# NOTES ON INTERNAL AND TRANSMITTED FORCES IN VIBRATING MULTI-DEGREE-OF-FREEDOM SYSTEMS <br> Revision A 

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## Introduction

The internal and transmitted forces of a vibrating system can be determined via a finite element model. This paper explores the uses and limitations of these forces via examples.

One of the findings is that the force transmitted between each nodal pair must be calculated on an elemental basis as a post-processing step using a transformation matrix. This matrix is referred to as the "force recovery matrix" in this paper. The matrix may be non-symmetric. Furthermore, the matrix will be shown to be non-square for a beam bending element.

## Example

Consider the longitudinal vibration of an aluminum, fixed-free, circular rod with the following properties.

| Length | $\mathrm{L}=24 \mathrm{inch}$ |
| :--- | :--- |
| Diameter | $\mathrm{D}=1 \mathrm{inch}$ |
| Area | $\mathrm{A}=0.785 \mathrm{inch}^{\wedge} 2$ |
| Area Moment of Inertia | $\mathrm{I}=0.0491 \mathrm{inch}^{\wedge} 4$ |
| Elastic Modulus | $\mathrm{E}=1.0 \mathrm{e}+07 \mathrm{lbf} / \mathrm{in}^{\wedge} 2$ |
| Mass Density | $\rho=0.1 \mathrm{lbm} / \mathrm{in}^{\wedge} 3$ |
| Speed of Sound in <br> Material | $\mathrm{c}=1.96 \mathrm{e}+05 \mathrm{in} / \mathrm{sec}$ |
| Viscous Damping Ratio | $\xi=0$ |

The fundamental frequency for the fixed-free case is

$$
\begin{equation*}
\mathrm{f}_{1}=\frac{\mathrm{c}}{4 \mathrm{~L}}=\frac{1.96 \mathrm{e}+05 \mathrm{in} / \mathrm{sec}}{4(24 \mathrm{in})}=2047 \mathrm{~Hz} \tag{1}
\end{equation*}
$$

Equation (1) is taken from Reference 1.
Model the rod as series of springs and masses. The modeling will follow the finite element approach in Reference 2 except that a lumped mass matrix will be used for simplicity.

The system is subjected to free vibration due to initial conditions.
The initial velocity is zero.
The initial displacement is equal to the first mode shape scaled so that the free end displacement is 0.001 inch. This initial displacement might be difficult to achieve in reality, but it is useful for demonstration purposes because the resulting response will be solely that of the fundamental mode.

The following analyses were performed using Matlab script: mdof_free.m

Constrained Model with Two-degrees-of-freedom


Figure 1. Unconstrained Model


Figure 2. Constrained Model

The unconstrained equation of motion is

$$
\begin{align*}
& {\left[\begin{array}{ccc}
\mathrm{m}_{1} & 0 & 0 \\
0 & \mathrm{~m}_{2} & 0 \\
0 & 0 & \mathrm{~m} 3
\end{array}\right]\left[\begin{array}{l}
\ddot{x}_{0} \\
\ddot{x}_{1} \\
\ddot{x}_{2}
\end{array}\right]+\left[\begin{array}{ccc}
\mathrm{k}_{1} & -\mathrm{k}_{1} & 0 \\
-\mathrm{k}_{1} & \mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
0 & -\mathrm{k}_{2} & \mathrm{k} 2
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{0} \\
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=0}  \tag{2}\\
& \frac{\rho A L}{4}\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
\ddot{x}_{0} \\
\ddot{x}_{1} \\
\ddot{x}_{2}
\end{array}\right]+\frac{E A}{h}\left[\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{array}\right]\left[\begin{array}{l}
x_{0} \\
\mathrm{x}_{1} \\
x_{2}
\end{array}\right]=0 \tag{3}
\end{align*}
$$

The nodes are equally spaced. The element length is $\mathrm{h}=12 \mathrm{in}$.
The constrained equation of motion is

$$
\begin{align*}
& {\left[\begin{array}{cc}
\mathrm{m} 2 & 0 \\
0 & \mathrm{~m} 3
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathrm{x}}_{1} \\
\ddot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=0}  \tag{4}\\
& \frac{\rho A L}{4}\left[\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathrm{x}}_{1} \\
\ddot{\mathrm{x}}_{2}
\end{array}\right]+\frac{\mathrm{EA}}{\mathrm{~h}}\left[\begin{array}{cc}
2 & -1 \\
-1 & 1
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=0 \tag{5}
\end{align*}
$$

Let $E_{1}$ and $E_{2}$ be the forces in springs 1 and 2, respectively. The axial forces transmitted through the springs for the constrained model are thus

$$
\left[\begin{array}{l}
\mathrm{E}_{1}  \tag{6}\\
\mathrm{E}_{2}
\end{array}\right]=\left[\begin{array}{cc}
\mathrm{k}_{1} & 0 \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]
$$

Note that tension is positive.
The elemental force recovery matrix $\mathrm{F}_{\mathrm{R}}$ is thus

$$
\begin{gather*}
\mathrm{F}_{\mathrm{R}}=\left[\begin{array}{cc}
-\mathrm{k}_{1} & 0 \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]  \tag{7}\\
\mathrm{F}_{\mathrm{R}}=\frac{\mathrm{EA}}{\mathrm{~h}}\left[\begin{array}{ll}
-1 & 0 \\
-1 & 1
\end{array}\right] \tag{8}
\end{gather*}
$$

The force recover matrix has some characteristics as the global stiffness matrix, but the matrices are fundamentally different.

The natural frequencies and mode shapes from the Matlab output are:

```
Natural Frequencies
    No. f(Hz)
1. 1994.4
2. 4814.8
    Modes Shapes (column format)
ModeShapes =
    14.3101 -14.3101
    20.2376 20.2376
```

The resulting forces are given in Table 1. The displacement results are given in Figure 3.


Figure 3.

Table 1. Two-degree-of-freedom Model, Force Results

| Element | Peak (lbf) | Type |
| :---: | :---: | :--- |
| Spring 1 | 463 | Elemental Force |
| Spring 2 | 192 | Elemental Force |
| Mass 1 | 271 | Nodal Net Force |
| Mass 2 | 192 | Nodal Net Force |

Note that the spring force is calculated from the relative displacement of the spring's end points multiplied by the spring stiffness, or equivalently by equation (6).

The net force on the point mass is equal to the mass times its acceleration.

## Constrained Model with Four-degrees-of-freedom

The previous example is repeated with additional degrees-of-freedom.


Figure 4. Unconstrained Model


Figure 5. Constrained Model

The unconstrained equation of motion is

$$
\frac{\rho A L}{10}\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0  \tag{9}\\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
\ddot{x}_{0} \\
\ddot{x}_{1} \\
\ddot{x}_{2} \\
\ddot{x}_{3} \\
\ddot{x}_{4} \\
\ddot{x}_{5}
\end{array}\right]+\frac{\mathrm{EA}}{\mathrm{~h}}\left[\begin{array}{cccccc}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 1
\end{array}\right]\left[\begin{array}{l}
x_{0} \\
x_{1} \\
x_{2} \\
x_{3} \\
x_{4} \\
x_{5}
\end{array}\right]=0
$$

The nodes are equally spaced. The element length is $\mathrm{h}=4.8 \mathrm{in}$.
The constrained equation of motion is obtained by removing the first row and column from equation (9). The resulting equation is omitted for brevity.

The elemental spring forces for the constrained model are calculated via

$$
\begin{align*}
& {\left[\begin{array}{l}
E_{1} \\
\mathrm{E}_{2} \\
\mathrm{E}_{3} \\
\mathrm{E}_{4}
\end{array}\right]=\left[\begin{array}{cccc}
\mathrm{k}_{1} & 0 & 0 & 0 \\
\mathrm{k}_{2} & -\mathrm{k}_{2} & 0 & 0 \\
0 & -\mathrm{k}_{3} & \mathrm{k}_{3} & 0 \\
0 & 0 & -k_{4} & \mathrm{k}_{4}
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2} \\
\mathrm{x}_{3} \\
\mathrm{x}_{4}
\end{array}\right]}  \tag{10}\\
& {\left[\begin{array}{l}
\mathrm{E}_{1} \\
\mathrm{E}_{2} \\
\mathrm{E}_{3} \\
\mathrm{E}_{4}
\end{array}\right]=\frac{\mathrm{EA}}{\mathrm{~h}}\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right]} \tag{11}
\end{align*}
$$

The natural frequencies and mode shapes from the Matlab output are:

```
Natural Frequencies
    No. f(Hz)
1. 2038.1
2. 5914.9
3. 9212.7
4. 11609
5. 12868
```

```
    Modes Shapes (column format)
ModeShapes =
\begin{tabular}{rrrrr}
6.2537 & 16.3725 & 20.2376 & -16.3725 & 6.2537 \\
11.8953 & 19.2471 & -0.0000 & 19.2471 & -11.8953 \\
16.3725 & 6.2537 & -20.2376 & -6.2537 & 16.3725 \\
19.2471 & -11.8953 & -0.0000 & -11.8953 & -19.2471 \\
20.2376 & -20.2376 & 20.2376 & 20.2376 & 20.2376
\end{tabular}
```

The resulting forces are given in Table 2. The displacements are given in Figure 6.

| Table 2. | Five-degree-of-freedom Model, Force Results |  |
| :---: | :---: | :--- |
| Element | Peak (lbf) | Type |
| Spring 1 | 505 | Elemental Force |
| Spring 2 | 456 | Elemental Force |
| Spring 3 | 362 | Elemental Force |
| Spring 4 | 232 | Elemental Force |
| Spring 5 | 80 | Elemental Force |
| Mass 1 | 49 | Nodal Net Force |
| Mass 2 | 94 | Nodal Net Force |
| Mass 3 | 130 | Nodal Net Force |
| Mass 4 | 152 | Nodal Net Force |
| Mass 5 | 80 | Nodal Net Force |



Figure 6.

## Comparison

| Table 3. Peak Net Force (lbf) on Point Masses |  |  |
| :---: | :---: | :---: |
| Location | Two-dof | Five-dof |
| Midpoint | 271 | 130 |
| Free End | 192 | 80 |

Table 4. Peak Transmitted Force (lbf) through Springs

| Two-dof | Five-dof |
| :---: | :---: |
| 463 | 505 |

The peak transmitted force is also the force transmitted to ground. The exact force from the continuous model in Appendix A is 514 lbf.

## Normal Stress

The following calculation is made using the Five-dof results.
The peak stress at the fixed boundary is 643 psi for the Five-dof result, as obtained by dividing the transmitted force to ground by the cross-sectional area.

The maximum stress can also be calculated using the method in Reference 3, as follows.
The peak velocity was $12.8 \mathrm{in} / \mathrm{sec}$, as measured at the free end.
The characteristic impedance of aluminum bar is $\rho \mathrm{c}=50.8 \mathrm{psi}$ sec/in. The peak stress is

$$
\begin{equation*}
\left[\sigma_{\mathrm{n}}\right]_{\max }=\rho \mathrm{c} \mathrm{v}_{\mathrm{n}, \max }=(50.8 \mathrm{psi} \mathrm{sec} / \mathrm{in})(12.8 \mathrm{in} / \mathrm{sec})=650 \mathrm{psi} \tag{12}
\end{equation*}
$$

The exact stress from the continuous model in Appendix A is 654 psi.

## Conclusions

## Net Force on Each Point Mass

The net force on any point mass depends on the number of nodes, or "mesh density."
The mass value of each point mass decreases as the number of nodes increases, for uniform spacing. Yet, the nodal point mass acceleration should ideally remain constant. ${ }^{1}$

Thus a higher nodal density yields lower net forces acting upon each nodal point mass, as shown in Table 3.

This can be thought of in terms of the familiar physics equation.

$$
\begin{equation*}
\text { Force }=\text { mass } \mathrm{x} \text { acceleration } \tag{5}
\end{equation*}
$$

The acceleration should ideally remain constant at a given point on the rod regardless of mesh density. Thus, a decrease in mass must have a corresponding decrease in the applied net force for constant acceleration.

The nodal net force thus has limited value as a parameter for a continuous system.

## Transmitted Force through Springs

The peak transmitted force to the ground should ideally be the same regardless of mess density. The results for the two models were within $10 \%$ for this force parameter, as shown in Figure 4.

The spring forces could also be readily used for stress calculations.

## Future Work

Further work is needed. Consideration should also be given to:

1. Systems with consistent mass matrices
2. Damping force
3. Forced vibration, both sine and random
4. Additional element types, such as plate and solid elements
5. Structures with complex geometries and mixed element types

Another topic which was not covered in this paper is the transformation from global to local displacement coordinates for the case of elements which do not align with the global axes.

[^0]
## References

1. T. Irvine, Longitudinal Natural Frequencies of Rods and Response to Initial Conditions, Revision D, Vibrationdata, 2010.
2. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Revision C, Vibrationdata, 2010.
3. T. Irvine, Shock and Vibration Stress as a Function of Velocity, Revision A, Vibrationdata, 2010.
4. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision E, Vibrationdata, 2008.
5. T. Irvine, Free Vibration of a Cantilever Beam, Vibrationdata, 2010.
6. T. Irvine, Calculating Transfer Functions from Normal Modes, Revision A, Vibrationdata, 2010.
7. T. Irvine, The Longitudinal Vibration Response of a Rod to an Applied Force, Vibrationdata, 2010.

## APPENDIX A

Continuous System Solution for the Longitudinal Vibration of a Fixed-Free Rod
The variables are

L is the length
c is the longitudinal wave speed
$\omega_{\mathrm{n}} \quad$ is the natural frequency of mode n $u(x, t)$ is the longitudinal displacement

The displacement for a fixed-free rod is per Reference 1 is

$$
\begin{equation*}
u(x, t)=\sum_{n=1,3,5, \ldots}^{\infty}\left\{\left[\sin \left(\frac{n \pi x}{2 L}\right)\right]\left[A_{n} \sin \left(\frac{n \pi c t}{2 L}\right)+B_{n} \cos \left(\frac{n \pi c t}{2 L}\right)\right]\right\} \tag{A-1}
\end{equation*}
$$

The velocity is

$$
\begin{equation*}
\dot{u}(x, t)=\sum_{n=1,3,5, \ldots}^{\infty}\left\{\left(\frac{\mathrm{n} \pi \mathrm{c}}{2 \mathrm{~L}}\right)\left[\sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right)\right]\left[\mathrm{A}_{\mathrm{n}} \cos \left(\frac{\mathrm{n} \pi \mathrm{ct}}{2 \mathrm{~L}}\right)-\mathrm{B}_{\mathrm{n}} \sin \left(\frac{\mathrm{n} \pi \mathrm{ct}}{2 \mathrm{~L}}\right)\right]\right\} \tag{A-2}
\end{equation*}
$$

The natural frequencies are

$$
\begin{equation*}
\omega_{\mathrm{n}}=\mathrm{n} \pi \frac{\mathrm{c}}{2 \mathrm{~L}} \tag{A-3}
\end{equation*}
$$

For the case of zero initial velocity across all locations, $\mathrm{A}_{\mathrm{n}}=0$.

Thus,

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, \mathrm{t})=\sum_{\mathrm{n}=1,3,5, \ldots}^{\infty}\left\{\left[\sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right)\right]\left[\mathrm{B}_{\mathrm{n}} \cos \left(\frac{\mathrm{n} \pi \mathrm{ct}}{2 \mathrm{~L}}\right)\right]\right\} \tag{A-4}
\end{equation*}
$$

Set the initial displacement as

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, 0)=\sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \tag{A-5}
\end{equation*}
$$

Thus at $\mathrm{t}=0$,

$$
\begin{equation*}
\sum_{\mathrm{n}=1,3,5, \ldots}^{\infty}\left\{\mathrm{B}_{\mathrm{n}} \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right)\right\}=\sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \tag{A-6}
\end{equation*}
$$

Multiply by $\sin \left(\frac{m \pi x}{2 L}\right), m=1,3,5, \ldots$.

$$
\begin{equation*}
\sum_{n=1,3,5, \ldots}^{\infty}\left\{B_{n} \sin \left(\frac{n \pi x}{2 L}\right) \sin \left(\frac{m \pi x}{2 L}\right)\right\}=\sin \left(\frac{\pi x}{2 L}\right) \sin \left(\frac{m \pi x}{2 L}\right), m=1,3,5, \ldots \tag{A-7}
\end{equation*}
$$

Integrate

$$
\begin{equation*}
\int_{0}^{\mathrm{L}} \sum_{\mathrm{n}=1,3,5, \ldots}^{\infty}\left\{\mathrm{B}_{\mathrm{n}} \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{m} \pi \mathrm{x}}{2 \mathrm{~L}}\right)\right\} \mathrm{dx}=\int_{0}^{\mathrm{L}} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{m} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx} \tag{A-8}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{\mathrm{n}=1,3,5, \ldots}^{\infty}\left\{\int_{0}^{\mathrm{L}} \mathrm{~B}_{\mathrm{n}} \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{m} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx}\right\}=\int_{0}^{\mathrm{L}} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{m} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx} \tag{A-9}
\end{equation*}
$$

Orthogonality requires that $\mathrm{m}=\mathrm{n}$.

$$
\begin{equation*}
\sum_{\mathrm{n}=1,3,5, \ldots}^{\infty}\left\{\int_{0}^{\mathrm{L}} \mathrm{~B}_{\mathrm{n}} \sin ^{2}\left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx}\right\}=\int_{0}^{\mathrm{L}} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx} \tag{A-10}
\end{equation*}
$$

For $\mathrm{n}=1$,

$$
\begin{align*}
& \int_{0}^{L} B_{1} \sin ^{2}\left(\frac{\pi x}{2 L}\right) d x=\int_{0}^{L} \sin ^{2}\left(\frac{\pi x}{2 L}\right) d x  \tag{A-11}\\
& B_{1}=1 \tag{A-12}
\end{align*}
$$

For $\mathrm{n} \neq 1$,

$$
\begin{gather*}
\int_{0}^{\mathrm{L}} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx}=\frac{1}{2} \int_{0}^{\mathrm{L}}\left[-\cos \left(\frac{\pi(\mathrm{n}+1) \mathrm{x}}{2 \mathrm{~L}}\right)+\cos \left(\frac{\pi(\mathrm{n}-1) \mathrm{x}}{2 \mathrm{~L}}\right)\right] \mathrm{dx}  \tag{A-13}\\
\left.\int_{0}^{\mathrm{L}} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx}=\frac{1}{2}\left\{-\left(\frac{2 \mathrm{~L}}{\pi(\mathrm{n}+1)}\right) \sin \left(\frac{\pi(\mathrm{n}+1) \mathrm{x}}{2 \mathrm{~L}}\right)+\left(\frac{2 \mathrm{~L}}{\pi(\mathrm{n}-1)}\right) \sin \left(\frac{\pi(\mathrm{n}-1) \mathrm{x}}{2 \mathrm{~L}}\right)\right)\right\}\left.\right|_{0} ^{\mathrm{L}} \tag{A-14}
\end{gather*}
$$

$$
\begin{equation*}
\int_{0}^{\mathrm{L}} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \sin \left(\frac{\mathrm{n} \pi \mathrm{x}}{2 \mathrm{~L}}\right) \mathrm{dx}=0 \tag{A-15}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\mathrm{B}_{\mathrm{n}}=0 \quad \text { for } \mathrm{n}=3,5,7, \ldots \tag{A-16}
\end{equation*}
$$

The resulting displacement equation is

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, \mathrm{t})=\sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \cos \left(\frac{\pi \mathrm{ct}}{2 \mathrm{~L}}\right) \tag{A-17}
\end{equation*}
$$

Now assume that the initial displacement was scaled such that

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, 0)=\mathrm{D} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \tag{A-18}
\end{equation*}
$$

where D is the initial displacement at the free end.
The solution becomes

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, \mathrm{t})=\mathrm{D} \sin \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \cos \left(\frac{\pi \mathrm{ct}}{2 \mathrm{~L}}\right) \tag{A-19}
\end{equation*}
$$

The strain is

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dx}} \mathrm{u}(\mathrm{x}, \mathrm{t})=\frac{\pi}{2 \mathrm{~L}} \mathrm{D} \cos \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \cos \left(\frac{\pi \mathrm{ct}}{2 \mathrm{~L}}\right) \tag{A-20}
\end{equation*}
$$

The normal stress is

$$
\begin{equation*}
\sigma(\mathrm{x}, \mathrm{t})=\frac{\pi \mathrm{E}}{2 \mathrm{~L}} \mathrm{D} \cos \left(\frac{\pi \mathrm{x}}{2 \mathrm{~L}}\right) \cos \left(\frac{\pi \mathrm{ct}}{2 \mathrm{~L}}\right) \tag{A-21}
\end{equation*}
$$

The maximum normal stress is

$$
\begin{equation*}
\sigma_{\max }=\frac{\pi \mathrm{E}}{2 \mathrm{~L}} \mathrm{D} \tag{A-22}
\end{equation*}
$$

For the sample problem in the main text,

$$
\begin{align*}
\sigma_{\max } & =\frac{\pi(1.0 \mathrm{e}+07 \mathrm{psi})}{2(24 \mathrm{in})}(0.001 \mathrm{in})  \tag{A-23}\\
\sigma_{\max } & =654 \mathrm{psi} \tag{A-24}
\end{align*}
$$

The peak transmitted force is thus

$$
\begin{equation*}
\mathrm{F}_{\max }=\left(\sigma_{\max }\right)(\mathrm{A})=514 \mathrm{lbf} \tag{A-25}
\end{equation*}
$$

Note that the critical static buckling load P for this rod is

$$
\begin{equation*}
\mathrm{P}=\frac{\pi^{2} \mathrm{EI}}{4 \mathrm{~L}^{2}}=\frac{\pi^{2}(1.0 \mathrm{e}+07 \mathrm{psi})\left(0.0491 \mathrm{in}^{4}\right)}{4(24 \mathrm{in})^{2}}=2103 \mathrm{lbf} \tag{A-26}
\end{equation*}
$$

The critical dynamic buckling load would be higher, but this is a subject for a future paper.

## APPENDIX B

## Rod, Steady-State Longitudinal Vibration due to an Applied Force

The following is taken from Reference 6.
The variables are:

| f | Excitation frequency |
| :---: | :--- |
| $\mathrm{f}_{\mathrm{r}}$ | Natural frequency for mode $r$ |
| N | Total degrees-of-freedom |
| $\mathrm{H}_{\mathrm{ij}}(\mathrm{f})$ | The steady state displacement at coordinate $i$ due to a harmonic force <br> excitation only at coordinate $j$ |
| $\xi_{\mathrm{r}}$ | Damping ratio for mode $r$ |
| $\phi_{\mathrm{ir}}$ | Mass-normalized eigenvector for physical coordinate $i$ and mode number $r$ |
| $\omega$ | Excitation frequency (rad/sec) |
| $\omega_{\mathrm{r}}$ | Natural frequency (rad/sec) for mode $r$ |

## Receptance

The steady-state displacement at coordinate $i$ due to a harmonic force excitation only at coordinate $j$ is:

$$
\begin{equation*}
\mathrm{H}_{\mathrm{ij}}(\mathrm{f})=\sum_{\mathrm{r}=1}^{\mathrm{N}}\left\{\frac{\phi_{\mathrm{ir}} \phi_{\mathrm{jr}}}{\omega_{\mathrm{r}}^{2}} \frac{1}{\left(1-\rho_{\mathrm{r}}^{2}\right)+\mathrm{j}\left(2 \xi_{\mathrm{r}} \rho_{\mathrm{r}}\right)}\right\} \tag{B-1}
\end{equation*}
$$

where

$$
\begin{gather*}
\rho_{\mathrm{r}}=\mathrm{f} / \mathrm{f}_{\mathrm{r}}  \tag{B-2}\\
\mathrm{j}=\sqrt{-1} \tag{B-3}
\end{gather*}
$$

Note that j is used both as an index and as an imaginary number in equation (B-1).

Note that the phase angle is typically represented as the angle by which force leads displacement. In terms of a C++ or Matlab type equation, the phase angle would be

$$
\begin{equation*}
\text { Phase }=-\operatorname{atan} 2(\operatorname{imag}(\mathrm{H}), \operatorname{real}(\mathrm{H})) \tag{B-4}
\end{equation*}
$$

Note that both the phase and the transfer function vary with frequency.
A more formal equation is

$$
\begin{equation*}
\operatorname{Phase}(\mathrm{f})=-\arctan \left\{\frac{\operatorname{imag}\left(\mathrm{H}_{\mathrm{ij}}(\mathrm{f})\right)}{\operatorname{real}\left(\mathrm{H}_{\mathrm{ij}}(\mathrm{f})\right)}\right\} \tag{B-5}
\end{equation*}
$$

## Relative Displacement

Consider two translational degrees-of-freedom $i$ and $j$. A force is applied at degree-of-freedom $k$.
The steady-state relative displacement transfer function $\mathrm{R}_{\mathrm{ij}}$ between $i$ and $j$ due to an applied force at $k$ is

$$
\begin{aligned}
\mathrm{R}_{\mathrm{ij}} & =\mathrm{H}_{\mathrm{ik}}(\mathrm{f})-\mathrm{H}_{\mathrm{jk}}(\mathrm{f}) \\
& =\sum_{\mathrm{r}=1}^{\mathrm{N}}\left\{\frac{\phi_{\mathrm{ir}} \phi_{\mathrm{kr}}}{\omega_{\mathrm{r}}^{2}} \frac{1}{\left(1-\rho_{\mathrm{r}}^{2}\right)+\mathrm{j}\left(2 \xi_{\mathrm{r}} \rho_{\mathrm{r}}\right)}\right\}-\sum_{\mathrm{r}=1}^{\mathrm{N}}\left\{\frac{\phi_{\mathrm{jr}} \phi_{\mathrm{kr}}}{\omega_{\mathrm{r}}^{2}} \frac{1}{\left(1-\rho_{\mathrm{r}}^{2}\right)+\mathrm{j}\left(2 \xi_{\mathrm{r}} \rho_{\mathrm{r}}\right)}\right\}
\end{aligned}
$$



Figure B-1.

Recall the two-degree-of-freedom model from the main text. Change the model so the modal damping is $5 \%$. Set all initial conditions to zero.

Apply a force on the second mass.

Frequency Response Function

DISPLACEMENT TRANSFER FUNCTION MAGNITUDE RESPONSE NODE 1 FORCE AT NODE 2


Figure B-2.


Figure B-3.

## Sine Excitation

Let the applied sinusoidal force be 100 lbf at 1994.4 Hz , which coincides with the fundamental frequency.

Table B-1. Force Transmitted via Springs

| Spring No. | (in) | (lbf) |
| :---: | :---: | :---: |
| 1 | $1.84 \mathrm{E}-05$ | 1207 |
| 2 | $7.85 \mathrm{E}-06$ | 513 |

The longitudinal rod continuous model in Reference 7 yielded a reaction force of 1272 lbf , which corresponds to the force transmitted via Spring no. 1.

Again, the two-degree-of-freedom system is a discrete model of the rod.

Random Excitation
To be included in a future revision.

## APPENDIX C

## Bernoulli-Euler Beam Bending, Free Vibration

Consider a beam, such as the cantilever beam in Figure 1.


Figure C-1.
where
E is the modulus of elasticity.
I is the area moment of inertia.
L is the length.
$\rho$ is mass per length.
The product EI is the bending stiffness.
The following equations are based on Reference 4.


Figure C-2.

The element free-body diagram is shown in Figure C-2.
The displacement vector for beam bending is

$$
\left[\begin{array}{c}
\mathrm{y}_{1}  \tag{C-1}\\
\theta_{1} \\
\mathrm{y}_{2} \\
\theta_{2}
\end{array}\right]
$$

The stiffness matrix for beam bending is

$$
\mathrm{K}_{\mathrm{j}}=\left(\frac{\mathrm{EI}}{\mathrm{~h}^{3}}\right)\left[\begin{array}{cccc}
12 & 6 \mathrm{~h} & -12 & 6 \mathrm{~h}  \tag{C-2}\\
& 4 \mathrm{~h}^{2} & -6 \mathrm{~h} & 2 \mathrm{~h}^{2} \\
& & 12 & -6 \mathrm{~h} \\
& & & 4 h^{2}
\end{array}\right]
$$

The mass matrix for beam bending is

$$
M_{j}=\left(\frac{\mathrm{h} \rho}{420}\right)\left[\begin{array}{cccc}
156 & 22 \mathrm{~h} & 54 & -13 \mathrm{~h}  \tag{C-3}\\
& 4 \mathrm{~h}^{2} & 13 & -3 \mathrm{~h}^{2} \\
& & 156 & -22 \mathrm{~h} \\
& & & 4 \mathrm{~h}^{2}
\end{array}\right]
$$

Note that h is the element length. Also, j is the node number in the following formulas.
The following limits apply to the next set of equations.

$$
(\mathrm{j}-1) \mathrm{h} \leq \mathrm{x} \leq \mathrm{jh}, \quad \xi=\mathrm{j}-\mathrm{x} / \mathrm{h}, \quad 0 \leq \xi \leq 1
$$

The transverse displacement $\mathrm{Y}(\mathrm{x})$ for a given element is

$$
\begin{align*}
Y(x)= & +\left\{3 \xi^{2}-2 \xi^{3}\right\} y_{j-1}+\left\{\xi^{2}-\xi^{3}\right\} h \theta_{j-1} \\
& +\left\{1-3 \xi^{2}+2 \xi^{3}\right\} y_{j}+\left\{-\xi+2 \xi^{2}-\xi^{3}\right\} h \theta_{j} \tag{C-4}
\end{align*}
$$

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{dx}} \mathrm{Y}(\mathrm{x}) & =\{-1 / \mathrm{h}\}\left\{\left[6 \xi-6 \xi^{2}\right] \mathrm{y}_{\mathrm{j}-1}+\left[2 \xi-3 \xi^{2}\right] \mathrm{h} \theta_{\mathrm{j}-1}\right. \\
& \left.+\left[1-6 \xi+6 \xi^{2}\right] \mathrm{y}_{\mathrm{j}}+\left[-1+4 \xi-3 \xi^{2}\right] \mathrm{h} \theta_{\mathrm{j}}\right\} \tag{C-5}
\end{align*}
$$

$$
\begin{align*}
& \frac{d^{2}}{d^{2}} \mathrm{Y}(\mathrm{x})=\left\{1 / \mathrm{h}^{2}\right\}\left\{[6-12 \xi] \mathrm{y}_{\mathrm{j}-1}+[2-6 \xi] \mathrm{h} \theta_{\mathrm{j}-1}\right. \\
& \left.\quad+[-6+12 \xi]_{\mathrm{y}}+[4-6 \xi] \mathrm{h} \theta_{\mathrm{j}}\right\} \tag{C-6}
\end{align*}
$$

The bending moment $M(x)$ is

$$
\begin{gather*}
M(x)=E I \frac{d^{2}}{d x^{2}} Y(x)  \tag{C-8}\\
M(x)=E I\left\{1 / h^{2}\right\}\left\{[6-12 \xi] y_{j-1}+[2-6 \xi] h \theta_{j-1}+[-6+12 \xi] y_{j}+[4-6 \xi] h \theta_{j}\right\} \tag{C-9}
\end{gather*}
$$

The bending moment at the starting point $(\xi=0)$ is

$$
\begin{equation*}
M_{j-1}=\frac{E I}{h^{2}}\left\{6 y_{j-1}+2 h \theta_{j-1}-6 y_{j}+4 h \theta_{j}\right\} \tag{C-10}
\end{equation*}
$$

The bending moment at the ending point $(\xi=1)$ is

$$
\begin{equation*}
\mathrm{M}_{\mathrm{j}}=\frac{\mathrm{EI}}{\mathrm{~h}^{2}}\left\{-6 \mathrm{y}_{\mathrm{j}-1}-4 \mathrm{~h} \theta_{\mathrm{j}-1}+6 \mathrm{y}_{\mathrm{j}}-2 \mathrm{~h} \theta_{\mathrm{j}}\right\} \tag{C-11}
\end{equation*}
$$

The shear force $\mathrm{V}(\mathrm{x})$ is

$$
\begin{align*}
& V(x)=E I \frac{d^{3}}{d x^{3}} Y(x)  \tag{C-12}\\
& V(x)=\frac{E I}{h^{3}}\left\{12 y_{j-1}+6 h \theta_{j-1}-12 y_{j}+6 h \theta_{j}\right\} \tag{C-13}
\end{align*}
$$

Note that the shear force is constant over the element per the modeling method.
The elemental shear force and moments can be arranged as

$$
\left[\begin{array}{c}
\mathrm{V}  \tag{C-14}\\
\mathrm{M}_{\mathrm{j}-1} \\
\mathrm{M}_{\mathrm{j}}
\end{array}\right]=\left(\frac{\mathrm{EI}}{\mathrm{~h}^{2}}\right)\left[\begin{array}{cccc}
12 / \mathrm{h} & 6 & -12 / \mathrm{h} & 6 \\
6 & 2 \mathrm{~h} & -6 & 4 \mathrm{~h} \\
-6 & -4 \mathrm{~h} & 6 & -2 \mathrm{~h}
\end{array}\right]\left[\begin{array}{c}
\mathrm{y}_{\mathrm{j}-1} \\
\theta_{\mathrm{j}-1} \\
y_{\mathrm{j}} \\
\theta_{j}
\end{array}\right]
$$

The shear and normal stresses can then be calculated from the shear force and bending moments in equation (C-2). Note that the force recovery matrix is embedded in equation (C-14).

## Example

Consider the rod from the main text.
The rod will have rotation and transverse displacement due to bending for this example. Longitudinal motion will be excluded.

The system is subjected to free vibration due to initial conditions.
The initial velocity is zero.
The initial displacement is equal to the first mode shape scaled so that the free end displacement is 0.010 inch. This initial displacement might be difficult to achieve in reality, but it is useful for demonstration purposes because the resulting response will be solely that of the fundamental mode.

The following analyses were performed using Matlab scripts: beam.m \& mdof_free.m.
Two models are analyzed. The first has 24 elements. The second has 48 elements.
The first, four modal frequencies from the 48 -element model are:

|  | Natural | Participation | Effective |
| :--- | ---: | :---: | :--- |
| Mode | Frequency | Factor | Modal Mass |
| 1 | 47.72 Hz | 0.05469 | 0.002991 |
| 2 | 299 Hz | 0.03035 | 0.0009209 |
| 3 | 837.3 Hz | 0.01775 | 0.0003151 |
| 4 | 1641 Hz | 0.01273 | 0.0001621 |

The moment and shear forces for the two models are shown in Table C-1 as calculated from equation (C-14). The results from two other methods are shown for comparison. The calculation details for the other methods are given after Table C-1.

| Table C-1. Maximum, Moment and Shear Results |  |  |  |
| :---: | :---: | :---: | :---: |
| Model | Maximum <br> Bending <br> Moment <br> (in lbf) | Maximum <br> Normal Stress <br> (psi) | Maximum <br> Shear Force <br> (lbf) |
| 24 Elements | 26.5 | 270 | 1.72 |
| 48 Elements | 28.2 | 287 | 1.72 |
| Stress-Velocity | - | 304 | - |
| Equivalent Static | 25.6 | 260 | 1.1 |
| Exact | 30.0 | 305 | 1.72 |

Notes

1. The maximum values occur at the fixed boundary.
2. The exact solution is taken from the continuous model in Reference 5.
3. The static deflection shape differs from the fundamental mode shape.

The 48-element results agree reasonably well with the exact results. Again, the results for the two finite element models were obtained via the force recovery matrix.


Figure C-3.

The first mode shape of the rod from the 48 -element model is shown. The absolute displacement is uncalibrated.


Figure C-4.


Figure C-5.

## Stress-Velocity Method

The maximum velocity is $3.0 \mathrm{in} / \mathrm{sec}$, which occurs at the free end of the rod.
The maximum stress can be calculated per the method in Reference 3. The characteristic impedance of aluminum bar is $\rho c=50.8 \mathrm{psi} \mathrm{sec} / \mathrm{in}$.

The peak stress is

$$
\begin{equation*}
\sigma_{\max }==\hat{\mathrm{k}} \rho \mathrm{c} \mathrm{v}_{\mathrm{n}, \max } \tag{C-15}
\end{equation*}
$$

Values for the $\hat{\mathrm{k}}$ constant for typical cross-sections are:

| Cross-section | $\hat{\mathrm{k}}$ |
| :--- | :---: |
| Solid Circular | 2 |
| Rectangular | $\sqrt{3}$ |

$$
\begin{equation*}
\left[\sigma_{\mathrm{n}}\right]_{\max }=\rho \mathrm{c} \mathrm{v}_{\mathrm{n}, \max }=2(50.8 \mathrm{psi} \mathrm{sec} / \mathrm{in})(3.0 \mathrm{in} / \mathrm{sec})=304 \mathrm{psi} \tag{C-16}
\end{equation*}
$$

## Equivalent Static Method

The stiffness at the free end of the rod is

$$
\begin{equation*}
\mathrm{k}=\frac{3 \mathrm{EI}}{\mathrm{~L}^{3}}=\frac{3\left(1.0 \mathrm{e}+07 \mathrm{lbf} / \mathrm{in}^{\wedge} 2\right)\left(0.0491 \mathrm{in}^{\wedge} 4\right)}{(24 \mathrm{in})^{3}}=106.5 \mathrm{lbf} / \mathrm{in} \tag{C-17}
\end{equation*}
$$

The equivalent force at the rod's free end for a 0.01 inch displacement at the free end is thus 1.07 inch.

The reaction moment at the rod's fixed end is 25.6 in-lbf, which is the force multiplied by the 24 inch length.


[^0]:    ${ }^{1}$ In practice, mesh density has an effect on the solution accuracy. A greater number of degrees-of-freedom tends to yield a more accurate solution until numerical error overcomes the benefit.

