MODAL TRANSIENT RESPONSE TO ARBITRARY
BASE EXCITATION VIA CONVOLUTION
Revision A

By Tom Irvine
Email: tomirvine@aol.com

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The relative displacement substitution method is used for the case of a single-degree-freedom system subjected to base excitation. This substitution does not appear suitable, however, for the case of a multi-degree-of-freedom system subject to input from multiple points. Thus, the following approach is needed as an intermediate step. Assume that the base input may vary arbitrarily with time. Furthermore, assume that the base excitation energy is transmitted via spring elements only.

The non-homogeneous equation is

$$\ddot{\eta} + 2\,\xi\,\omega\,\dot{\eta} + \omega_n{}^2\ \eta = \quad \beta\, y(t) \tag{1}$$

Take the Laplace transform

$$L\left\{\ddot{\eta} + 2\,\xi\,\omega\,\dot{\eta} + \omega_n{}^2\ \eta\right\} = \beta L\{\, y(t)\} \tag{2}$$

$$\begin{aligned} & s^2 H(s) - s\eta(0) - \dot{\eta}(0) \\ & + 2\xi\omega_n s H(s) - 2\xi\omega_n \eta(0) \\ & + \omega_n{}^2 H(s) \ = \alpha L\{\,\dot{y}(t)\} + \beta L\{\, y(t)\} \end{aligned} \tag{3}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\} H(s) + \{-1\}\dot{\eta}(0) + \{-s - 2\xi\omega_n\}\eta(0) = \beta L\{\, y(t)\} \tag{4}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\} H(s) = \dot{\eta}(0) + \{s + 2\xi\omega_n\}\eta(0) \ + \beta L\{\, y(t)\} \tag{5}$$

$$H(s) = \left\{ \frac{\dot{\eta}(0) + [s + 2\xi\omega_n]\eta(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} + \left\{ \frac{\beta L\{ y(t)\}}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{6}$$

Let

$$H(s) = H_n(s) + H_f(s) \tag{7}$$

The natural response Laplace transform is

$$H_n(s) = \left\{ \frac{\dot{\eta}(0) + [s + 2\xi\omega_n]\eta(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{8}$$

The natural time history response is

$$\eta_n(t) = \eta(0)\exp(-\xi\omega_n t)\cos(\omega_d t) + \left\{ \frac{\dot{\eta}(0) + (\xi\omega_n)\eta(0)}{\omega_d} \right\} \exp(-\xi\omega_n t)\sin(\omega_d t) \tag{9}$$

where

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{10}$$

The forced response Laplace transform is

$$H_f(s) = \frac{\beta L\{ y(t)\}}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{11}$$

The forced response time history via a convolution integral is

$$\eta_f(t) = \left\{\frac{\beta}{\omega_d}\right\}\int_0^t \exp(-\xi\omega_n(t-\tau))\sin(\omega_d(t-\tau))y(\tau)d\tau \tag{12}$$

The integrals are changed to series for digital data.

$$\eta_f(t) = \left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m} \exp(-\xi\omega_n(t_m-\tau_i))\sin(\omega_d(t_m-\tau_i))\, y(\tau_i)\Delta\tau \tag{13}$$

The total modal displacement response time history is

$$\eta(t) =$$

$$\eta(0)\exp(-\xi\omega_n t)\cos(\omega_d t) + \left\{\frac{\dot{\eta}(0) + (\xi\omega_n)\eta(0)}{\omega_d}\right\}\exp(-\xi\omega_n t)\sin(\omega_d t)$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m} \exp(-\xi\omega_n(t_m-\tau_i))\sin(\omega_d(t_m-\tau_i))\, y(\tau_i)\Delta\tau \tag{14}$$

<u>References</u>

1. T. Irvine, Response of a Single-degree-of-Freedom System Subjected to a Classical Pulse Base Excitation, Rev A, Vibrationdata, 1999.
2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata, Rev E, 2008.
3. T. Irvine, Table of Laplace Transforms, Rev E, Vibrationdata, 2000.
4. T. Irvine, An Introduction to the Shock Response Spectrum, Rev P, Vibrationdata, 2002.

APPENDIX A

Modal Velocity

Again, the total modal displacement response time history is

$$\eta(t) =$$

$$\eta(0)\exp\left(-\xi\omega_n t\right)\cos\left(\omega_d t\right) + \left\{\frac{\dot{\eta}(0) + \left(\xi\omega_n\right)\eta(0)}{\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp\left(-\xi\omega_n (t_m - \tau_i)\right)\sin\left(\omega_d (t_m - \tau_i)\right) y(\tau_i)\Delta\tau$$

(A-1)

The total modal velocity response time history is

$$\frac{d}{dt}\eta(t) =$$

$$\eta(0)\exp\left(-\xi\omega_n t\right)\left[-\xi\omega_n\cos\left(\omega_d t\right) - \omega_d\sin\left(\omega_d t\right)\right]$$

$$+\left\{\frac{\dot{\eta}(0) + \left(\xi\omega_n\right)\eta(0)}{\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\left[-\xi\omega_n\sin\left(\omega_d t\right) + \omega_d\cos\left(\omega_d t\right)\right]$$

$$+\frac{d}{dt}\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp\left(-\xi\omega_n (t_m - \tau_i)\right)\sin\left(\omega_d (t_m - \tau_i)\right) y(\tau_i)\Delta\tau$$

(A-2)

$$\frac{d}{dt}\eta(t) =$$

$$\eta(0)\exp(-\xi\omega_n t)\left[-\xi\omega_n \cos(\omega_d t) - \omega_d \sin(\omega_d t)\right]$$

$$+\left\{\frac{\dot{\eta}(0) + (\xi\omega_n)\eta(0)}{\omega_d}\right\}\exp(-\xi\omega_n t)\left[-\xi\omega_n \sin(\omega_d t) + \omega_d \cos(\omega_d t)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m - \tau_i))\left\{-\xi\omega_n \sin(\omega_d(t_m - \tau_i)) + \omega_d \cos(\omega_d(t_m - \tau_i))\right\} y(\tau_i)\Delta\tau$$

(A-3)

$$\frac{d}{dt}\eta(t) =$$

$$\exp(-\xi\omega_n t)\left[\dot{\eta}(0)\cos(\omega_d t) - \left[\omega_d\eta(0) + \frac{\xi\omega_n}{\omega_d}(\dot{\eta}(0) + (\xi\omega_n)\eta(0))\right]\sin(\omega_d t)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m - \tau_i))\left\{-\xi\omega_n \sin(\omega_d(t_m - \tau_i)) + \omega_d \cos(\omega_d(t_m - \tau_i))\right\} y(\tau_i)\Delta\tau$$

(A-4)

APPENDIX B

Modal Acceleration

Again, the total velocity displacement response time history is

$$\frac{d}{dt}\eta(t) =$$

$$\exp(-\xi\omega_n t)\left[\dot{\eta}(0)\cos(\omega_d t) - \left[\omega_d \eta(0) + \frac{\xi\omega_n}{\omega_d}\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\sin(\omega_d t)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m-\tau_i))\left\{-\xi\omega_n\sin(\omega_d(t_m-\tau_i)) + \omega_d\cos(\omega_d(t_m-\tau_i))\right\}y(\tau_i)\Delta\tau$$

(B-1)

The total modal acceleration response time history is

$$\frac{d^2}{dt^2}\eta(t) =$$

$$-\xi\omega_n\exp(-\xi\omega_n t)\left[\dot{\eta}(0)\cos(\omega_d t) - \left[\omega_d \eta(0) + \frac{\xi\omega_n}{\omega_d}\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\sin(\omega_d t)\right]$$

$$+\exp(-\xi\omega_n t)\left[-\omega_d\dot{\eta}(0)\sin(\omega_d t) - \left[\omega_d^2\eta(0) + \xi\omega_n\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\cos(\omega_d t)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\frac{d}{dt}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m-\tau_i))\left\{-\xi\omega_n\sin(\omega_d(t_m-\tau_i)) + \omega_d\cos(\omega_d(t_m-\tau_i))\right\}y(\tau_i)\Delta\tau$$

(B-2)

$$\frac{d^2}{dt^2}\eta(t) =$$

$$-\xi\omega_n \exp(-\xi\omega_n t)\left[\dot{\eta}(0)\cos(\omega_d t) - \left[\omega_d \eta(0) + \frac{\xi\omega_n}{\omega_d}\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\sin(\omega_d t)\right]$$

$$+\exp(-\xi\omega_n t)\left[-\omega_d \dot{\eta}(0)\sin(\omega_d t) - \left[\omega_d{}^2\eta(0) + \xi\omega_n\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\cos(\omega_d t)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m-\tau_i))\{-\xi\omega_n\}\{-\xi\omega_n\sin(\omega_d(t_m-\tau_i)) + \omega_d\cos(\omega_d(t_m-\tau_i))\}\,y(\tau_i)\Delta\tau$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m-\tau_i))\left\{-\xi\omega_n\omega_d\cos(\omega_d(t_m-\tau_i)) - \omega_d{}^2\sin(\omega_d(t_m-\tau_i))\right\}y(\tau_i)\Delta\tau$$

(B-3)

$$\frac{d^2}{dt^2}\eta(t) =$$

$$-\xi\omega_n \exp(-\xi\omega_n t)\left[\dot{\eta}(0)\cos(\omega_d t) - \left[\omega_d \eta(0) + \frac{\xi\omega_n}{\omega_d}\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\sin(\omega_d t)\right]$$

$$+\exp(-\xi\omega_n t)\left[-\omega_d \dot{\eta}(0)\sin(\omega_d t) - \left[\omega_d{}^2\eta(0) + \xi\omega_n\left(\dot{\eta}(0) + (\xi\omega_n)\eta(0)\right)\right]\cos(\omega_d t)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m-\tau_i))\left\{\xi^2\omega_n{}^2\sin(\omega_d(t_m-\tau_i)) - \xi\omega_n\omega_d\cos(\omega_d(t_m-\tau_i))\right\}y(\tau_i)\Delta\tau$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp(-\xi\omega_n(t_m-\tau_i))\left\{-\xi\omega_n\omega_d\cos(\omega_d(t_m-\tau_i)) - \omega_d{}^2\sin(\omega_d(t_m-\tau_i))\right\}y(\tau_i)\Delta\tau$$

(B-4)

$$\frac{d^2}{dt^2}\eta(t) =$$

$$-\xi\omega_n \exp\left(-\xi\omega_n t\right)\left[\dot{\eta}(0)\cos\left(\omega_d t\right) - \left[\omega_d \eta(0) + \frac{\xi\omega_n}{\omega_d}\left(\dot{\eta}(0) + \left(\xi\omega_n\right)\eta(0)\right)\right]\sin\left(\omega_d t\right)\right]$$

$$+\exp\left(-\xi\omega_n t\right)\left[-\omega_d \dot{\eta}(0)\sin\left(\omega_d t\right) - \left[\omega_d{}^2 \eta(0) + \xi\omega_n\left(\dot{\eta}(0) + \left(\xi\omega_n\right)\eta(0)\right)\right]\cos\left(\omega_d t\right)\right]$$

$$+\left\{\frac{\beta}{\omega_d}\right\}\sum_{i=1}^{m}\exp\left(-\xi\omega_n\left(t_m - \tau_i\right)\right)\left\{\left[\left(\xi\omega_n\right)^2 - \omega_d{}^2\right]\sin\left(\omega_d\left(t_m - \tau_i\right)\right) - 2\xi\omega_n\omega_d\cos\left(\omega_d\left(t_m - \tau_i\right)\right)\right\}y(\tau_i)\Delta\tau$$

(B-5)

An alternate method is to calculate the modal acceleration via:

$$\ddot{\eta} = -2\xi\omega\dot{\eta} - \omega_n{}^2\,\eta + \beta\,y(t) \qquad \text{(B-6)}$$