MODAL GAIN FACTORS IN LAUNCH VEHICLE AUTOPILOT STABILITY ANALYSIS

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Variables

$[\mathrm{I}]$	=	Identity matrix
ξ	=	Modal damping ratio
ω	=	Natural frequency (rad/sec)
Ω	=	Forcing frequency (rad/sec)
q	=	Modal displacement
x	=	Physical displacement
$[\phi]$	=	Mass normalized eigenvector matrix
F	=	Applied force
H	=	Mobility transfer function
t	=	Time

Equation of Motion

Consider a launch vehicle represented in terms of a linearized structural dynamics equation. The equation includes bending modes primarily, but it may also contain slosh modes if the booster has liquid propellant.

The equation in decoupled modal coordinates is

$$[\mathrm{I}]\{\ddot{q}\}+[2\xi\omega]\{\dot{q}\}+\lfloor \omega^2 \rfloor\{q\} = [\phi]^T F \quad (1)$$

Assume a harmonic force.

$$F = \overline{F}(\Omega)\ \exp(j\,\Omega\, t) \tag{2}$$

The modal displacement is represented as.

$$q = \overline{q}\,(\Omega)\ \exp(j\,\Omega\, t) \tag{3}$$

The modal velocity is

$$\dot{q} = j\,\Omega\,\overline{q}\,(\Omega)\ \exp(j\,\Omega\, t) \tag{4}$$

The modal acceleration is

$$\ddot{q} = -\Omega^2\ \overline{q}\,(\Omega)\ \exp(j\,\Omega\, t) \tag{5}$$

By substitution,

$$\left\{-\,\Omega^2 + j2\xi\omega\Omega + \omega^2\right\}\left\{\overline{q}\,(\Omega)\ \exp(j\,\Omega\, t)\right\} = [\,\phi\,]^T\ \overline{F}(\Omega)\ \exp(j\,\Omega\, t) \tag{6}$$

$$\left\{\omega^2 - \Omega^2 + j2\xi\omega\Omega\right\}\overline{q}\,(\Omega)\ =\ [\,\phi\,]^T\,\overline{F}(\Omega) \tag{7}$$

$$\overline{q}\,(\Omega)\ =\ \frac{[\,\phi\,]^T\ \overline{F}(\Omega)}{\left\{\omega^2 - \Omega^2 + j2\xi\omega\Omega\right\}} \tag{8}$$

The largest displacement response occurs at resonance where $\omega = \Omega$.

$$\overline{q}_{\ max}\ =\ \frac{[\,\phi\,]^T\ \overline{F}}{j2\xi\omega^2} \tag{9}$$

Note that

$$\dot{\bar{q}}(\Omega) = j\,\Omega\,\bar{q}\,(\Omega) \tag{10}$$

At resonance,

$$\dot{\bar{q}}(\omega) = j\,\omega\,\bar{q}\,(\omega) \tag{11}$$

The corresponding velocity at resonance is

$$\dot{q}_{\,max} = \frac{[\,\phi\,]^T\,\bar{F}}{2\,\xi\omega} \tag{12}$$

The physical velocity is related to the modal velocity by

$$\{\dot{x}\} = [\,\phi\,]\{\dot{q}\} \tag{13}$$

By substitution,

$$\{\dot{x}\} = \frac{[\,\phi\,][\,\phi\,]^T\,\bar{F}}{2\,\xi\omega} \tag{14}$$

The modal gain factor matrix for a given mode r driven at resonance is

$$\frac{\phi_{\,r}\,\phi_r{}^T}{2\,\xi_r\,\omega_r} \tag{15}$$

Note that the gain factor varies with each pair of physical degrees-of-freedom.

The variation may be spatial. The variation may also be between a translational and rotational physical degree-of-freedom.

Furthermore, modes with higher gain factors tend to be more significant for control system design. Slosh modes are an exception, however.

Information for a specific application is given in Reference 1. The approach in Reference 1 shows the type of degrees-of-freedom to be included in the eigenvectors for a practical example.

The corresponding magnification factor for a given mode is

$$\frac{1}{2\xi} \tag{16}$$

Reference

1. T. Irvine, Notes on Mode Shapes and Bending Gain Units in Nastran, Vibrationdata, 2003.

APPENDIX A

Mobility Transfer Function

$$\bar{q}(\Omega) = \frac{[\phi]^T \,\bar{F}(\Omega)}{\left\{\omega^2 - \Omega^2 + j2\xi\omega\Omega\right\}} \tag{A-1}$$

$$\dot{\bar{q}}(\Omega) = \frac{j\,\Omega[\phi]^T \,\bar{F}(\Omega)}{\left\{\omega^2 - \Omega^2 + j2\xi\omega\Omega\right\}} \tag{A-2}$$

Furthermore,

$$\left\{\dot{\bar{x}}\right\} = [\phi]\left\{\dot{\bar{q}}\right\} = \frac{j\,\Omega[\phi][\phi]^T \,\bar{F}(\Omega)}{\omega^2 - \Omega^2 + j2\xi\omega\Omega} \tag{A-3}$$

The mobility transfer function (velocity/force) for between any pair physical locations is

$$H_{ij}(\Omega) = \sum_{r=1}^{n} \left\{\frac{j\,\Omega\,\phi_{ir}\,\phi_{jr}}{\omega_r{}^2 - \Omega^2 + j2\xi_r\omega_r\,\Omega}\right\} \tag{A-4}$$

The subscript r is mode index.

The number of modes is n.

The subscripts i and j refer to physical locations. Note that there is reciprocity. So i is the response location if j is the force input location, and vice-versa.

The mobility transfer function simplifies to

$$H_{ij}(\Omega) = j\,\Omega \sum_{r=1}^{n} \left\{\frac{\phi_{ir}\,\phi_{jr}}{\omega_r{}^2 - \Omega^2 + j2\xi_r\omega_r\,\Omega}\right\} \tag{A-5}$$