THE STEADY-STATE FREQUENCY RESPONSE FUNCTION OF A
MULTI-DEGREE-OF-FREEDOM SYSTEM TO HARMONIC FORCE EXCITATION
Revision F

By Tom Irvine
Email: tomirvine@aol.com

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Introduction

The Frequency Response Function (FRF) method is demonstrated by an example. Consider the system in Figure 1.

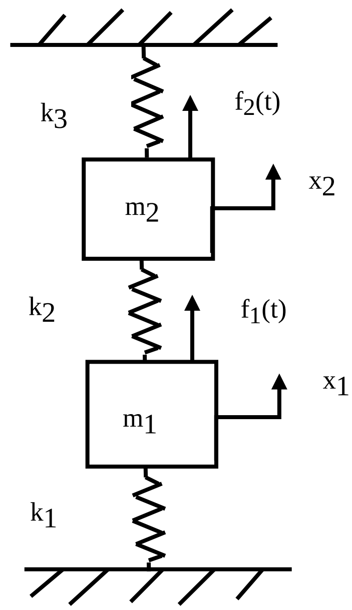

Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

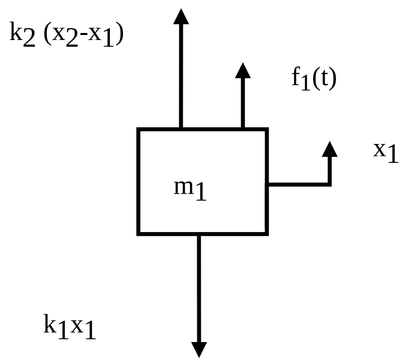

Figure 2.

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1 x_1 \tag{2}$$

$$m_1 \ddot{x}_1 - k_2(x_2 - x_1) + k_1 x_1 = f_1(t) \tag{3}$$

$$m_1 \ddot{x}_1 + k_2(-x_2 + x_1) + k_1 x_1 = f_1(t) \tag{4}$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = f_1(t) \tag{5}$$

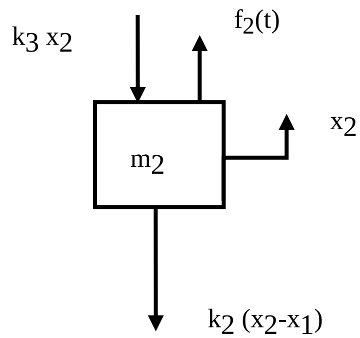

Figure 3.

Derive the equation of motion for mass 2.

$$\sum F = m_2\,\ddot{x}_2 \tag{6}$$

$$m_2\ddot{x}_2 = f_2(t) - k_2(x_2 - x_1) - k_3x_2 \tag{7}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = f_2(t) \tag{8}$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = f_2(t) \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{10}$$

Decoupling

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{13}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\ddot{\bar{x}} + K\bar{x} = \bar{0} \tag{16}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega \bar{q} \exp(j\omega t) \tag{18}$$

$$\ddot{\bar{x}} = -\omega^2 \bar{q} \exp(j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M \bar{q} \exp(j\omega t) + K \bar{q} \exp(j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M \bar{q} + K \bar{q}\right\} \exp(j\omega t) = \bar{0} \tag{21}$$

$$-\omega^2 M \bar{q} + K \bar{q} = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M + K\right\} \bar{q} = \bar{0} \tag{23}$$

$$\left\{K - \omega^2 M\right\} \bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = \bar{0} \tag{25}$$

$$\det\left\{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{26}$$

$$\det\left\{\begin{bmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix}\right\} = 0 \tag{27}$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - {k_2}^2 = 0 \tag{28}$$

$$(k_1+k_2)(k_2+k_3)-\omega^2 m_1(k_2+k_3)-\omega^2 m_2(k_1+k_2)+\omega^4 m_1 m_2-k_2{}^2=0 \tag{29}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1 k_3+(k_1+k_3)k_2+k_2{}^2-k_2{}^2=0 \tag{30}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1 k_3+k_1 k_2+k_2 k_3=0 \tag{31}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2=\frac{-b-\sqrt{b^2-4ac}}{2a} \tag{32}$$

$$\omega_2^2=\frac{-b+\sqrt{b^2-4ac}}{2a} \tag{33}$$

where

$$a=m_1 m_2 \tag{34}$$

$$b=[-m_1(k_2+k_3)-m_2(k_1+k_2)] \tag{35}$$

$$c=k_1 k_2+k_1 k_3+k_2 k_3 \tag{36}$$

The eigenvectors are found via the following equations.

$$\left\{K-\omega_1{}^2\,M\right\}\bar{q}_1=\bar{0} \tag{37}$$

$$\left\{K-\omega_2{}^2\,M\right\}\bar{q}_2=\bar{0} \tag{38}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{39}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{40}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \begin{bmatrix} \bar{q}_1 & \bar{q}_2 \end{bmatrix} \tag{41}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{42}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{43}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{44}$$

where

I	is the identity matrix
Ω	is a diagonal matrix of eigenvalues

The superscript T represents transpose.

Note the mass-normalized forms

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{45}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{46}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial.

Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{47}$$

Substitute equation (47) into the equation of motion, equation (11).

$$M\hat{Q}\,\ddot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = \bar{F} \tag{48}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T\,\bar{F} \tag{49}$$

The orthogonality relationships yield

$$I\,\ddot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T\,\bar{F} \tag{50}$$

The equations of motion along with an added damping matrix become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix}+\begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}+\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}=\begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{51}$$

Note that the two equations are decoupled in terms of the generalized coordinate.

Equation (51) yields two equations

$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = \hat{v}_1 f_1(t) + \hat{v}_2 f_2(t) \tag{52}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = \hat{w}_1 f_1(t) + \hat{w}_2 f_2(t) \tag{53}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{54}$$

Thus

$$\bar{x}(0) = \hat{Q}\,\bar{\eta}(0) \tag{55}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\,\bar{x}(0) = \hat{Q}^T M\hat{Q}\,\bar{\eta}(0) \tag{56}$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \tag{57}$$

$$\hat{Q}^T M \bar{x}(0) = I \bar{\eta}(0) \tag{58}$$

$$\hat{Q}^T M \bar{x}(0) = \bar{\eta}(0) \tag{59}$$

Finally, the transformed initial displacement is

$$\bar{\eta}(0) = \hat{Q}^T M \bar{x}(0) \tag{60}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^T M \bar{\dot{x}}(0) \tag{61}$$

<u>Displacement Response to Applied Forces</u>

Consider two forces:

$$f_1(t) = B_1 \sin(\alpha_1 t + \phi_1) \tag{62}$$

$$f_1(t) = B_1 \{\cos(\phi_1)\sin(\alpha_1 t) + \sin(\phi_1)\cos(\alpha_1 t)\} \tag{63}$$

$$f_2(t) = B_2 \sin(\alpha_2 t + \phi_2) \tag{64}$$

$$f_2(t) = B_2 \{\cos(\phi_2)\sin(\alpha_2 t) + \sin(\phi_2)\cos(\alpha_2 t)\} \tag{65}$$

By substitution,

$$\begin{aligned} \ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = {} & \hat{v}_1\{B_1[\cos(\phi_1)\sin(\alpha_1 t) + \sin(\phi_1)\cos(\alpha_1 t)]\} \\ & + \hat{v}_2\{B_2[\cos(\phi_2)\sin(\alpha_2 t) + \sin(\phi_2)\cos(\alpha_2 t]\} \end{aligned} \tag{66}$$

$$\begin{aligned} \ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = {} & \hat{w}_1\{B_1[\cos(\phi_1)\sin(\alpha_1 t) + \sin(\phi_1)\cos(\alpha_1 t)]\} \\ & + \hat{w}_2\{B_2[\cos(\phi_2)\sin(\alpha_2 t) + \sin(\phi_2)\cos(\alpha_2 t]\} \end{aligned} \tag{67}$$

Assume

$$\eta_1 = N_{11}\sin(\alpha_1 t) + P_{11}\cos(\alpha_1 t) + N_{12}\sin(\alpha_2 t) + P_{12}\cos(\alpha_2 t) \tag{68}$$

$$\dot{\eta}_1 = \alpha_1 N_{11}\cos(\alpha_1 t) - \alpha_1 P_{11}\sin(\alpha_1 t) + \alpha_2 N_{12}\cos(\alpha_2 t) - \alpha_2 P_{12}\sin(\alpha_2 t) \tag{69}$$

$$\ddot{\eta}_1 = -\alpha_1{}^2 N_{11}\sin(\alpha_1 t) - \alpha_1{}^2 P_{11}\cos(\alpha_1 t) - \alpha_2{}^2 N_{12}\sin(\alpha_2 t) - \alpha_2{}^2 P_{12}\cos(\alpha_2 t) \tag{70}$$

Also,

$$\eta_2 = N_{21}\sin(\alpha_1 t) + P_{21}\cos(\alpha_1 t) + N_{22}\sin(\alpha_2 t) + P_{22}\cos(\alpha_2 t) \tag{71}$$

$$\dot{\eta}_2 = \alpha_1 N_{21}\cos(\alpha_1 t) - \alpha_1 P_{21}\sin(\alpha_1 t) + \alpha_2 N_{22}\cos(\alpha_2 t) - \alpha_2 P_{22}\sin(\alpha_2 t) \tag{72}$$

$$\ddot{\eta}_2 = -\alpha_1{}^2 N_{21}\sin(\alpha_1 t) - \alpha_1{}^2 P_{21}\cos(\alpha_1 t) - \alpha_2{}^2 N_{22}\sin(\alpha_2 t) - \alpha_2{}^2 P_{22}\cos(\alpha_2 t) \tag{73}$$

By substitution,

$$\begin{aligned}
&\left[-\alpha_1{}^2 N_{11}\sin(\alpha_1 t) - \alpha_1{}^2 P_{11}\cos(\alpha_1 t) - \alpha_2{}^2 N_{12}\sin(\alpha_2 t) - \alpha_2{}^2 P_{12}\cos(\alpha_2 t)\right] \\
&+ 2\xi_1\omega_1\left[\alpha_1 N_{11}\cos(\alpha_1 t) - \alpha_1 P_{11}\sin(\alpha_1 t) + \alpha_2 N_{12}\cos(\alpha_2 t) - \alpha_2 P_{12}\sin(\alpha_2 t)\right] \\
&+ \omega_1^2\left[N_{11}\sin(\alpha_1 t) + P_{11}\cos(\alpha_1 t) + N_{12}\sin(\alpha_2 t) + P_{12}\cos(\alpha_2 t)\right] \\
&\qquad = \hat{v}_1\{B_1[\cos(\phi_1)\sin(\alpha_1 t) + \sin(\phi_1)\cos(\alpha_1 t)]\} \\
&\qquad\qquad + \hat{v}_2\{B_2[\cos(\phi_2)\sin(\alpha_2 t) + \sin(\phi_2)\cos(\alpha_2 t]\}
\end{aligned} \tag{74}$$

$$+N_{11}\left[+\omega_1^2-\alpha_1^2\right]\sin(\alpha_1 t)+N_{11}[2\xi_1\omega_1\alpha_1]\cos(\alpha_1 t)$$
$$+P_{11}[-2\xi_1\omega_1\alpha_1]\sin(\alpha_1 t)+P_{11}\left[+\omega_1^2-\alpha_1^2\right]\cos(\alpha_1 t)$$
$$+N_{12}\left[+\omega_1^2-\alpha_2^2\right]\sin(\alpha_2 t)+N_{12}[2\xi_1\omega_1\alpha_2]\cos(\alpha_2 t)$$
$$+P_{12}[-2\xi_1\omega_1\alpha_2]\sin(\alpha_2 t)+P_{12}\left[+\omega_1^2-\alpha_2^2\right]\cos(\alpha_2 t)$$

$$\begin{aligned} = \quad & \hat{v}_1\{B_1[\cos(\phi_1)\sin(\alpha_1 t)+\sin(\phi_1)\cos(\alpha_1 t)]\} \\ & +\hat{v}_2\{B_2[\cos(\phi_2)\sin(\alpha_2 t)+\sin(\phi_2)\cos(\alpha_2 t)]\} \end{aligned} \tag{75}$$

Equation (75) is satisfied by the following set of four equations.

$$\left\{+N_{11}\left[+\omega_1^2-\alpha_1^2\right]+P_{11}[-2\xi_1\omega_1\alpha_1]\right\}\sin(\alpha_1 t)=\hat{v}_1\{B_1\cos(\phi_1)\sin(\alpha_1 t)\} \tag{76}$$

$$\left\{+N_{11}[2\xi_1\omega_1\alpha_1]+P_{11}\left[+\omega_1^2-\alpha_1^2\right]\right\}\cos(\alpha_1 t)=\hat{v}_1\{B_1\sin(\phi_1)\cos(\alpha_1 t)\} \tag{77}$$

$$\left\{+N_{12}\left[+\omega_1^2-\alpha_2^2\right]+P_{12}[-2\xi_1\omega_1\alpha_2]\right\}\sin(\alpha_2 t)=\hat{v}_2\{B_2\cos(\phi_2)\sin(\alpha_2 t)\} \tag{78}$$

$$\left\{+N_{12}[2\xi_1\omega_1\alpha_2]+P_{12}\left[+\omega_1^2-\alpha_2^2\right]\right\}\cos(\alpha_2 t)=\hat{v}_2\{B_2\sin(\phi_2)\cos(\alpha_2 t)\} \tag{79}$$

The four equations can be group into two subsystem as in matrix format as follows.

$$\begin{bmatrix} +\omega_1^2-\alpha_1^2 & -2\xi_1\omega_1\alpha_1 \\ 2\xi_1\omega_1\alpha_1 & +\omega_1^2-\alpha_1^2 \end{bmatrix}\begin{bmatrix} N_{11} \\ P_{11} \end{bmatrix}=\begin{bmatrix} \hat{v}_1 B_1\cos(\phi_1) \\ \hat{v}_1 B_1\sin(\phi_1) \end{bmatrix} \tag{80}$$

$$\begin{bmatrix} +\omega_1^2-\alpha_2^2 & -2\xi_1\omega_1\alpha_2 \\ 2\xi_1\omega_1\alpha_2 & +\omega_1^2-\alpha_2^2 \end{bmatrix}\begin{bmatrix} N_{12} \\ P_{12} \end{bmatrix}=\begin{bmatrix} \hat{v}_2 B_2\cos(\phi_2) \\ \hat{v}_2 B_2\sin(\phi_2) \end{bmatrix} \tag{81}$$

Similarly,

$$\begin{bmatrix} +\omega_2^2-\alpha_1{}^2 & -2\xi_2\,\omega_2\alpha_1 \\ 2\xi_2\,\omega_2\alpha_1 & +\omega_2^2-\alpha_1{}^2 \end{bmatrix}\begin{bmatrix} N_{21} \\ P_{21} \end{bmatrix}=\begin{bmatrix} \hat{w}_2 B_1\cos(\phi_1) \\ \hat{w}_2 B_1\sin(\phi_1) \end{bmatrix} \tag{82}$$

$$\begin{bmatrix} +\omega_2^2-\alpha_2{}^2 & -2\xi_2\,\omega_2\alpha_2 \\ 2\xi_2\,\omega_2\alpha_2 & +\omega_2^2-\alpha_2{}^2 \end{bmatrix}\begin{bmatrix} N_{22} \\ P_{22} \end{bmatrix}=\begin{bmatrix} \hat{w}_2 B_2\cos(\phi_2) \\ \hat{w}_2 B_2\sin(\phi_2) \end{bmatrix} \tag{83}$$

Recall

$$\overline{x}=\hat{Q}\,\overline{\eta} \tag{84}$$

$$\hat{Q}=\begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{85}$$

$$x_1(t)=\hat{v}_1\eta_1(t)+\hat{w}_1\eta_2(t) \tag{86}$$

$$x_2(t)=\hat{v}_2\eta_1(t)+\hat{w}_2\eta_2(t) \tag{87}$$

$$\eta_1=N_{11}\sin(\alpha_1 t)+P_{11}\cos(\alpha_1 t)+N_{12}\sin(\alpha_2 t)+P_{12}\cos(\alpha_2 t) \tag{88}$$

$$\eta_2=N_{21}\sin(\alpha_1 t)+P_{21}\cos(\alpha_1 t)+N_{22}\sin(\alpha_2 t)+P_{22}\cos(\alpha_1 t) \tag{89}$$

$$x_1(t) = \hat{v}_1\{N_{11}\sin(\alpha_1 t) + P_{11}\cos(\alpha_1 t) + N_{12}\sin(\alpha_2 t) + P_{12}\cos(\alpha_2 t)\}$$
$$+ \hat{w}_1\{N_{21}\sin(\alpha_1 t) + P_{21}\cos(\alpha_1 t) + N_{22}\sin(\alpha_2 t) + P_{22}\cos(\alpha_2 t)\} \tag{90}$$

$$\begin{aligned} x_1(t) = &+[\hat{v}_1 N_{11} + \hat{w}_1 N_{21}]\sin(\alpha_1 t) \\ &+[\hat{v}_1 P_{11} + \hat{w}_1 P_{21}]\cos(\alpha_1 t) \\ &+[\hat{v}_1 N_{12} + \hat{w}_1 N_{22}]\sin(\alpha_2 t) \\ &+[\hat{v}_1 P_{12} + \hat{w}_1 P_{22}]\cos(\alpha_2 t) \end{aligned} \tag{91}$$

Define spectral magnitude and phase components for the coordinate x_1 displacement as follows:

$$M_{d,11} = \sqrt{\{\hat{v}_1 N_{11} + \hat{w}_1 N_{21}\}^2 + \{\hat{v}_1 P_{11} + \hat{w}_1 P_{21}\}^2} \tag{92}$$

$$M_{d,12} = \sqrt{\{\hat{v}_1 N_{12} + \hat{w}_1 N_{22}\}^2 + \{\hat{v}_1 P_{12} + \hat{w}_1 P_{22}\}^2} \tag{93}$$

$$\phi_{d,11} = \arctan\left(\frac{\hat{v}_1 N_{11} + \hat{w}_1 N_{21}}{\hat{v}_1 P_{11} + \hat{w}_1 P_{21}}\right) \tag{94}$$

$$\phi_{d,12} = \arctan\left(\frac{\hat{v}_1 N_{12} + \hat{w}_1 N_{22}}{\hat{v}_1 P_{12} + \hat{w}_1 P_{22}}\right) \tag{95}$$

The x_1 coordinate has spectral magnitude $M_{d,11}$ and phase $\phi_{d,11}$ at frequency α_1.

The x_1 coordinate has spectral magnitude $M_{d,12}$ and phase $\phi_{d,12}$ at frequency α_2.

The phase angle is the arctan of the sine amplitude divided by the cosine amplitude.

Similarly, for coordinate x_2,

$$x_2(t) = \hat{v}_2\{N_{11}\sin(\alpha_1 t) + P_{11}\cos(\alpha_1 t) + N_{12}\sin(\alpha_2 t) + P_{12}\cos(\alpha_2 t)\}$$
$$+ \hat{w}_2\{N_{21}\sin(\alpha_1 t) + P_{21}\cos(\alpha_1 t) + N_{22}\sin(\alpha_2 t) + P_{22}\cos(\alpha_2 t)\} \tag{96}$$

$$\begin{aligned} x_2(t) = &+[\hat{v}_2 N_{11} + \hat{w}_2 N_{21}]\sin(\alpha_1 t) \\ &+[\hat{v}_2 P_{11} + \hat{w}_2 P_{21}]\cos(\alpha_1 t) \\ &+[\hat{v}_2 N_{12} + \hat{w}_2 N_{22}]\sin(\alpha_2 t) \\ &+[\hat{v}_2 P_{12} + \hat{w}_2 P_{22}]\cos(\alpha_2 t) \end{aligned} \tag{97}$$

Define spectral magnitude and phase components for the coordinate x_2 displacement as follows:

$$M_{d,21} = \sqrt{\{\hat{v}_2 N_{11} + \hat{w}_2 N_{21}\}^2 + \{\hat{v}_2 P_{11} + \hat{w}_2 P_{21}\}^2} \tag{98}$$

$$M_{d,22} = \sqrt{\{\hat{v}_2 N_{12} + \hat{w}_2 N_{22}\}^2 + \{\hat{v}_2 P_{12} + \hat{w}_2 P_{22}\}^2} \tag{99}$$

$$\phi_{d,21} = \arctan\left(\frac{\hat{v}_2 N_{11} + \hat{w}_2 N_{21}}{\hat{v}_2 P_{11} + \hat{w}_2 P_{21}}\right) \tag{100}$$

$$\phi_{d,22} = \arctan\left(\frac{\hat{v}_2 N_{12} + \hat{w}_2 N_{22}}{\hat{v}_2 P_{12} + \hat{w}_2 P_{22}}\right) \tag{101}$$

The x_2 coordinate displacement has spectral magnitude $M_{d,21}$ and phase $\phi_{d,21}$ at frequency α_1.

The x_2 coordinate displacement has spectral magnitude $M_{d,22}$ and phase $\phi_{d,22}$ at frequency α_2.

The phase angle is the arctan of the sine amplitude divided by the cosine amplitude.

Velocity Response to Applied Forces

For coordinate x_1,

$$\dot{\eta}_1=\alpha_1 N_{11}\cos(\alpha_1 t)-\alpha_1 P_{11}\sin(\alpha_1 t)+\alpha_2 N_{12}\cos(\alpha_2 t)-\alpha_2 P_{12}\sin(\alpha_2 t) \tag{102}$$

$$\dot{\eta}_2=\alpha_1 N_{21}\cos(\alpha_1 t)-\alpha_1 P_{21}\sin(\alpha_1 t)+\alpha_2 N_{22}\cos(\alpha_2 t)-\alpha_2 P_{22}\sin(\alpha_2 t) \tag{103}$$

$$\begin{aligned}\dot{x}_1(t)=\hat{v}_1&\{\alpha_1 N_{11}\cos(\alpha_1 t)-\alpha_1 P_{11}\sin(\alpha_1 t)+\alpha_2 N_{12}\cos(\alpha_2 t)-\alpha_2 P_{12}\sin(\alpha_2 t)\}\\ +\hat{w}_1&\{\alpha_1 N_{21}\cos(\alpha_1 t)-\alpha_1 P_{21}\sin(\alpha_1 t)+\alpha_2 N_{22}\cos(\alpha_2 t)-\alpha_2 P_{22}\sin(\alpha_2 t)\}\end{aligned} \tag{104}$$

$$\begin{aligned}\dot{x}_1(t)=&+\alpha_1[\hat{v}_1 N_{11}+\hat{w}_1 N_{21}]\cos(\alpha_1 t)\\ &-\alpha_1[\hat{v}_1 P_{11}+\hat{w}_1 P_{21}]\sin(\alpha_1 t)\\ &+\alpha_2[\hat{v}_1 N_{12}+\hat{w}_1 N_{22}]\cos(\alpha_2 t)\\ &-\alpha_2[\hat{v}_1 P_{12}+\hat{w}_1 P_{22}]\sin(\alpha_2 t)\end{aligned} \tag{105}$$

Define spectral magnitude and phase components for coordinate x_1 velocity as follows:

$$M_{v,11}=\alpha_1\sqrt{\{\hat{v}_1 N_{11}+\hat{w}_1 N_{21}\}^2+\{\hat{v}_1 P_{11}+\hat{w}_1 P_{21}\}^2} \tag{106}$$

$$M_{v,12}=\alpha_2\sqrt{\{\hat{v}_1 N_{12}+\hat{w}_1 N_{22}\}^2+\{\hat{v}_1 P_{12}+\hat{w}_1 P_{22}\}^2} \tag{107}$$

$$\phi_{v,11}=\arctan\left(\frac{-\hat{v}_1 P_{11}-\hat{w}_1 P_{21}}{+\hat{v}_1 N_{11}+\hat{w}_1 N_{21}}\right) \tag{108}$$

$$\phi_{v,12}=\arctan\left(\frac{-\hat{v}_1 P_{12}-\hat{w}_1 P_{22}}{+\hat{v}_1 N_{12}+\hat{w}_1 N_{22}}\right) \tag{109}$$

The x_1 coordinate velocity has spectral magnitude $M_{v,11}$ and phase $\phi_{v,11}$ at frequency α_1.

The x_1 coordinate velocity has spectral magnitude $M_{v,12}$ and phase $\phi_{v,12}$ at frequency α_2.

The phase angle is the arctan of the sine amplitude divided by the cosine amplitude.

Similarly, for coordinate x_2,

$$\dot{\eta}_1 = \alpha_1 N_{11}\cos(\alpha_1 t) - \alpha_1 P_{11}\sin(\alpha_1 t) + \alpha_2 N_{12}\cos(\alpha_2 t) - \alpha_2 P_{12}\sin(\alpha_2 t) \tag{110}$$

$$\dot{\eta}_2 = \alpha_1 N_{21}\cos(\alpha_1 t) - \alpha_1 P_{21}\sin(\alpha_1 t) + \alpha_2 N_{22}\cos(\alpha_2 t) - \alpha_2 P_{22}\sin(\alpha_2 t) \tag{111}$$

$$\begin{aligned} \dot{x}_2(t) = {} & \hat{v}_2\{\alpha_1 N_{11}\cos(\alpha_1 t) - \alpha_1 P_{11}\sin(\alpha_1 t) + \alpha_2 N_{12}\cos(\alpha_2 t) - \alpha_2 P_{12}\sin(\alpha_2 t)\} \\ & + \hat{w}_2\{\alpha_1 N_{21}\cos(\alpha_1 t) - \alpha_1 P_{21}\sin(\alpha_1 t) + \alpha_2 N_{22}\cos(\alpha_2 t) - \alpha_2 P_{22}\sin(\alpha_2 t)\} \end{aligned} \tag{112}$$

$$\begin{aligned} \dot{x}_2(t) = {} & +\alpha_1[\hat{v}_2 N_{11} + \hat{w}_2 N_{21}]\cos(\alpha_1 t) \\ & - \alpha_1[\hat{v}_2 P_{11} + \hat{w}_2 P_{21}]\sin(\alpha_1 t) \\ & + \alpha_2[\hat{v}_2 N_{12} + \hat{w}_2 N_{22}]\cos(\alpha_2 t) \\ & - \alpha_2[\hat{v}_2 P_{12} + \hat{w}_2 P_{22}]\sin(\alpha_2 t) \end{aligned} \tag{113}$$

Define spectral magnitude and phase components for coordinate x_1 velocity as follows:

$$M_{v,21} = \alpha_1\sqrt{\{\hat{v}_2 N_{11} + \hat{w}_2 N_{21}\}^2 + \{\hat{v}_2 P_{11} + \hat{w}_2 P_{21}\}^2} \tag{114}$$

$$M_{v,22} = \alpha_2\sqrt{\{\hat{v}_2 N_{12} + \hat{w}_2 N_{22}\}^2 + \{\hat{v}_2 P_{12} + \hat{w}_2 P_{22}\}^2} \tag{115}$$

$$\phi_{v,21} = \arctan\left(\frac{-\hat{v}_2 P_{11} - \hat{w}_2 P_{21}}{+\hat{v}_2 N_{11} + \hat{w}_2 N_{21}}\right) \tag{116}$$

$$\phi_{v,22} = \arctan\left(\frac{-\hat{v}_2 P_{12} - \hat{w}_2 P_{22}}{+\hat{v}_2 N_{12} + \hat{w}_2 N_{22}}\right) \tag{117}$$

The x_2 coordinate velocity has spectral magnitude $M_{v,21}$ and phase $\phi_{v,21}$ at frequency α_1.

The x_2 coordinate velocity has spectral magnitude $M_{v,22}$ and phase $\phi_{v,22}$ at frequency α_2.

The phase angle is the arctan of the sine amplitude divided by the cosine amplitude.

Acceleration Response to Applied Forces

For coordinate x_1,

$$\ddot{\eta}_1 = -\alpha_1^2 N_{11}\sin(\alpha_1 t) - \alpha_1^2 P_{11}\cos(\alpha_1 t) - \alpha_2^2 N_{12}\sin(\alpha_2 t) - \alpha_2^2 P_{12}\cos(\alpha_2 t) \quad (118)$$

$$\ddot{\eta}_2 = -\alpha_1^2 N_{21}\sin(\alpha_1 t) - \alpha_1^2 P_{21}\cos(\alpha_1 t) - \alpha_2^2 N_{22}\sin(\alpha_2 t) - \alpha_2^2 P_{22}\cos(\alpha_2 t) \quad (119)$$

$$\begin{aligned}\ddot{x}_1(t) = &-\alpha_1^2\left[\hat{v}_1 N_{11} + \hat{w}_1 N_{21}\right]\sin(\alpha_1 t)\\ &-\alpha_1^2\left[\hat{v}_1 P_{11} + \hat{w}_1 P_{21}\right]\cos(\alpha_1 t)\\ &-\alpha_2^2\left[\hat{v}_1 N_{12} + \hat{w}_1 N_{22}\right]\sin(\alpha_2 t)\\ &-\alpha_2^2\left[\hat{v}_1 P_{12} + \hat{w}_1 P_{22}\right]\cos(\alpha_2 t)\end{aligned} \quad (120)$$

Define spectral magnitude and phase components for coordinate x_1 acceleration as follows:

$$M_{a,11} = \alpha_1^2\sqrt{\left\{\hat{v}_1 N_{11} + \hat{w}_1 N_{21}\right\}^2 + \left\{\hat{v}_1 P_{11} + \hat{w}_1 P_{21}\right\}^2} \quad (121)$$

$$M_{a,12} = \alpha_2^2\sqrt{\left\{\hat{v}_1 N_{12} + \hat{w}_1 N_{22}\right\}^2 + \left\{\hat{v}_1 P_{12} + \hat{w}_1 P_{22}\right\}^2} \quad (122)$$

$$\phi_{a,11} = \arctan\left(\frac{-\hat{v}_1 N_{11} - \hat{w}_1 N_{21}}{-\hat{v}_1 P_{11} - \hat{w}_1 P_{21}}\right) \quad (123)$$

$$\phi_{a,12} = \arctan\left(\frac{-\hat{v}_1 N_{12} - \hat{w}_1 N_{22}}{-\hat{v}_1 P_{12} - \hat{w}_1 P_{22}}\right) \quad (124)$$

The x_1 coordinate velocity has spectral magnitude $M_{a,11}$ and phase $\phi_{a,11}$ at frequency α_1.

The x_1 coordinate velocity has spectral magnitude $M_{a,12}$ and phase $\phi_{a,12}$ at frequency α_2.

The phase angle is the arctan of the sine amplitude divided by the cosine amplitude.

Similarly, for coordinate x_2,

$$\ddot{\eta}_1 = -\alpha_1^2 N_{11} \sin(\alpha_1 t) - \alpha_1^2 P_{11} \cos(\alpha_1 t) - \alpha_2^2 N_{12} \sin(\alpha_2 t) - \alpha_2^2 P_{12} \cos(\alpha_2 t) \tag{125}$$

$$\ddot{\eta}_2 = -\alpha_1^2 N_{21} \sin(\alpha_1 t) - \alpha_1^2 P_{21} \cos(\alpha_1 t) - \alpha_2^2 N_{22} \sin(\alpha_2 t) - \alpha_2^2 P_{22} \cos(\alpha_2 t) \tag{126}$$

$$\begin{aligned} \ddot{x}_2(t) = & -\alpha_1^2 [\hat{v}_2 N_{11} + \hat{w}_2 N_{21}] \sin(\alpha_1 t) \\ & -\alpha_1^2 [\hat{v}_2 P_{11} + \hat{w}_2 P_{21}] \cos(\alpha_1 t) \\ & -\alpha_2^2 [\hat{v}_2 N_{12} + \hat{w}_2 N_{22}] \sin(\alpha_2 t) \\ & -\alpha_2^2 [\hat{v}_2 P_{12} + \hat{w}_2 P_{22}] \cos(\alpha_2 t) \end{aligned} \tag{127}$$

Define spectral magnitude and phase components for coordinate x_1 acceleration as follows:

$$M_{a,21} = \alpha_1^2 \sqrt{\{\hat{v}_1 N_{11} + \hat{w}_1 N_{21}\}^2 + \{\hat{v}_1 P_{11} + \hat{w}_1 P_{21}\}^2} \tag{128}$$

$$M_{a,22} = \alpha_2^2 \sqrt{\{\hat{v}_1 N_{12} + \hat{w}_1 N_{22}\}^2 + \{\hat{v}_1 P_{12} + \hat{w}_1 P_{22}\}^2} \tag{129}$$

$$\phi_{a,21} = \arctan\left(\frac{-\hat{v}_1 N_{11} - \hat{w}_1 N_{21}}{-\hat{v}_1 P_{11} - \hat{w}_1 P_{21}} \right) \tag{130}$$

$$\phi_{a,22} = \arctan\left(\frac{-\hat{v}_1 N_{12} - \hat{w}_1 N_{22}}{-\hat{v}_1 P_{12} - \hat{w}_1 P_{22}} \right) \tag{131}$$

The x_2 coordinate velocity has spectral magnitude $M_{a,21}$ and phase $\phi_{a,21}$ at frequency α_1.

The x_2 coordinate velocity has spectral magnitude $M_{a,22}$ and phase $\phi_{a,22}$ at frequency α_2.

The phase angle is the arctan of the sine amplitude divided by the cosine amplitude.

References

1. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Revision D, Vibrationdata, 2010.

2. T. Irvine, An Introduction to Spectral Functions, Revision B, Vibrationdata, 2000.

3. T. Irvine, Calculating Transfer Functions from Normal Modes, Vibrationdata, 2010.

APPENDIX A

Example

Consider the system in Figure 1 with the values in Table 2.

Assume 5% damping for each mode. Assume zero initial conditions.

Table A-1. Parameters		
Variable	Value	Unit
m_1	3.0	lbf sec^2/in
m_2	2.0	lbf sec^2/in
k_1	400,000	lbf/in
k_2	300,000	lbf/in
k_3	100,000	lbf/in

Table A-2. Forces			
Force	Amp (lbf)	Freq (Hz)	Phase (deg)
1	100	55	30
2	200	100	60

The mass matrix is

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{(A-1)}$$

The stiffness matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 700{,}000 & -300{,}000 \\ -300{,}000 & 400{,}000 \end{bmatrix} \quad \text{(A-2)}$$

The analysis is performed using a Matlab script.

```
>> two_dof_force_frf_alt

    two_dof_force_frf_alt.m   ver 1.5  June 11, 2010

    by Tom Irvine   Email: tomirvine@aol.com

 This program finds the frequency response of a
 two-degree-of-freedom system subjected to applied forces.
 The equation of motion is:   M (d^2x/dt^2) + K x = F(t)

 Assume symmetric mass and stiffness matrices.

 Select units: 1=English  2=metric
 1
 Select mass units: 1=lbm   2=lbf sec^2/in
 2
 Enter m11 (lbf sec^2/in)
 3
 Enter m22 (lbf sec^2/in)
 2

 Enter k11 (lbf/in)
 700000
 Enter k12 (lbf/in)
 -300000
 Enter k22 (lbf/in)
 400000
 Enter the force applied to dof 1 (lbf)
 100
 Enter the frequency (Hz) of dof 1 force
 55
 Enter the phase (deg) of dof 1 force
 30
 Enter the force applied to dof 2 (lbf)
 200
 Enter the frequency (Hz) of dof 2 force
 100
 Enter the phase (deg) of dof 2 force
 60

 Enter modal damping ratio 1
 0.05
 Enter modal damping ratio 2
 0.05
```

```
 The mass matrix is

m =

     3     0
     0     2


 The stiffness matrix is

k =

      700000     -300000
     -300000      400000

 Natural Frequencies
 No.       f(Hz)
1.        48.552
2.        92.839

  Modes Shapes (column format)

ModeShapes =

    0.3797   -0.4349
    0.5326    0.4651


 Phase angle convention:  arctan( sine amplitude/cosine amplitude)

Displacement Results

 Response  Excitation    Freq    Magnitude    Phase
    dof       dof         (Hz)     (inch)      (deg)
     1         1        55.00     4.337e-004    213.1
     1         2       100.00     5.048e-004    348.4
     2         1        55.00     7.948e-004    221.2
     2         2       100.00     8.260e-004    182.7

 dof   RMS(inch)  Peak Magnitude(inch)
   1   0.0004706    0.0009385
   2   0.0008106    0.001621

Velocity Results

 Response  Excitation    Freq    Magnitude    Phase
    dof       dof         (Hz)    (in/sec)     (deg)
     1         1        55.00     1.499e-001    123.1
     1         2       100.00     3.172e-001    258.4
     2         1        55.00     2.747e-001    131.2
     2         2       100.00     5.190e-001     92.7
```

dof	RMS(in/sec)	Peak Magnitude(in/sec)
1	0.248	0.467
2	0.4152	0.7937

Acceleration Results

Response dof	Excitation dof	Freq (Hz)	Magnitude (in/sec^2)	Phase (deg)
1	1	55.00	5.179e+001	33.1
1	2	100.00	1.993e+002	168.4
2	1	55.00	9.492e+001	41.2
2	2	100.00	3.261e+002	2.7

dof	RMS(in/sec^2)	Peak Magnitude(in/sec^2)
1	145.6	251.1
2	240.2	421

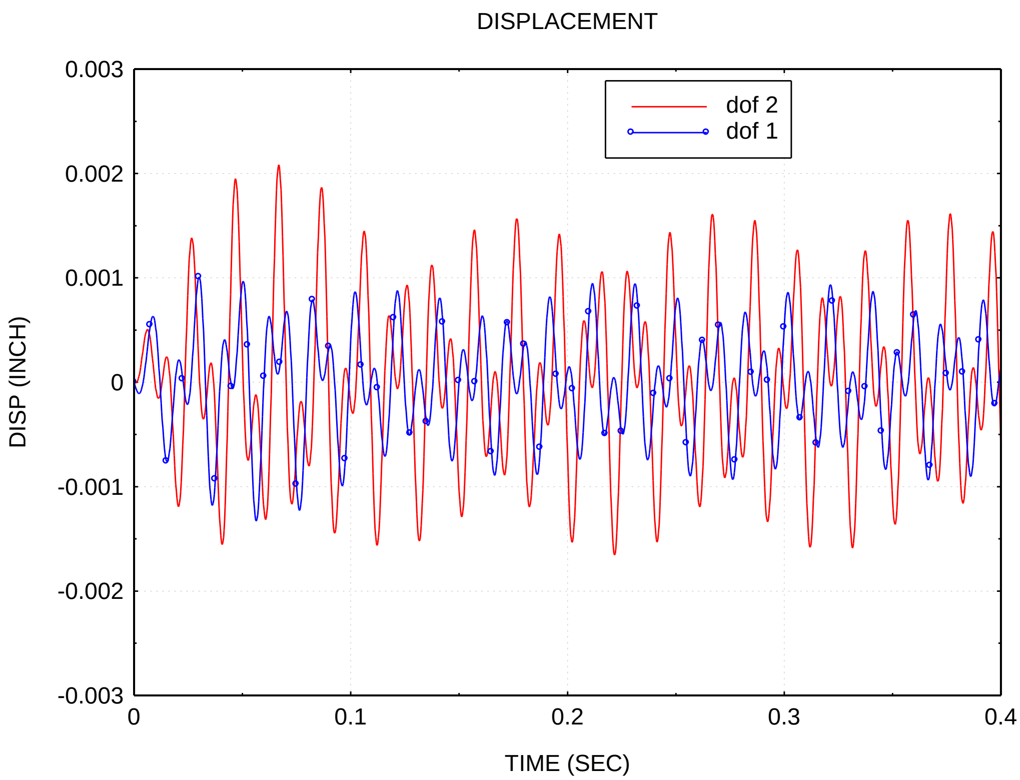

Figure A-1.
Time History Calculated in Reference 1, via Matlab script: twodof_sine_force.m

The peak magnitude values from the previously given frequency response function results agree with the steady-state portion of the corresponding modal transient analysis from Reference 1, as shown in Figure A-1.

The steady-state portion begins approximately at 0.1 seconds.

Again, the peak values from the steady-state frequency response function analysis are

```
 dof    Peak Magnitude(inch)
  1        0.0009385
  2        0.001621
```

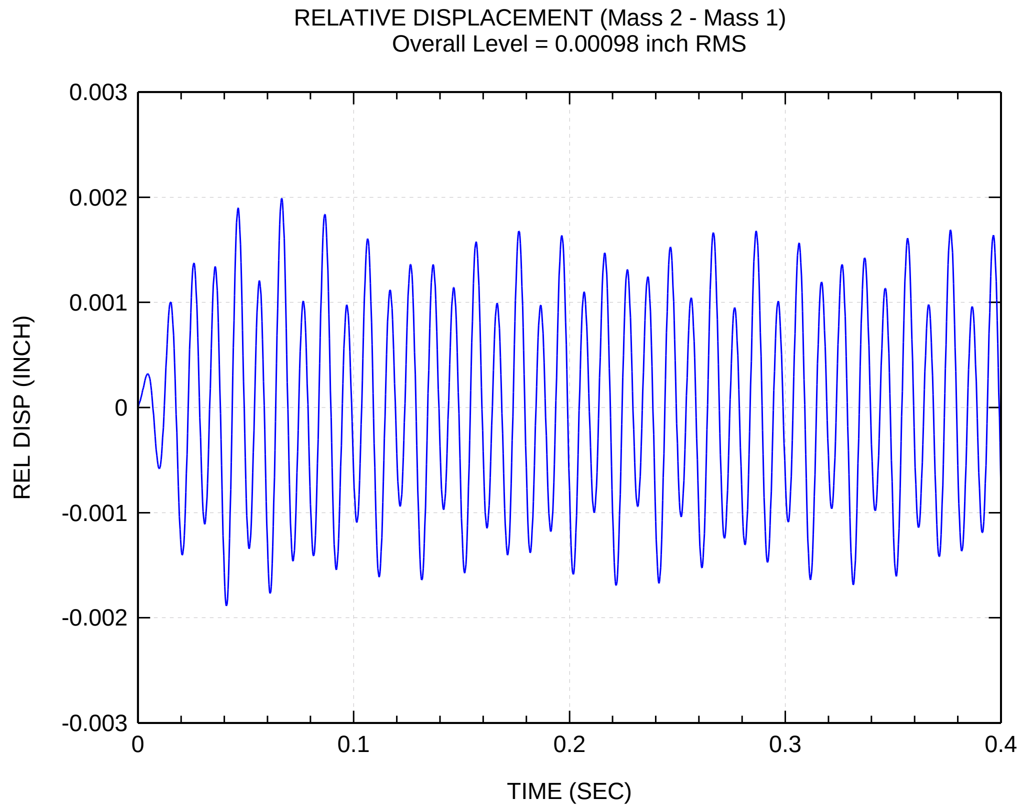

Figure A-2. Relative Displacement via Matlab script: twodof_sine_force.m

The absolute peak after 0.1 seconds is 0.00169 inch.

APPENDIX B

Transfer Functions from Normal Modes

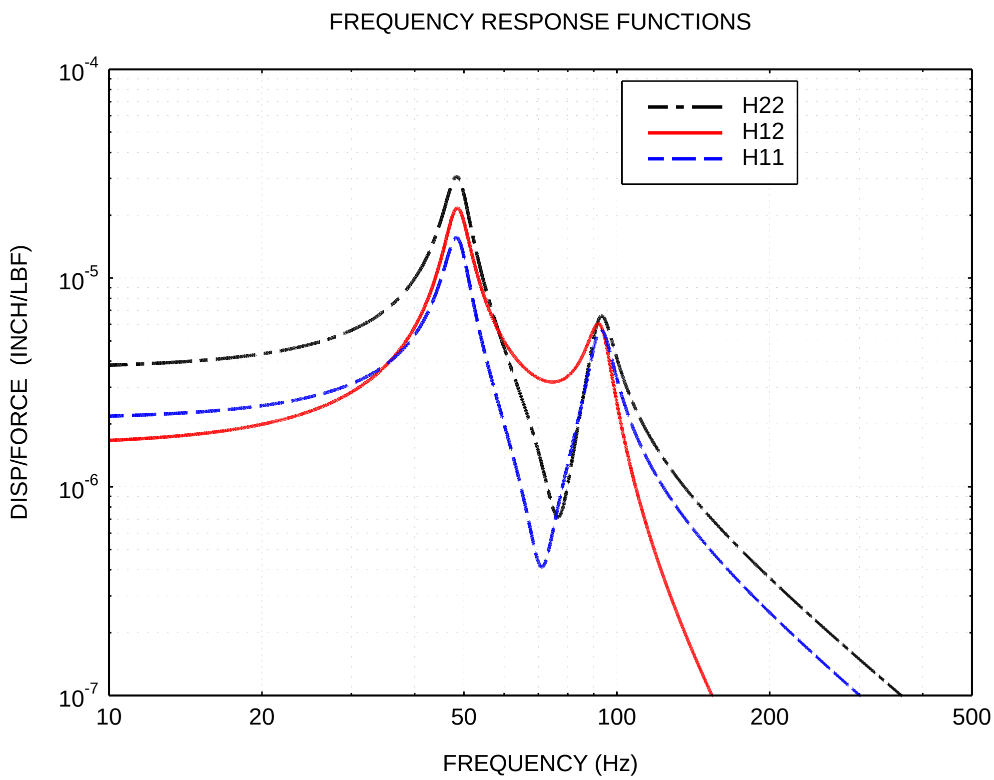

Figure B-1.

The corresponding frequency response functions in Figure B-1 are calculated from the natural frequencies, modal damping ratios and mode shapes via Reference 3.

The Matlab script was: transfer_from_modes.m

Consider the response at coordinate 1.

Function	Hz	in/lbf	Lbf	Inch
H11	55	4.33E-06	100	0.00043
H12	100	2.52E-06	200	0.00050
			Total	0.00093

Consider the response at coordinate 2.

Function	Hz	in/lbf	Lbf	Inch
H21	55	7.95E-06	100	0.00079
H22	100	4.13E-06	200	0.00083
			Total	0.00162

The results in the above tables agree with the results in Appendix A.

The following relative displacement responses were obtained via transfer functions using Reference 3. The results agree with Figure A-2.

```
>> transfer_from_modes_rd

 transfer_from_modes_rd.m   ver 1.0   September 20, 2010
 by Tom Irvine

  Natural Frequencies (Hz)
     48.55
     92.84

  Modes Shapes (column format)

QE =

   -0.3797   -0.4349
   -0.5326    0.4651

  RDfreq1 = 0.0003706 inch
  RDfreq2 = 0.001321 inch

    Peak Level = 0.001692 inch
 Overall Level = 0.0009702 inch
```