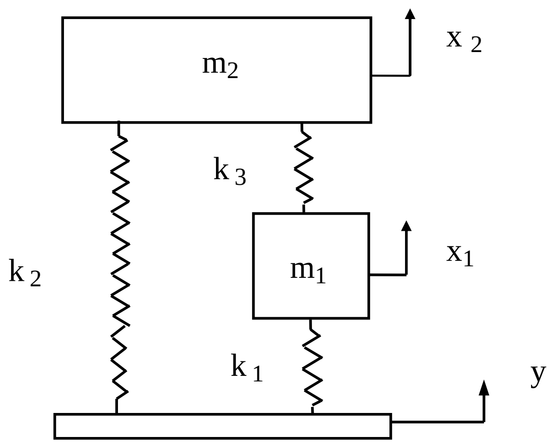

Figure A-1.

Consider the system in Figure A-1. Assign the values in Table A-1.

Table A-1. Parameters	
Variable	Value
m_1	2.0 kg
m_2	1.0 kg
k_1	100,000 N/m
k_2	200,000 N/m
k_3	300,000 N/m

Furthermore, assume that each mode has a damping value of 5%.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_3 & -k_3 \\ -k_3 & k_2+k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{A-1}$$

Solve for the acceleration response time histories. The homogeneous, undamped problem is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix}+\begin{bmatrix} 400{,}000 & -300{,}000 \\ -300{,}000 & 500{,}000 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-2}$$

The eigenvalue problem is

$$\begin{bmatrix} 400{,}000-2\omega^2 & -300{,}000 \\ -300{,}000 & 500{,}000-\omega^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-3}$$

Set the determinant equal to zero

$$\det\begin{bmatrix} 400{,}000-2\omega^2 & -300{,}000 \\ -300{,}000 & 500{,}000-\omega^2 \end{bmatrix}=0 \tag{A-4}$$

The roots of the polynomial are

$$\omega_1 = 300.3 \ \text{rad/sec} \tag{A-5}$$

$$\omega_2 = 780.9 \ \text{rad/sec} \tag{A-6}$$

$$f_1 = 47.8 \ \text{Hz} \tag{A-7}$$

$$f_2 = 124.3 \ \text{Hz} \tag{A-8}$$

The corresponding eigenvector matrix is

$$Q=\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{A-9}$$

$$Q=\begin{bmatrix} 1.366 & -0.366 \\ 1 & 1 \end{bmatrix} \tag{A-10}$$

The next goal is to obtain a normalized eigenvector matrix $\hat{Q}$ such that

$$\hat{Q}^{T} M \hat{Q} = I \tag{A-11}$$

The normalized eigenvector matrix is

$$\hat{Q} = \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{A-12}$$

Note that the eigenvector in the first column has a uniform polarity. Thus, the two masses vibrate in phase for the first mode.

The eigenvector in the second column has two components with opposite polarity. The two masses vibration 180 degrees out of phase for the second mode.

FRF Analysis

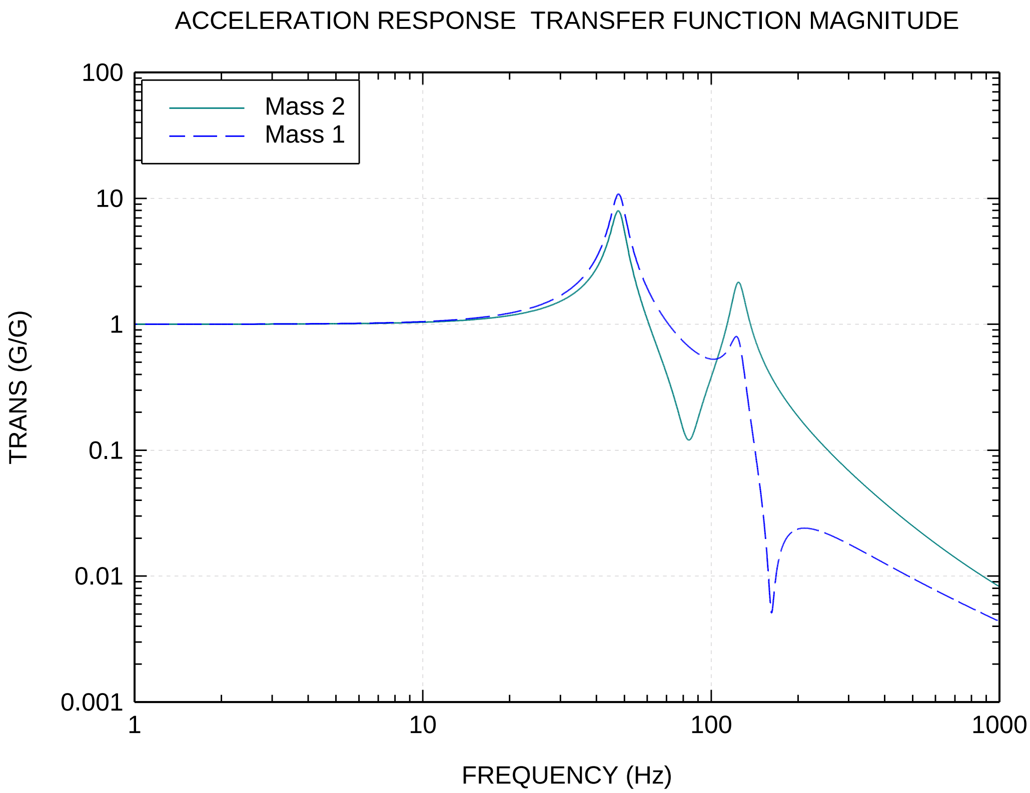

Figure A-2.

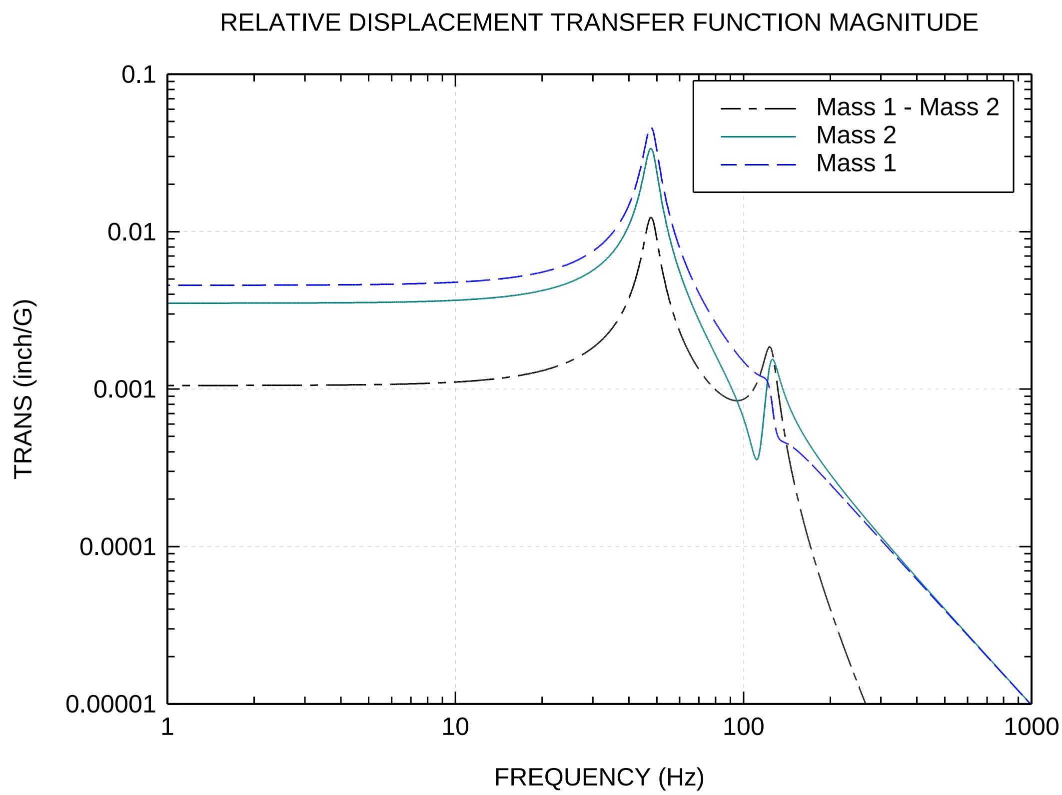

Figure A-3.

Table A-2. Magnitude Values at Fundamental Mode, Frequency = 47.8 Hz		
Magnitude Parameter	Mass 1	Mass 2
Acceleration Response (G/G)	10.9	8.0
Relative Displacement (inch/G)	0.046	0.034

Note that:

$$\text{Acceleration Response} = \text{Relative Displacement} / \omega_1^{\,2}$$

This equation is at least true for a system where the modal frequencies are well separated.

APPENDIX B

<u>Two-degree-of-freedom System with CG Offset, Example 2</u>

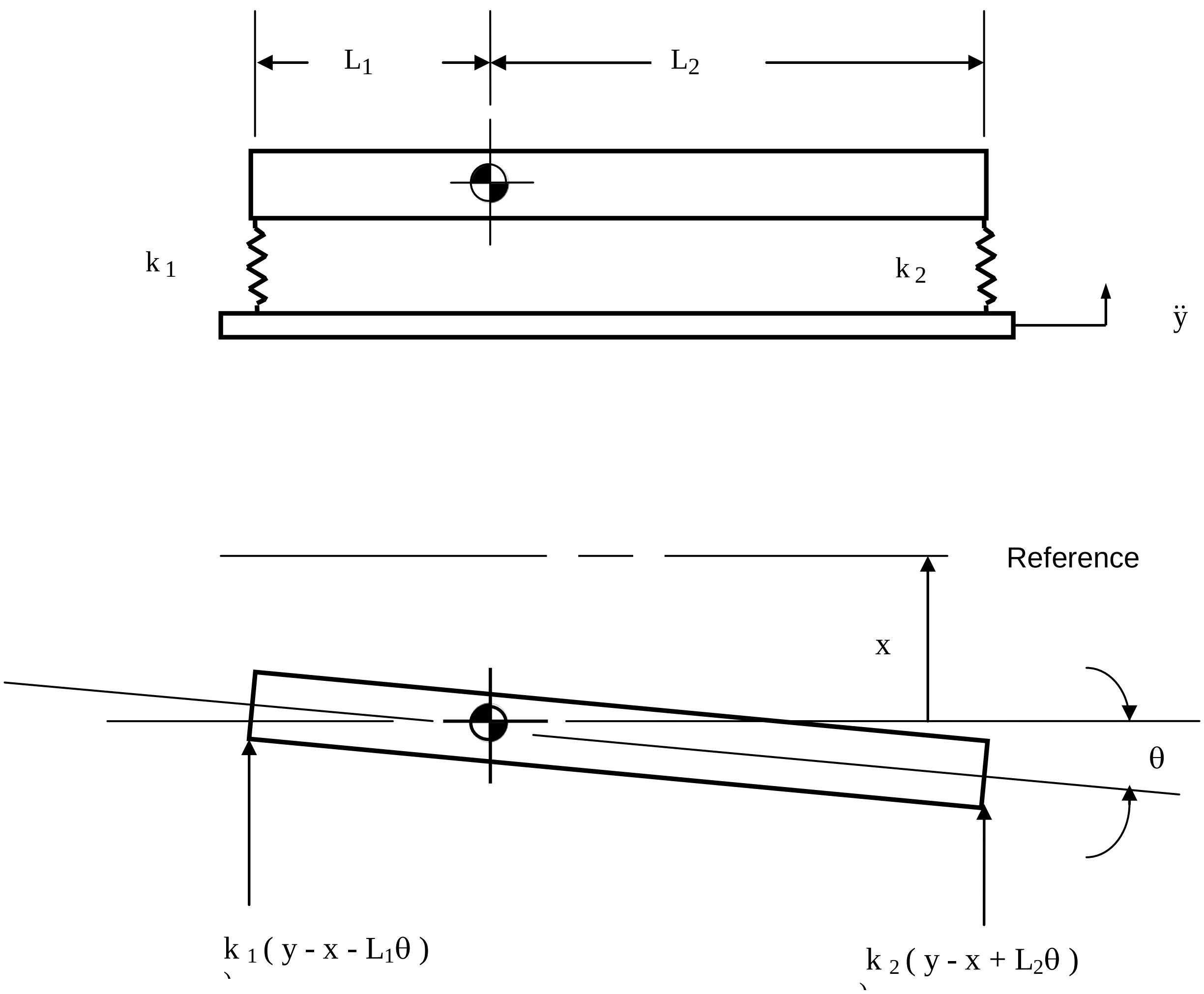

Figure I-1.

Sign Convention:

Translation: upward in vertical axis is positive.
Rotation: clockwise is positive.

Sum the forces in the vertical direction.

$$\sum F = m\ddot{x} \tag{B-1}$$

$$m\ddot{x} = k_1 (y - x - L_1 \theta) + k_2 (y - x + L_2 \theta) \tag{B-2}$$

$$m\ddot{x} + k_1 (x + L_1 \theta) + k_2 (x - L_2 \theta) = (k_1 + k_2) y \tag{B-3}$$

$$m\ddot{x} + (k_1 + k_2) x + (k_1 L_1 - k_2 L_2) \theta = (k_1 + k_2) y \tag{B-4}$$

Sum the moments about the center of mass.

$$\sum M = J\ddot{\theta} \tag{B-5}$$

$$J\ddot{\theta} = k_1 L_1 (y - x - L_1 \theta) - k_2 L_2 (y - x + L_2 \theta) \tag{B-6}$$

$$J\ddot{\theta} - k_1 L_1 (- x - L_1 \theta) + k_2 L_2 (- x + L_2 \theta) = (k_1 L_1 - k_2 L_2) y \tag{B-7}$$

$$J\ddot{\theta} + \left(k_1 L_1^2 + k_2 L_2^2\right) \theta + (k_1 L_1 - k_2 L_2) x = (k_1 L_1 - k_2 L_2) y \tag{B-8}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1 L_1 - k_2 L_2 \end{bmatrix} [\, y \,]$$

(B-9)

Define a relative displacement z such that

$$x = z + y \tag{B-10}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{z}+\ddot{y} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} k_1+k_2 & k_1L_1-k_2L_2 \\ k_1L_1-k_2L_2 & k_1L_1^2+k_2L_2^2 \end{bmatrix}\begin{bmatrix} z+y \\ \theta \end{bmatrix}=\begin{bmatrix} k_1+k_2 \\ k_1L_1-k_2L_2 \end{bmatrix}\left[y\right]$$

(B-11)

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} k_1+k_2 & k_1L_1-k_2L_2 \\ k_1L_1-k_2L_2 & k_1L_1^2+k_2L_2^2 \end{bmatrix}\begin{bmatrix} z \\ \theta \end{bmatrix}=-\begin{bmatrix} m\ddot{y} \\ 0 \end{bmatrix}$$

(B-12)

Use the method in the main text to decouple the equations and to add damping.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix}+\begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}+\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}=-\begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix}\begin{bmatrix} m\ddot{y} \\ 0 \end{bmatrix}$$

(B-13)

$$\ddot{\eta}_1+2\xi_1\omega_1+\omega_1^2\,\eta_1=-\hat{q}_{11}m\,\ddot{y} \tag{B-14}$$

$$\ddot{\eta}_2+2\xi_2\omega_2+\omega_2^2\,\eta_2=-\hat{q}_{12}m\,\ddot{y} \tag{B-15}$$

Now assume a harmonic base input, where A is the acceleration.

$$\ddot{y}=A\exp\left(j\omega t\right) \tag{B-16}$$

Assume a harmonic modal displacement.

$$\eta_i=\psi_i\,\exp\left(j\omega t\right) \tag{B-17}$$

$$\dot{\eta}_i=j\,\omega_i\,\psi_i\,\exp\left(j\omega t\right) \tag{B-18}$$

$$\ddot{\eta}_i=-\omega_i^2\,\psi_i\,\exp\left(j\omega t\right) \tag{B-19}$$

By substitution,

$$\left\{-\omega^2 + j\,2\,\xi_1\,\omega_1\,\omega + \omega_1^2\right\}\psi_1 \exp(j\omega t) = -\hat{q}_{11} m\,A \exp(j\omega t) \quad \text{(B-20)}$$

$$\left\{-\omega^2 + j\,2\,\xi_2\,\omega_2\,\omega + \omega_2^2\right\}\psi_2 \exp(j\omega t) = -\hat{q}_{21} m\,A \exp(j\omega t) \quad \text{(B-21)}$$

The pair of equations becomes

$$\left\{\left[\omega_1^2 - \omega^2\right] + j\,2\,\xi_1\,\omega_1\,\omega\right\}\psi_1 \exp(j\omega t) = -\hat{q}_{11} m\,A \exp(j\omega t) \quad \text{(B-22)}$$

$$\left\{\left[\omega_2^2 - \omega^2\right] + j\,2\,\xi_2\,\omega_2\,\omega\right\}\psi_2 \exp(j\omega t) = -\hat{q}_{21} m\,A \exp(j\omega t) \quad \text{(B-23)}$$

The modal displacements are

$$\eta_1 = \psi_1 \exp(j\omega t) = \frac{-\hat{q}_{11} m\,A}{[\omega_1^2 - \omega^2] + j\,2\,\xi_1\,\omega_1\,\omega} \exp(j\omega t) \quad \text{(B-24)}$$

$$\eta_2 = \psi_2 \exp(j\omega t) = \frac{-\hat{q}_{21} m\,A}{[\omega_2^2 - \omega^2] + j\,2\,\xi_2\,\omega_2\,\omega} \exp(j\omega t) \quad \text{(B-25)}$$

The modal velocities are

$$\dot{\eta}_1 = j\omega\psi_1 \exp(j\omega t) = \frac{-j\omega\hat{q}_{11} m\,A}{[\omega_1^2 - \omega^2] + j\,2\,\xi_1\,\omega_1\,\omega} \exp(j\omega t) \quad \text{(B-26)}$$

$$\dot{\eta}_2 = j\omega\psi_2 \exp(j\omega t) = \frac{-j\omega\,\hat{q}_{21} m\,A}{[\omega_2^2 - \omega^2] + j\,2\,\xi_2\,\omega_2\,\omega} \exp(j\omega t) \quad \text{(B-27)}$$

The modal accelerations are

$$\ddot{\eta}_1 = -\omega^2 \psi_1 \exp(j\omega t) = \frac{\omega^2 \, \hat{q}_{11} m \, A}{[\omega_1{}^2 - \omega^2] + j\,2\,\xi_1\,\omega_1\,\omega} \exp(j\omega t) \tag{B-28}$$

$$\ddot{\eta}_2 = -\omega^2 \psi_2 \exp(j\omega t) = \frac{\omega^2 \, \hat{q}_{21} m \, A}{[\omega_2{}^2 - \omega^2] + j\,2\,\xi_2\,\omega_2\,\omega} \exp(j\omega t) \tag{B-29}$$

Note that

$$z = \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{B-30}$$

$$\theta = \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{B-31}$$

$$\ddot{z} = \hat{q}_{11}\,\ddot{\eta}_1 + \hat{q}_{12}\,\ddot{\eta}_2 \tag{B-32}$$

$$\ddot{\theta} = \hat{q}_{21}\,\ddot{\eta}_1 + \hat{q}_{22}\,\ddot{\eta}_2 \tag{B-33}$$

The Fourier transform for the relative displacement of the CG is

$$Z = m\,A \left\{ \frac{\hat{q}_{11}{}^2}{[\omega_1{}^2 - \omega^2] + j\,2\,\xi_1\,\omega_1\,\omega} + \frac{\hat{q}_{12}\,\hat{q}_{21}}{[\omega_2{}^2 - \omega^2] + j\,2\,\xi_2\,\omega_2\,\omega} \right\} \tag{B-34}$$

The Fourier transform for the CG rotation is

$$\Theta_1 = m\,A \left\{ \frac{\hat{q}_{11}\,\hat{q}_{21}}{[\omega_1{}^2 - \omega^2] + j\,2\,\xi_1\,\omega_1\,\omega} + \frac{\hat{q}_{22}\,\hat{q}_{21}}{[\omega_2{}^2 - \omega^2] + j\,2\,\xi_2\,\omega_2\,\omega} \right\} \tag{B-35}$$

The Fourier transform for the absolute acceleration of the CG is

$$X_a = A\left\{1 + m\omega^2\left\{\frac{\hat{q}_{11}{}^2}{[\omega_1{}^2 - \omega^2] + j\,2\xi_1\omega_1\omega} + \frac{\hat{q}_{12}\,\hat{q}_{21}}{[\omega_2{}^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega}\right\}\right\} \tag{B-36}$$

The relative displacement for spring 1 is

$$z_1 = z + L_1\theta \tag{B-37}$$

The Fourier transform for the relative displacement in spring 1 is

$$Z_1 = m\ A\left\{\frac{\hat{q}_{11}{}^2}{[\omega_1{}^2 - \omega^2] + j\,2\xi_1\omega_1\omega} + \frac{\hat{q}_{12}\,\hat{q}_{21}}{[\omega_2{}^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega}\right\}$$

$$+ L_1\,m\ A\left\{\frac{\hat{q}_{11}\,\hat{q}_{21}}{[\omega_1{}^2 - \omega^2] + j\,2\xi_1\omega_1\omega} + \frac{\hat{q}_{22}\,\hat{q}_{21}}{[\omega_2{}^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega}\right\} \tag{B-38}$$

$$Z_1 = m\ A\left\{\frac{\hat{q}_{11}{}^2 + L_1\hat{q}_{11}\,\hat{q}_{21}}{[\omega_1{}^2 - \omega^2] + j\,2\xi_1\omega_1\omega} + \frac{\hat{q}_{12}\,\hat{q}_{21} + L_1\hat{q}_{22}\,\hat{q}_{21}}{[\omega_2{}^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega}\right\} \tag{B-39}$$

$$Z_1 = m\ A\left\{\hat{q}_{11}\frac{\hat{q}_{11} + L_1\,\hat{q}_{21}}{[\omega_1{}^2 - \omega^2] + j\,2\xi_1\omega_1\omega} + \hat{q}_{21}\frac{\hat{q}_{12} + L_1\hat{q}_{22}}{[\omega_2{}^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega}\right\} \tag{B-40}$$

The relative displacement for spring 2 is

$$z_2 = z - L_2\theta \tag{B-41}$$

The Fourier transform for the relative displacement in spring 2 is

$$Z_2 = m\ A\left\{\hat{q}_{11}\frac{\hat{q}_{11} - L_2\,\hat{q}_{21}}{[\omega_1{}^2 - \omega^2] + j\,2\xi_1\,\omega_1\,\omega} + \hat{q}_{21}\frac{\hat{q}_{12} - L_2\hat{q}_{22}}{[\omega_2{}^2 - \omega^2] + j\,2\xi_2\,\omega_2\,\omega}\right\} \tag{B-42}$$