MULTI-DEGREE-OF-FREEDOM SYSTEM
SUBJECTED TO AN ARBITRARY FORCE
Revision A

By Tom Irvine
Email: tomirvine@aol.com

January 20, 2010

Two-degree-of-freedom System

A two-degree-of-freedom system is used an example, as shown in Figure 1.

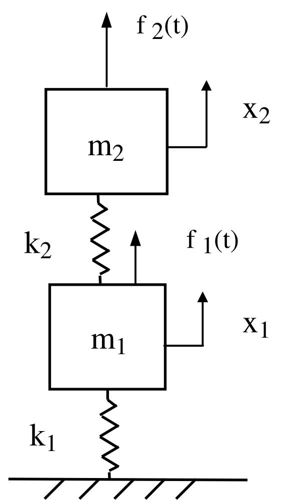

Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

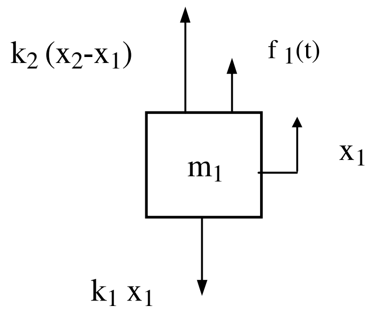

Figure 2.

Determine the equation of motion for mass 1.

$$\Sigma F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1 x_1 \tag{2}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = f_1(t) \tag{3}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = f_1(t) \tag{4}$$

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t) \tag{5}$$

m_2

$k_2\,(x_2\text{-}x_1)$

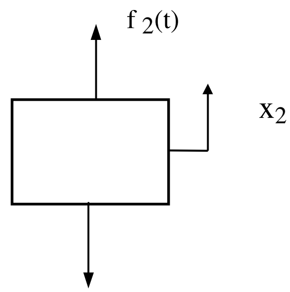

Figure 3.

Derive the equation of motion for mass 2.

$$\Sigma F = m_2 \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = f_2(t) - k_2(x_2 - x_1) \tag{7}$$

$$m_2 \ddot{x}_2 + k_2(x_2 - x_1) = f_2(t) \tag{8}$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = f_2(t) \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{10}$$

Decoupling

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M \overline{\ddot{x}} + K \overline{x} = \overline{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{13}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\,\ddot{\bar{x}} + K\,\bar{x} = \bar{0} \tag{16}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(\,j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega\,\bar{q}\exp(\,j\omega t) \tag{18}$$

$$\ddot{\bar{x}} = -\omega^2\;\bar{q}\exp(\,j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M\;\bar{q}\exp(\,j\omega t) + K\,\bar{q}\exp(\,j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M\;\bar{q} + K\,\bar{q}\right\}\exp(\,j\omega t) = \bar{0} \tag{21}$$

$$-\omega_n{}^2 M\;\bar{q} + K\,\bar{q} = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M\; + K\right\}\bar{q} = \bar{0} \tag{23}$$

$$\left\{ K - \omega^2 M \right\} \bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det \left\{ K - \omega^2 M \right\} = 0 \tag{25}$$

$$\det \left\{ \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} = 0 \tag{26}$$

$$\det \begin{bmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} = 0 \tag{27}$$

$$\left[(k_1 + k_2) - \omega^2 m_1 \right] \left[k_2 - \omega^2 m_2 \right] - k_2{}^2 = 0 \tag{28}$$

$$\omega^4 m_1 m_2 - \omega^2 [m_1 k_2 + m_2 (k_1 + k_2)] - k_2{}^2 = 0 \tag{29}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1{}^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{30}$$

$$\omega_2{}^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{31}$$

where

$$a = m_1 m_2 \tag{32}$$

$$b = -[m_1 k_2 + m_2 (k_1 + k_2)] \tag{33}$$

$$c = -k_2{}^2 \tag{34}$$

The eigenvectors are found via the following equations.

$$\left\{ K - \omega_1{}^2 M \right\} \bar{q}_1 = \bar{0} \tag{35}$$

$$\left\{ K - \omega_2{}^2 M \right\} \bar{q}_2 = \bar{0} \tag{36}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{37}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{38}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\, \bar{q}_1 \quad | \quad \bar{q}_2 \,] \tag{39}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{40}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{41}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{42}$$

where

T	Superscript, represents transpose
I	is the identity matrix
Ω	is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{43a}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{43b}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\ \bar{\eta} \tag{44}$$

Substitute equation (43) into the equation of motion, equation (11).

$$M\hat{Q}\ \bar{\ddot{\eta}} + K\hat{Q}\ \bar{\eta} = \bar{F} \tag{45}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T\ M\hat{Q}\ \bar{\ddot{\eta}}\ +\ \hat{Q}^T K\hat{Q}\ \bar{\eta} =\ \hat{Q}^T\bar{F} \tag{46}$$

The orthogonality relationships yield

$$I\ \bar{\ddot{\eta}}\ +\ \Omega\ \bar{\eta} =\ \hat{Q}^T\bar{F} \tag{47}$$

For the sample problem, equation (46) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{48}$$

Note that the two equations are decoupled in terms of the generalized coordinate.

Add modal damping.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1{}^2 & 0 \\ 0 & \omega_2{}^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{49}$$

Equation (49) yields two equations

$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1{}^2\,\eta_1 = \hat{v}_1\,f_1(t) + \hat{v}_2\,f_2(t) \tag{50}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2{}^2\,\eta_2 = \hat{w}_1\,f_1(t) + \hat{w}_2\,f_2(t) \tag{51}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{x} = \hat{Q}\ \bar{\eta} \tag{52}$$

Thus,

$$\bar{x}(0) = \hat{Q}\ \bar{\eta}(0) \tag{53}$$

Premultiply by $\hat{Q}^T\ M$.

$$\hat{Q}^T\ M\,\bar{x}(0) = \hat{Q}^T\ M\,\hat{Q}\,\eta(0) \tag{54}$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \tag{55}$$

$$\hat{Q}^T\ M\,\bar{x}(0) = I\ \eta(0) \tag{56}$$

$$\hat{Q}^T\ M\,\bar{x}(0) = \eta(0) \tag{57}$$

Finally, the transformed initial displacement is

$$\eta(0) = \hat{Q}^T\ M\,\bar{x}(0) \tag{58}$$

Similarly, the transformed initial velocity is

$$\dot{\eta}(0) = \hat{Q}^T \; M \, \bar{\dot{x}}(0) \tag{59}$$

A basis for a solution is thus derived.

Arbitrary Force

Now consider the case of arbitrary forcing functions.

$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1{}^2 \, \eta_1 = \hat{v}_1 \, f_1(t) + \hat{v}_2 \, f_2(t) \tag{60}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2{}^2 \, \eta_2 = \hat{w}_1 \, f_1(t) + \hat{w}_2 \, f_2(t) \tag{61}$$

Apply the convolution integral per Reference 1.

$$\begin{aligned} \eta_1(t) = & -\frac{\hat{v}_1}{\omega_{d,1}} \int_0^t f_1(\tau) \left[\exp\left\{-\xi_1\omega_{n,1}(t-\tau)\right\}\right]\left[\sin\omega_{d,1}(t-\tau)\right] d\tau \\ & -\frac{\hat{v}_2}{\omega_{d,1}} \int_0^t f_2(\tau) \left[\exp\left\{-\xi_1\omega_{n,1}(t-\tau)\right\}\right]\left[\sin\omega_{d,1}(t-\tau)\right] d\tau \end{aligned} \tag{62}$$

$$\begin{aligned} \eta_2(t) = & -\frac{\hat{w}_1}{\omega_{d,2}} \int_0^t f_1(\tau) \left[\exp\left\{-\xi_2\omega_{n,2}(t-\tau)\right\}\right]\left[\sin\omega_{d,2}(t-\tau)\right] d\tau \\ & -\frac{\hat{w}_2}{\omega_{d,2}} \int_0^t f_2(\tau) \left[\exp\left\{-\xi_2\omega_{n,2}(t-\tau)\right\}\right]\left[\sin\omega_{d,2}(t-\tau)\right] d\tau \end{aligned} \tag{63}$$

$$\eta_{1,i} = +\left\{2\exp\left[-\xi_1\omega_{n,1}\Delta t\right]\cos\left[\omega_{d,1}\Delta t\right]\eta_{i-1}\right\}$$

$$-\left\{\exp\left[-2\xi_1\omega_{n,1}\Delta t\right]\eta_{i-2}\right\}$$

$$+\left\{\hat{v}_1\, f_{1,i-1} + \hat{v}_2\, f_{2,i-1}\right\}\left\{\Delta t/\omega_{d,1}\right\}\exp\left[-\xi_1\omega_{n,1}\Delta t\right]\left\{\sin\left[\omega_{d,1}\Delta t\right]\right\}$$

(64)

$$\eta_{2,i} = +\left\{2\exp\left[-\xi_2\omega_{n,2}\Delta t\right]\cos\left[\omega_{d,2}\Delta t\right]\eta_{i-1}\right\}$$

$$-\left\{\exp\left[-2\xi_2\omega_{n,2}\Delta t\right]\eta_{i-2}\right\}$$

$$+\left\{\hat{v}_1\, f_{1,i-1} + \hat{v}_2\, f_{2,i-1}\right\}\left\{\Delta t/\omega_{d,2}\right\}\exp\left[-\xi_2\omega_{n,2}\Delta t\right]\left\{\sin\left[\omega_{d,2}\Delta t\right]\right\}$$

(65)

The modal velocity is

$$\dot{\eta}_{1,i} = +\left\{2\exp\left[-\xi_1\omega_{n,1}\Delta t\right]\cos\left[\omega_{d,1}\Delta t\right]\dot{\eta}_{i-1}\right\}$$

$$-\left\{\exp\left[-2\xi_1\omega_{n,1}\Delta t\right]\dot{\eta}_{i-2}\right\}$$

$$+\Delta t\left\{\hat{v}_1 f_{1,i} + \hat{v}_2\, f_{2,i}\right\}$$

$$+\left\{\hat{v}_1\, f_{1,i-1} + \hat{v}_2\, f_{2,i-1}\right\}\left\{\Delta t\right\}\exp\left[-\xi_1\omega_{n,1}\Delta t\right]\left\{\left[\frac{-\xi_1\omega_{n,1}}{\omega_{d,1}}\right]\sin\left[\omega_{d,1}\Delta t\right]-\cos\left[\omega_{d,1}\Delta t\right]\right\}$$

(66)

$$\dot{\eta}_{2,i} = +\left\{2\exp\left[-\xi_2\omega_{n,2}\Delta t\right]\cos\left[\omega_{d,2}\Delta t\right]\dot{\eta}_{i-1}\right\}$$

$$-\left\{\exp\left[-2\xi_2\omega_{n,2}\Delta t\right]\dot{\eta}_{i-2}\right\}$$

$$+\Delta t\left\{\hat{w}_1 f_{1,i} + \hat{w}_2 f_{2,i}\right\}$$

$$+\left\{\hat{w}_1 f_{1,i-1} + \hat{w}_2 f_{2,i-1}\right\}\left\{\Delta t\right\}\exp\left[-\xi_1\omega_{n,1}\Delta t\right]\left\{\left[\frac{-\xi_2\omega_{n,2}}{\omega_{d,2}}\right]\sin\left[\omega_{d,2}\Delta t\right]-\cos\left[\omega_{d,2}\Delta t\right]\right\}$$

(67)

The acceleration values can then be found via

$$\ddot{\eta}_1 = -2\xi_1\omega_1\dot{\eta}_1 - \omega_1{}^2\,\eta_1 + \hat{v}_1\,f_1(t) + \hat{v}_2\,f_2(t) \tag{68}$$

$$\ddot{\eta}_2 = -2\xi_2\omega_2\dot{\eta}_2 - \omega_2{}^2\,\eta_2 + \hat{w}_1\,f_1(t) + \hat{w}_2\,f_2(t) \tag{69}$$

The physical displacements are then found as

$$\bar{x} = \hat{Q}\;\bar{\eta} \tag{70}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{71}$$

The velocity is

$$\bar{\dot{x}} = \hat{Q}\;\bar{\dot{\eta}} \tag{72}$$

The acceleration is

$$\bar{\ddot{x}} = \hat{Q}\;\bar{\ddot{\eta}} \tag{73}$$

<u>Reference</u>

1. T. Irvine, An Introduction to the Shock Response Spectrum Rev P, Vibrationdata, 2002.

APPENDIX A

Example

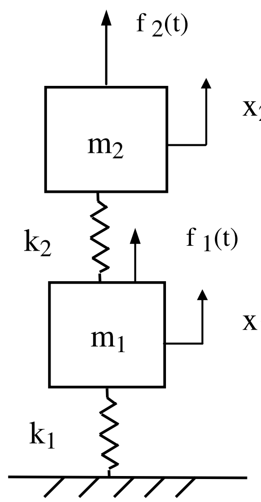

Figure A-1.

Let

1. Each mass = 1 lbm
2. Each spring = 2500 lbf/in
3. Each modal damping = 0.05
4. f1 = 0
5. f2 = 1 lbf, 0.010 sec, half-sine pulse

The response is calculated using the formulas derived in the main text as implemented via Matlab script: mdof_arb_force.m

The response time histories are shown in the following figures.

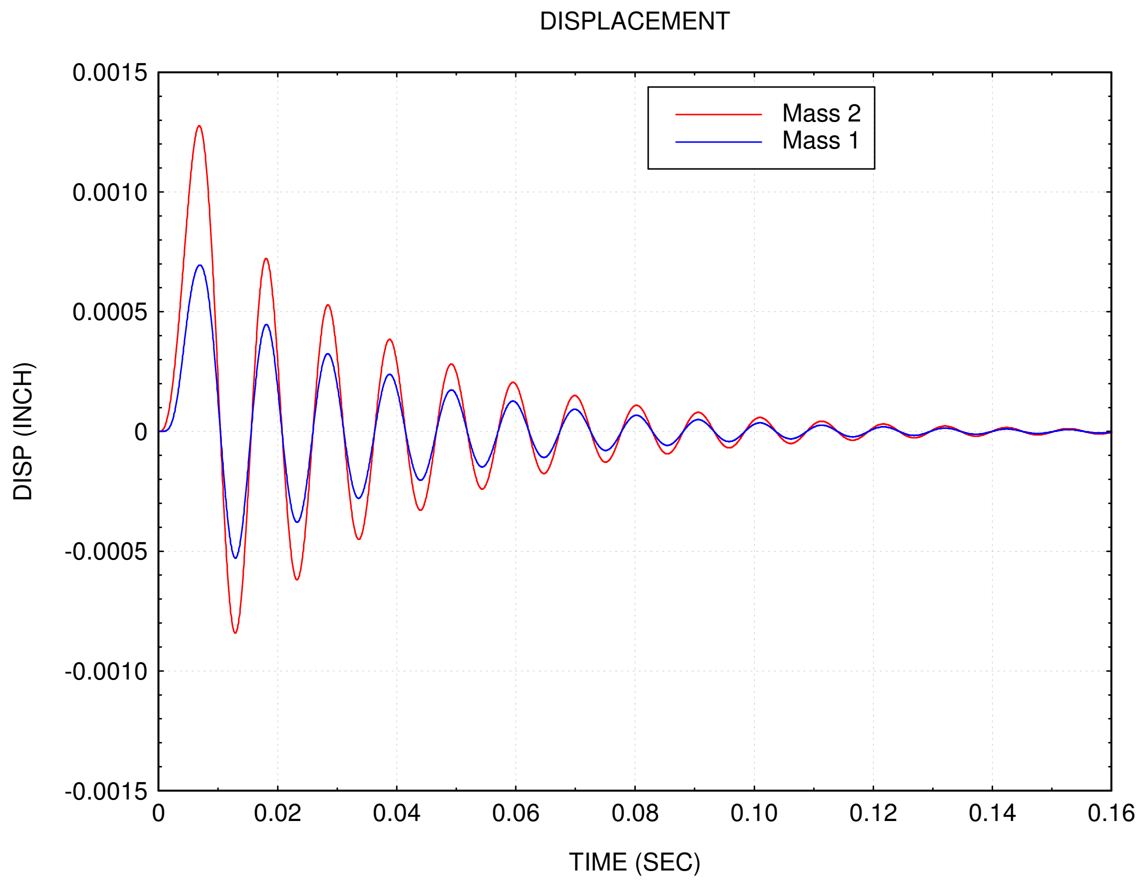

Figure A-2.

Natural Frequencies =

96.63 Hz
253 Hz

Modes Shapes (column format) =

-10.33	-16.71
-16.71	10.33

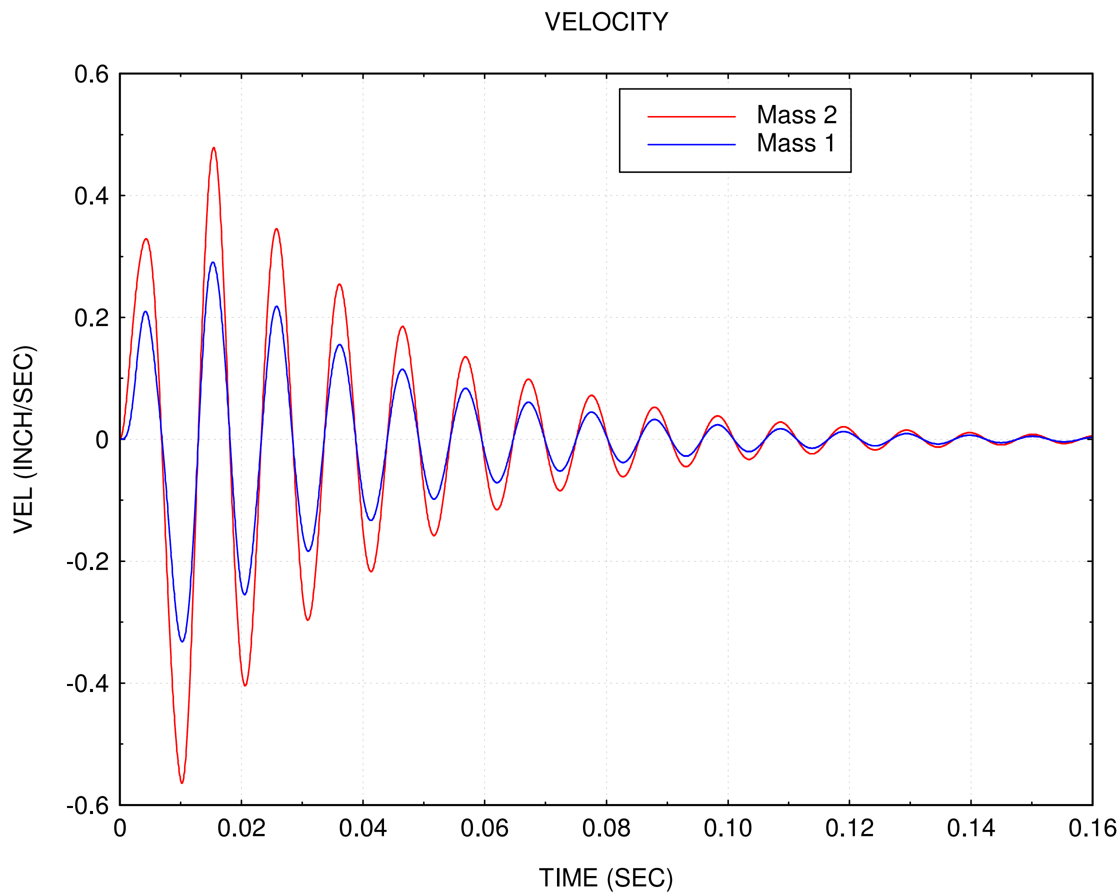
VELOCITY
Mass 2
Mass 1
VEL (INCH/SEC)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
TIME (SEC)

Figure A-3.

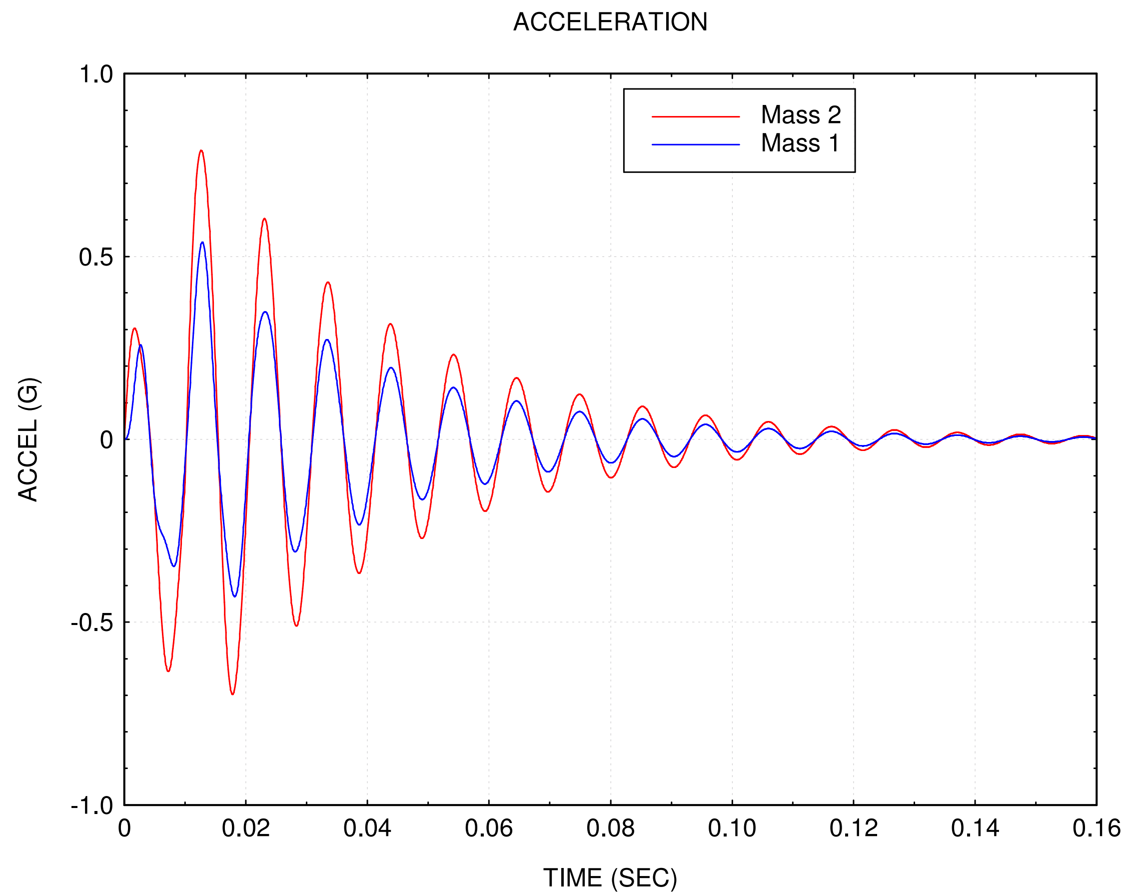
ACCELERATION
Mass 2
Mass 1
ACCEL (G)
1.0
0.5
0
-0.5
-1.0
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
TIME (SEC)

Figure A-4.