SHOCK RESPONSE OF MULTI-DEGREE-OF-FREEDOM SYSTEMS   Revision F

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Introduction

The primary purpose of this tutorial is to present the Modal Transient method for calculating the response of a multi-degree-of-freedom system to an arbitrary base input.

A secondary purpose is to compare the results of this method to simplified methods for multi-degree-of-freedom shock response.

Shock Response Spectrum

The shock response spectrum is inherently a single-degree-of-freedom concept, as discussed in Reference 1. Nevertheless, there is an occasional need to determine the response of a multi-degree-of-freedom system to shock response spectrum input for "engineering purposes."

The modal transient method is the formal method for carrying out this calculation, but it may require intensive calculations depending on the complexity of the model and the base input time history.

Over the years, a number of simplified modal combination methods have been proposed for estimating the multi-degree-of-freedom response. Each requires the eigenvalues, eigenvectors, and the modal participation factors of the system, or at least estimates of these parameters.

Equation of Motion

The governing equation of motion for the displacement η_i of mode i within a multi-degree-of-freedom system subjected to base excitation is

$$\ddot{\eta}_i + 2\xi_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = -\Gamma_i \ddot{y} \tag{1}$$

Γ_i is the modal participation factor for mode i , as given in Appendix C

$\ddot{y}$ is the base excitation acceleration

ω_i is the natural frequency

ξ_i is the viscous damping ratio

The response of a physical coordinate in a multiple-degree-of-freedom system can be approximated as the summation response of a series of single-degree-of-freedom systems.

The method assumes that the modal frequencies are well-separated.
The corresponding single-degree-of-freedom response for the relative displacement D_j is

$$\ddot{D}_j + 2\xi_i \omega_i \dot{D}_j + \omega_i^2 D_j = -\ddot{y} \tag{2}$$

The subscript j denotes the connection of D with a particular mode.

Thus,

$$\eta_j(t) = \Gamma_j D_j(t) \tag{3}$$

The advantage of this approach is that D_j can be calculated with relative ease, thereby providing an indirect solution for η_j. On the other hand, a direct calculation of η_j requires intensive calculations.

<u>Absolute Sum (ABSSUM)</u>

The absolute sum method is a conservative approach because it assumes that the maxima of all modes appear at the same instant of time.

The maximum relative displacement $(z_i)_{max}$ is calculated from the modal coordinates as

$$(z_i)_{max} \leq \sum_{j=1}^{N} \left|\hat{q}_{ij}\right| \left|\eta_{j\,max}\right| \tag{4}$$

where $\hat{q}_{ij}$ is the mass-normalized eigenvector coefficient for coordinate i and mode j.

The corresponding ABSSUM method is

$$(z_i)_{max} \leq \sum_{j=1}^{N} \left|\Gamma_j\right| \left|\hat{q}_{ij}\right| \left|D_{j,max}\right| \tag{5}$$

Equation (5) is taken from Reference 2, equation (7.53). Again, the maximum relative displacement D_j is taken as the single-degree-of-freedom response.

Square Root of the Sum of the Squares (SRSS)

The square root of the sum of the squares equation is

$$(z_i)_{max} \approx \sqrt{\sum_{j=1}^{N} \left[\hat{q}_{ij} \ \eta_{j,\, max} \right]^2} \tag{6}$$

The SRSS method with the single-degree-of-freedom response D_j is

$$(z_i)_{max} \approx \sqrt{\sum_{j=1}^{N} \left[\Gamma_j \ \hat{q}_{ij} \ D_{j,max} \right]^2} \tag{7}$$

Naval Research Laboratories (NRL) Method

The maximum relative displacement is calculated from the modal coordinates as

$$(z_i)_{max} = \left| \hat{q}_{ij} \eta_{j,\, max} \right| + \sum_{k=1,\ k \neq j}^{N} \left| (\hat{q}_{ik})(\eta_{k,\, max}) \right| \tag{8}$$

where the j-th mode is the mode that has the largest magnitude of the product $\left| \hat{q}_{ij} \eta_{j\, max} \right|$.

Note that the NRL method is the same as the ABSSUM method for N=2.

The simplified formula is

$$(z_i)_{max} = \left| (\hat{q}_{ij})(\Gamma_j \ D_j)_{,\, max} \right| + \sum_{k=1,\ k \neq j}^{N} \left| (\hat{q}_{ik})(\Gamma_k \ D_{k\, ,\, max}) \right| \tag{9}$$

Acceleration Response

Equations (6) through (7) can be extended for the absolute acceleration response, by making the appropriate substitutions.

References

1. T. Irvine, An Introduction to the Shock Response Spectrum Revision P Vibrationdata, 2002.
2. R. Kelly and G. Richman, Principles and Techniques of Shock Data Analysis, SVM-5; The Shock and Vibration Information Center, United States Department of Defense, Washington D.C., 1969.
3. A. Chopra, Dynamics of Structures, Prentice-Hall, New Jersey, 2001.
4. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.
5. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987.
6. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Revision A, Vibrationdata, 1999.

APPENDIX A

<u>Two-degree-of-freedom System, Modal Analysis</u>

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure A-1.

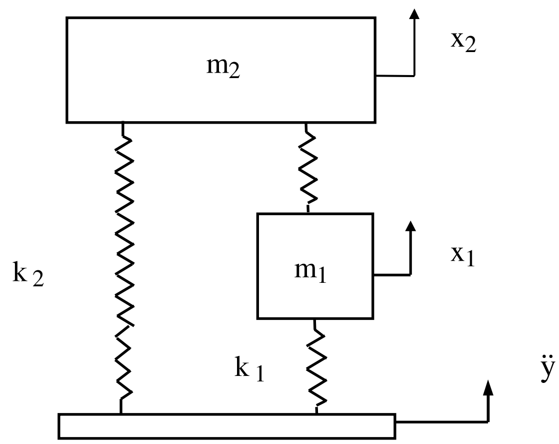

Figure A-1.

The system also has damping, but it is modeled as modal damping.

A free-body diagram of mass 1 is given in Figure A-2. A free-body diagram of mass 2 is given in Figure A-3.

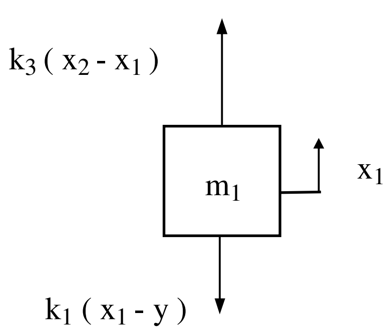

Figure A-2.

Determine the equation of motion for mass 1.

$$\Sigma F = m_1 \ddot{x}_1 \quad \text{(A-1)}$$

$$m_1 \ddot{x}_1 = k_3(x_2 - x_1) - k_1(x_1 - y) \quad \text{(A-2)}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_3(x_2 - x_1) = k_1 y \quad \text{(A-3)}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_3(x_1 - x_2) = k_1 y \quad \text{(A-4)}$$

$$m_1 \ddot{x}_1 + (k_1 + k_3)x_1 - k_3 x_2 = k_1 y \quad \text{(A-5)}$$

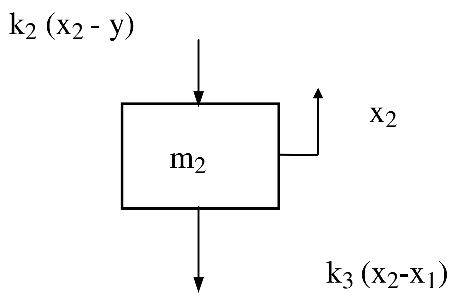

Figure A-3.

Derive the equation of motion for mass 2.

$$\Sigma F = m_2 \ddot{x}_2 \quad \text{(A-6)}$$

$$m_2 \ddot{x}_2 = -k_3(x_2 - x_1) - k_2(x_2 - y) \quad \text{(A-7)}$$

$$m_2 \ddot{x}_2 + k_2 x_2 + k_3(x_2 - x_1) = k_2 y \quad \text{(A-8)}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_3 x_1 = k_2 y \quad \text{(A-9)}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \quad \text{(A-10)}$$

Define a relative displacement z such that

$$x_1 = z_1 + y \quad \text{(A-11)}$$

$$x_2 = z_2 + y \quad \text{(A-12)}$$

Substitute equations (A-11) and (A-12) into (A-10).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 + \ddot{y} \\ \ddot{z}_2 + \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 + y \\ z_2 + y \end{bmatrix} = \begin{bmatrix} k_1\, y \\ k_2\, y \end{bmatrix} \quad \text{(A-13)}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1\ddot{y} \\ m_2\ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} k_1\, y \\ k_2\, y \end{bmatrix}$$

(A-14)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1\ddot{y} \\ m_2\ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} = \begin{bmatrix} k_1\, y \\ k_2\, y \end{bmatrix} \quad \text{(A-15)}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \quad \text{(A-16)}$$

Decoupling

Equation (A-16) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\,\ddot{\bar{z}} + K\,\bar{z} = \bar{F} \quad \text{(A-17)}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{A-18}$$

$$K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{A-19}$$

$$\bar{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{A-20}$$

$$\bar{F} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{A-21}$$

Consider the homogeneous form of equation (A-17).

$$M\,\ddot{\bar{z}} + K\,\bar{z} = \bar{0} \tag{A-22}$$

Seek a solution of the form

$$\bar{z} = \bar{q}\exp(j\omega t) \tag{A-23}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{z}} = j\omega\,\bar{q}\exp(j\omega t) \tag{A-24}$$

$$\ddot{\bar{z}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{A-25}$$

Substitute equations (A-23) through (A-25) into equation (A-22).

$$-\omega^2 M\,\bar{q}\exp(j\omega t) + K\,\bar{q}\exp(j\omega t) = \bar{0} \tag{A-26}$$

$$\left\{-\omega^2 M\,\bar{q} + K\,\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{A-27}$$

$$-\omega_n{}^2 M\,\bar{q} + K\,\bar{q} = \bar{0} \tag{A-28}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{A-29}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{A-30}$$

Equation (A-30) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{A-31}$$

$$\det\left\{\begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{A-32}$$

$$\det\begin{bmatrix} (k_1 + k_3) - \omega^2 m_1 & -k_3 \\ -k_3 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix} = 0 \tag{A-33}$$

$$\left[(k_1 + k_3) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_3{}^2 = 0 \tag{A-34}$$

$$\omega^4 m_1 m_2 - \omega^2\left[m_1(k_2 + k_3) + m_2(k_1 + k_3)\right] - k_3{}^2 = 0 \tag{A-35}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1{}^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{A-36}$$

$$\omega_2{}^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{A-37}$$

where

$$a = m_1 m_2 \tag{A-38}$$

$$b = -\left[m_1(k_2 + k_3) + m_2(k_1 + k_3)\right] \tag{A-39}$$

$$c = -k_3^2 \tag{A-40}$$

The eigenvectors are found via the following equations.

$$\left\{K - \omega_1^2 M\right\}\bar{q}_1 = \bar{0} \tag{A-41}$$

$$\left\{K - \omega_2^2 M\right\}\bar{q}_2 = \bar{0} \tag{A-42}$$

where

$$\bar{q}_1 = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \tag{A-34}$$

$$\bar{q}_2 = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \tag{A-44}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\ \bar{q}_1 \ \ | \ \ \bar{q}_2\ ] \tag{A-45}$$

$$Q = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \tag{A-46}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{A-47}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{A-48}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix} \quad \text{(A-49a)}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \quad \text{(A-49b)}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 5 and 6.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{z} = \hat{Q}\ \bar{\eta} \quad \text{(A-50a)}$$

$$z_1 = \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \quad \text{(A-50b)}$$

$$z_2 = \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \quad \text{(A-50c)}$$

Recall

$$x_1 = z_1 + y \quad \text{(A-51a)}$$

$$x_2 = z_2 + y \quad \text{(A-51b)}$$

The displacement terms are

$$x_1 = y + \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \quad \text{(A-51a)}$$

$$x_2 = y + \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \quad \text{(A-52b)}$$

The velocity terms are

$$\dot{x}_1 = \dot{y} + \hat{q}_{11}\,\dot{\eta}_1 + \hat{q}_{12}\,\dot{\eta}_2 \quad \text{(A-53a)}$$

$$\dot{x}_2 = \dot{y} + \hat{q}_{21}\,\dot{\eta}_1 + \hat{q}_{22}\,\dot{\eta}_2 \quad \text{(A-53b)}$$

The acceleration terms are

$$\ddot{x}_1 = \ddot{y} + \hat{q}_{11}\,\ddot{\eta}_1 + \hat{q}_{12}\,\ddot{\eta}_2 \tag{A-54a}$$

$$\ddot{x}_2 = \ddot{y} + \hat{q}_{21}\,\ddot{\eta}_1 + \hat{q}_{22}\,\ddot{\eta}_2 \tag{A-54b}$$

Substitute equation (50a) into the equation of motion, equation (A-17).

$$M\hat{Q}\,\ddot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = \bar{F} \tag{A-55}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T\bar{F} \tag{A-56}$$

The orthogonality relationships yield

$$I\,\ddot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T\bar{F} \tag{A-57}$$

For the sample problem, equation (A-57) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{A-58}$$

Note that the two equations are decoupled in terms of the modal coordinate.

Now assume modal damping by adding an uncoupled damping matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\,\omega_1 & 0 \\ 0 & 2\xi_2\,\omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix}$$

(A-59)

Now consider the initial conditions. Recall

$$\bar{z} = \hat{Q}\,\bar{\eta} \tag{A-60}$$

Thus,

$$\bar{z}(0) = \hat{Q}\,\bar{\eta}(0) \tag{A-61}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M \bar{z}(0) = \hat{Q}^T M \hat{Q} \eta(0) \tag{A-62}$$

Recall

$$\hat{Q}^T M \hat{Q} = I \tag{A-63}$$

$$\hat{Q}^T M \bar{z}(0) = I \ \eta(0) \tag{A-64}$$

$$\hat{Q}^T M \bar{z}(0) = \eta(0) \tag{A-65}$$

Finally, the transformed initial displacement is

$$\eta(0) = \hat{Q}^T M \bar{z}(0) \tag{A-66}$$

Similarly, the transformed initial velocity is

$$\dot{\eta}(0) = \hat{Q}^T M \dot{\bar{z}}(0) \tag{A-67}$$

The product of the first two matrices on the left side of equation (A-59) equals a vector of participation factors.

$$\begin{bmatrix} -\Gamma_1 \\ -\Gamma_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{22} \end{bmatrix} \tag{A-68}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1 \omega_1 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 \ddot{y} \\ -\Gamma_2 \ddot{y} \end{bmatrix} \tag{A-69}$$

Equation (A-69) can be solved in terms of Laplace transforms, or some other differential equation solution method. As an example, the solution for a half-sine input is given in Reference 6.

APPENDIX B

EXAMPLE 1

Normal Modes Analysis

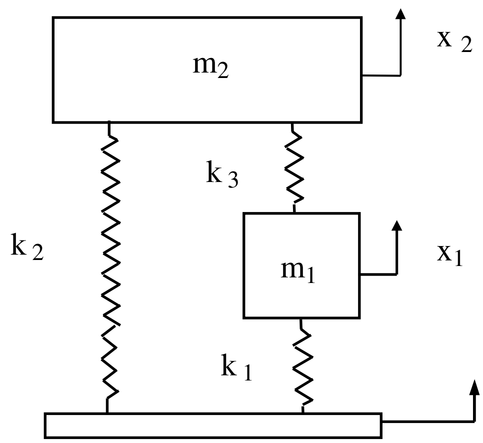

Figure B-1.

Consider the system in Figure B-1. Assign the values in Table B-1.

Table B-1. Parameters

Variable	Value
m_1	3.0 kg
m_2	2.0 kg
k_1	400,000 N/m
k_2	300,000 N/m
k_3	100,000 N/m

Furthermore, assume

1. Each mode has a damping value of 5%.
2. Zero initial conditions

Next, assume that the base input function is a half-sine pulse.

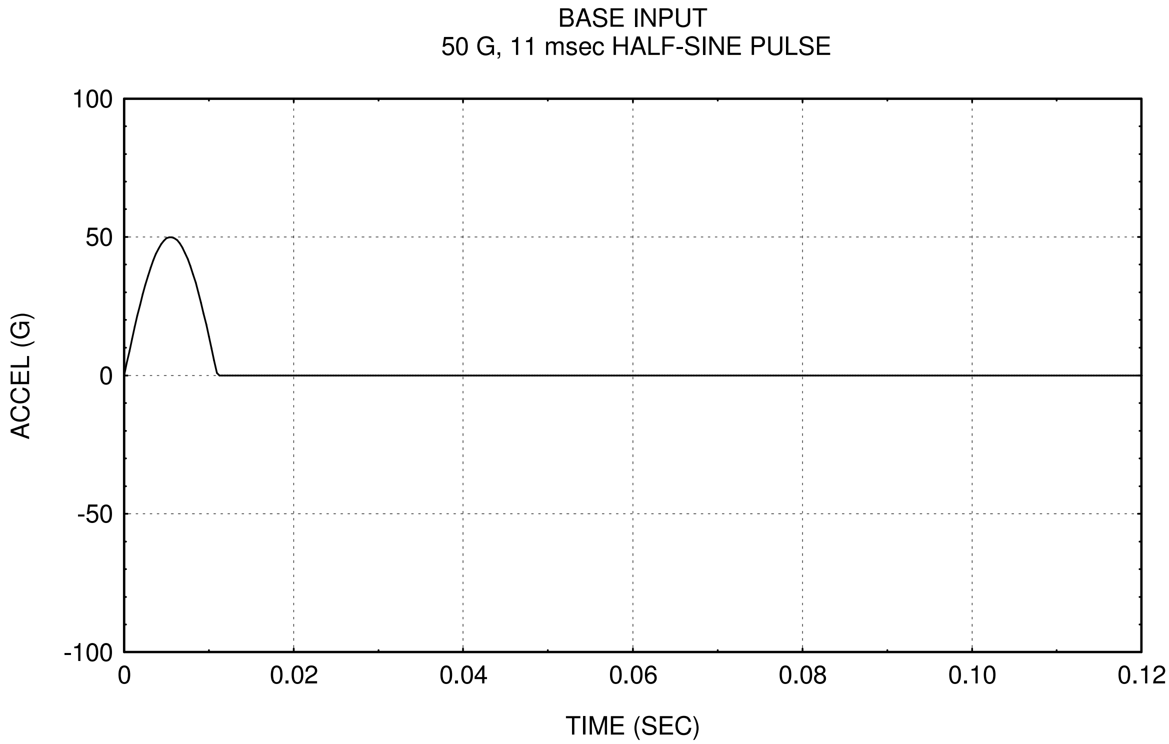

Figure B-2.

$$\ddot{y}(t) = \begin{cases} A\sin\left(\frac{\pi t}{T}\right), & 0 \le t \le T \\ 0\,, & t > T \end{cases} \quad \text{(B-1)}$$

Assign A = 50 G and T = 0.011 seconds.

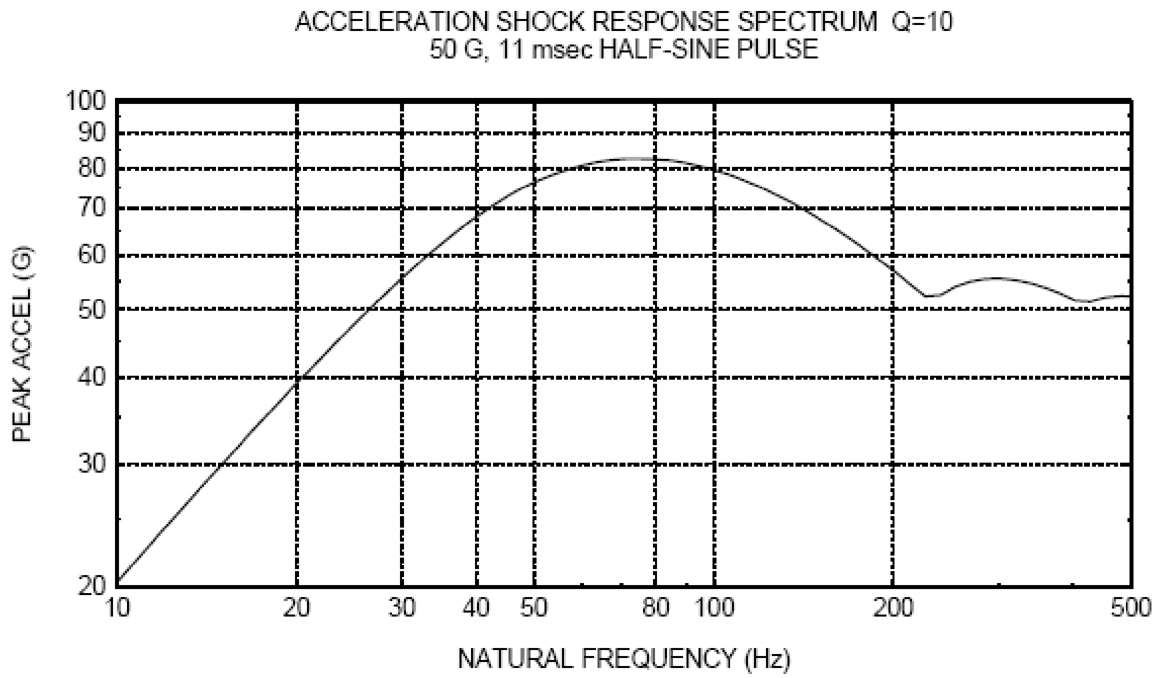

Figure B-3.

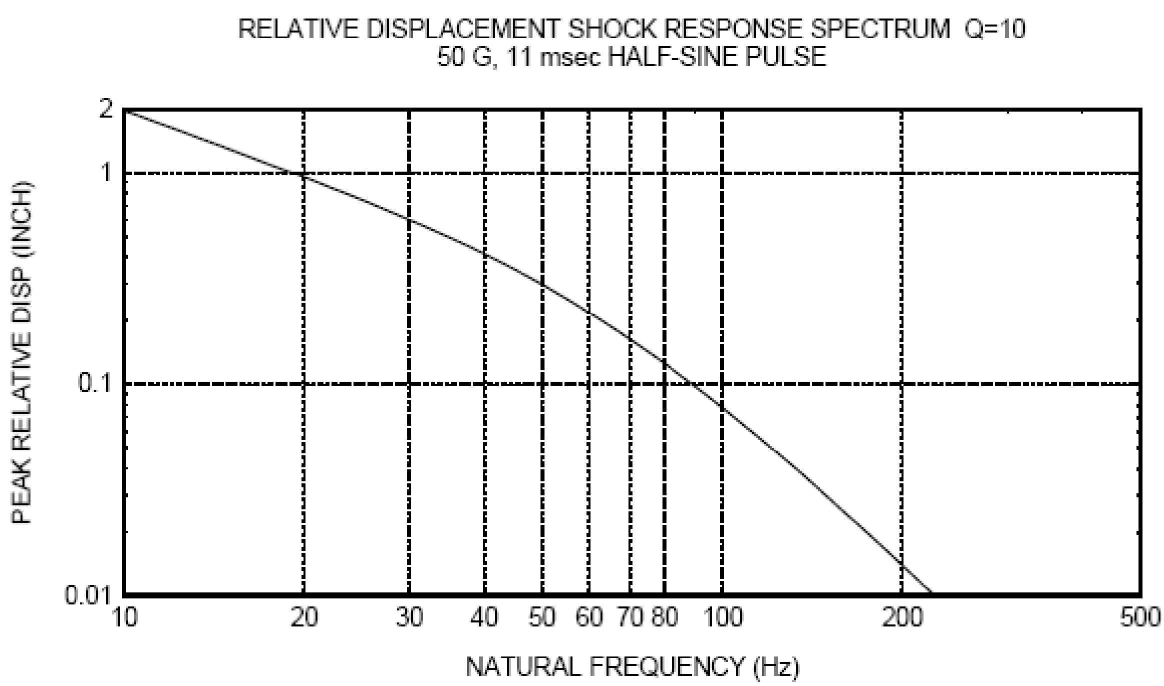

Figure B-4.

Solve for the acceleration response time histories. The homogeneous, undamped problem is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_3 & -k_3 \\ -k_3 & k_2+k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \quad \text{(B-2)}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix}+\begin{bmatrix} 500{,}000 & -100{,}000 \\ -100{,}000 & 400{,}000 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(B-3)}$$

The eigenvalue problem is

$$\begin{bmatrix} 500{,}000-2\omega^2 & -100{,}000 \\ -100{,}000 & 400{,}000-\omega^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(B-4)}$$

Set the determinant equal to zero

$$\det\begin{bmatrix} 500{,}000-2\omega^2 & -100{,}000 \\ -100{,}000 & 400{,}000-\omega^2 \end{bmatrix}=0 \quad \text{(B-5)}$$

The roots of the polynomial are

$$\omega_1 = 373.1 \ \text{rad/sec} \quad \text{(B-6)}$$

$$\omega_2 = 476.9 \ \text{rad/sec} \quad \text{(B-7)}$$

$$f_1 = 59.4 \ \text{Hz} \quad \text{(B-8)}$$

$$f_2 = 75.9 \ \text{Hz} \quad \text{(B-9)}$$

The frequencies are rather close together, which is a concern relative to an assumption behind the modal combination methods.

The corresponding eigenvector matrix is

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \quad \text{(B-10)}$$

$$Q = \begin{bmatrix} 1.215 & -0.549 \\ 1 & 1 \end{bmatrix} \quad \text{(B-11)}$$

The next goal is to obtain a normalized eigenvector matrix $\hat{Q}$ such that

$$\hat{Q}^T M \hat{Q} = I \quad \text{(B-12)}$$

The normalized eigenvector matrix is

$$\hat{Q} = \begin{bmatrix} 0.4792 & -0.3220 \\ 0.3943 & 0.5869 \end{bmatrix} \quad \text{(B-13a)}$$

$$\hat{Q}^T = \begin{bmatrix} 0.4792 & 0.3943 \\ -0.3220 & 0.5869 \end{bmatrix} \quad \text{(B-13b)}$$

Note that the eigenvector in the first column has a uniform polarity. Thus, the two masses vibrate in phase for the first mode.

The eigenvector in the second column has two components with opposite polarity. The two masses vibration 180 degrees out of phase for the second mode.

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{(B-14)}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \quad \text{(B-15)}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} 0.4792 & 0.3943 \\ -0.3220 & 0.5869 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \tag{B-16}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} 2.2264 \\ 0.2079 \end{bmatrix} \tag{B-17}$$

Modal Transient Analysis

The combination of equations (B-1) and (B-6) are solved using the method in Reference 6, as modified for a multi-degree-of-freedom system. The modifications are made via equations (A-69) and (A-50a).

The resulting relative displacement maxima are given in Table B-2. The time history responses are plotted in Figure B-5.

Table B-2. Modal Transient Analysis, Relative Displacement

Parameter	Mass 1 (inch)	Mass 2 (inch)
Maximum Absolute	0.229	0.211

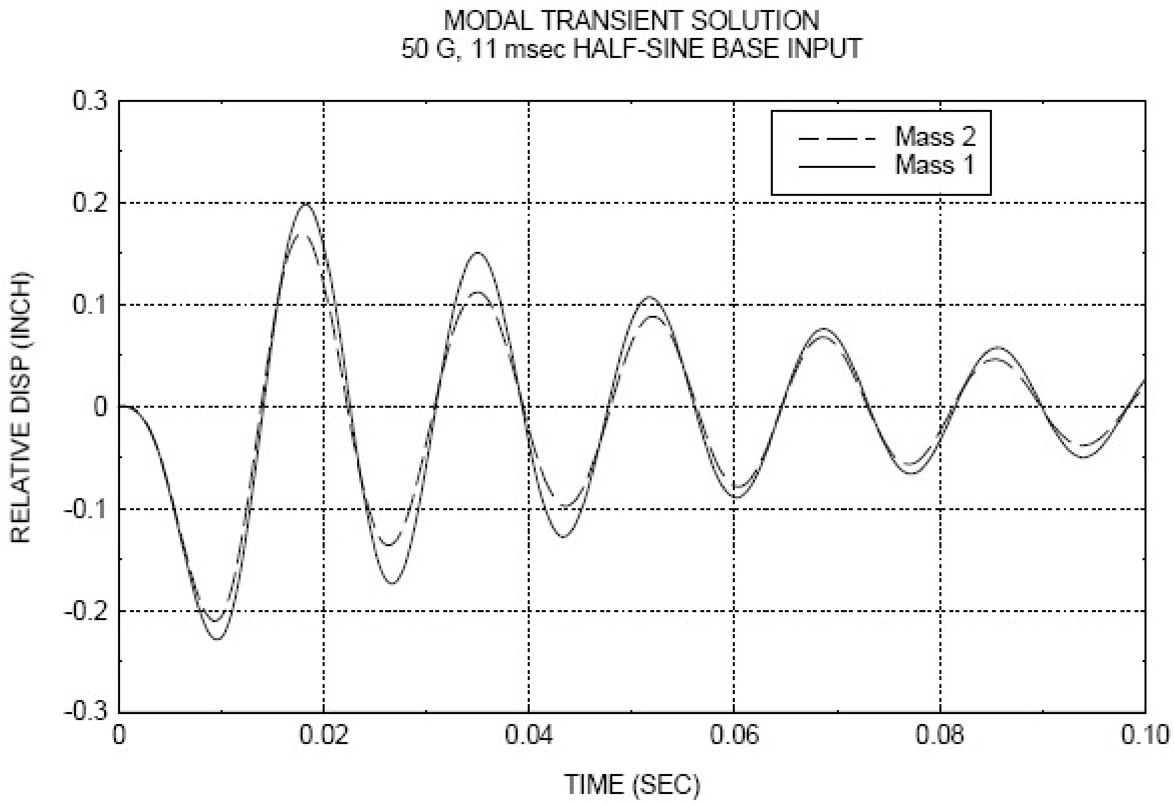

Figure B-5.

SDOF Response

The peak acceleration value for a single-degree-of-freedom can be calculated per the method in Reference 6. The resulting maxima are shown in Table B-3.

Table B-3. SDOF Response Values, Q=10		
Mode	Natural Frequency (Hz)	Peak Relative Displacement (inch)
1	59.4	0.222
2	75.9	0.140

Note that the coordinates in Table B-3 occur on the response spectrum curve in Figure B-4.

SRSS Method

Again, the SRSS equation is

$$(z_i)_{max} \approx \sqrt{\sum_{j=1}^{N} \left[\Gamma_j \ \ \hat{q}_{ij} \ D_{j,max}\right]^2} \quad \text{(B-18)}$$

Equation (B-18) is solved using

1. The eigenvalues from equation (B-13a)
2. The participation factors from equation (B-17)
3. The displacement values from Table B-3

The results are given in Table B-4.

Table B-4.
Comparison of SRSS Results with Modal Transient Analysis, Relative Displacement

Mass	Modal Transient Analysis (inch)	SRSS Analysis (inch)	Error
1	0.229	0.237	3.5 %
2	0.211	0.196	-7.1 %

Sample Calculation

$$(z_1)_{max} \approx \sqrt{\left[\Gamma_1 \ \ \hat{q}_{11} \ D_{1,max}\right]^2 + \left[\Gamma_2 \ \ \hat{q}_{12} \ D_{2,max}\right]^2} \quad \text{(B-19a)}$$

$$(z_1)_{max} \approx \sqrt{\left[(2.2264)(0.4792)(0.222)\right]^2 + \left[(0.2079)(-0.3220)(0.140)\right]^2} \quad \text{(B-19b)}$$

$$(z_1)_{max} \approx 0.237 \ \text{inch} \quad \text{(B-19c)}$$

The acceleration results are

Table B-5. Comparison of SRSS Results with Modal Transient Analysis, Absolute Acceleration			
Mass	Modal Transient Analysis (G)	SRSS Analysis (G)	Error
1	81.0	86.1	6.2%
2	79.9	71.4	-10.7%

<u>ABSSUM Method</u>

Again, the ABSSUM equation is

$$(z_i)_{max} \leq \sum_{j=1}^{N} \left|\Gamma_j\right| \left|\hat{q}_{ij}\right| \left|D_{j,max}\right| \tag{B-20}$$

Equation (B-20) is solved using

1. The eigenvalues from equation (B-13a)
2. The participation factors from equation (B-17)
3. The displacement values from Table B-3

The results are given in Table B-6.

Table B-6. Comparison of the Sum of the Absolute Magnitudes with Modal Transient Analysis, Relative Displacement			
Mass	Modal Transient Analysis (inch)	ABSSUM (inch)	Error
1	0.229	0.246	7.4 %
2	0.211	0.212	0.5 %

$$(z_1)_{max} \leq \left|\Gamma_1 \ \hat{q}_{11} \ D_{1,max}\right| + \left|\Gamma_2 \ \hat{q}_{12} \ D_{2,max}\right| \tag{B-21a}$$

$$(z_1)_{max} \leq \left|(2.2264)(0.4792)(0.222)\right| + \left|(0.2079)(-0.3220)(0.140)\right| \tag{B-21b}$$

$$(z_1)_{max} \leq 0.246 \text{ inch} \tag{B-21c}$$

The acceleration results are:

Table B-7. Comparison of ABSSUM Results with Modal Transient Analysis, Absolute Acceleration

Mass	Modal Transient Analysis (G)	ABSSUM Analysis (G)	Error
1	81.0	91.4	12.8%
2	79.9	80.7	1.0%

APPENDIX C

Modal Participation Factor

Consider a discrete dynamic system governed by the following equation

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{C-1}$$

where

- M is the mass matrix
- K is the stiffness matrix
- $\ddot{\bar{x}}$ is the acceleration vector
- $\bar{x}$ is the displacement vector
- $\bar{F}$ is the forcing function or base excitation function

A solution to the homogeneous form of equation (1) can be found in terms of eigenvalues and eigenvectors. The eigenvectors represent vibration modes.

Let ϕ be the eigenvector matrix.

The system's generalized mass matrix $\hat{m}$ is given by

$$\hat{m} = \phi^T M \phi \tag{C-2}$$

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^T M \bar{r} \tag{C-3}$$

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{C-4}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\;2}}{\hat{m}_{ii}} \tag{C-5}$$