FREE VIBRATION OF A SINGLE-DEGREE-OF-FREEDOM
SYSTEM WITH NONLINEAR STIFFNESS
Revision B

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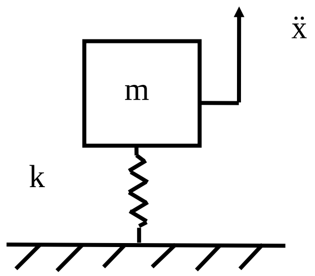
Consider a single-degree-of-freedom system.



where

m	is the mass
k	is the nonlinear stiffness term
x	is the absolute displacement of the mass

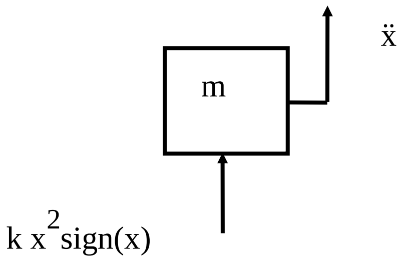
The spring force F for this example is

$$F = k\,x^2\,\mathrm{sign}(x) \tag{1}$$

where

$$\mathrm{sign}(x) = \frac{x}{|x|} \tag{2}$$

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{3}$$

$$m\ddot{x} = -k\,x^2\,\mathrm{sign}(x) \tag{4}$$

$$m\ddot{x} + k\,x^2\,\mathrm{sign}(x) = 0 \tag{5}$$

Divide through by m,

$$\ddot{x} + (k/m)\,x^2\,\mathrm{sign}(x) = 0 \tag{6}$$

$$\ddot{x} + \omega^2\,x^2\,\mathrm{sign}(x) = 0 \tag{7}$$

where

$$\omega = \sqrt{\frac{k}{m}} \tag{8}$$

A numerical method, such as the Runge-Kutta method, is required to solve equation (7), because a closed-form solution does not exist.

The following pair of equations for the period τ of a fully cycle are taken from Reference 1, equation (2.7).

$$\ddot{x} + \omega^2\,f(x) = 0 \tag{9a}$$

$$\tau = \frac{4}{\omega}\int_0^{u_m} \frac{du}{\sqrt{2\int_u^{u_m} f(u')du'}} \tag{9b}$$

where

u	is the displacement
u_m	is the maximum displacement

The maximum displacement must be found numerically solving equation (7).

For the sample problem,

$$f(u') = (u')^2 \operatorname{sign}(u') \tag{10}$$

The sign term can be omitted since the function will be positive values for the integration limits considered.

$$\int_u^{u_m} f(u')du' = \int_u^{u_m} (u')^2 du' \tag{11}$$

$$\int_u^{u_m} f(u')du' = \frac{1}{3}(u')^3 \Big|_u^{u_m} \tag{12}$$

$$\int_u^{u_m} f(u')du' = \frac{1}{3}\left[(u_m)^3 - (u)^3\right] \tag{13}$$

$$\tau = \frac{4}{\omega}\int_0^{u_m} \frac{du}{\sqrt{\frac{2}{3}\left[(u_m)^3 - (u)^3\right]}} \tag{14}$$

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{2}}\int_0^{u_m}\left\{\frac{du}{\sqrt{(u_m)^3-(u)^3}}\right\} \tag{15}$$

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{2}}\int_0^{u_m}\left\{\frac{du}{(u_m)^{3/2}\sqrt{1-(u/u_m)^3}}\right\} \tag{16}$$

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{2}}\,(u_m)^{-3/2}\int_0^{u_m}\left\{\frac{du}{\sqrt{1-(u/u_m)^3}}\right\} \tag{17}$$

Change the integration variable.

$$\xi = \left(\frac{u}{u_m}\right) \tag{18}$$

$$d\xi = \left(\frac{du}{u_m}\right) \tag{19}$$

$$u_m\, d\xi = du \tag{20}$$

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{2}}(u_m)^{-3/2}\int_0^1\left\{\frac{u_m d\xi}{\sqrt{1-\xi^3}}\right\} \tag{21}$$

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{2}}(u_m)^{-1/2}\int_0^1\left\{\frac{d\xi}{\sqrt{1-\xi^3}}\right\} \tag{22}$$

The integral is evaluated numerically,

$$\int_0^1 \left\{ \frac{d\xi}{\sqrt{1-\xi^3}} \right\} \approx 1.402 \tag{23}$$

Substitute equation (24) into (23).

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{2}}\frac{1}{\sqrt{u_m}}(1.402) \tag{24}$$

The natural period for the sample problem is thus

$$\tau \approx 6.87\ \frac{1}{\omega}\frac{1}{\sqrt{u_m}} \tag{25}$$

A larger maximum displacement yields a shorter period.

<u>Example</u>

A system is governed by equation (5).

m	= 1 lbm
k	= 10,000 lbf/in^2
damping	= 0.02

The system is subject to an initial displacement of 2 inches. (This may be unrealistic for a practical system, but it is useful for demonstration purposes.)

The resulting displacement is shown in Figure 1, as calculated via Matlab script: RK4_fv_nl1.m

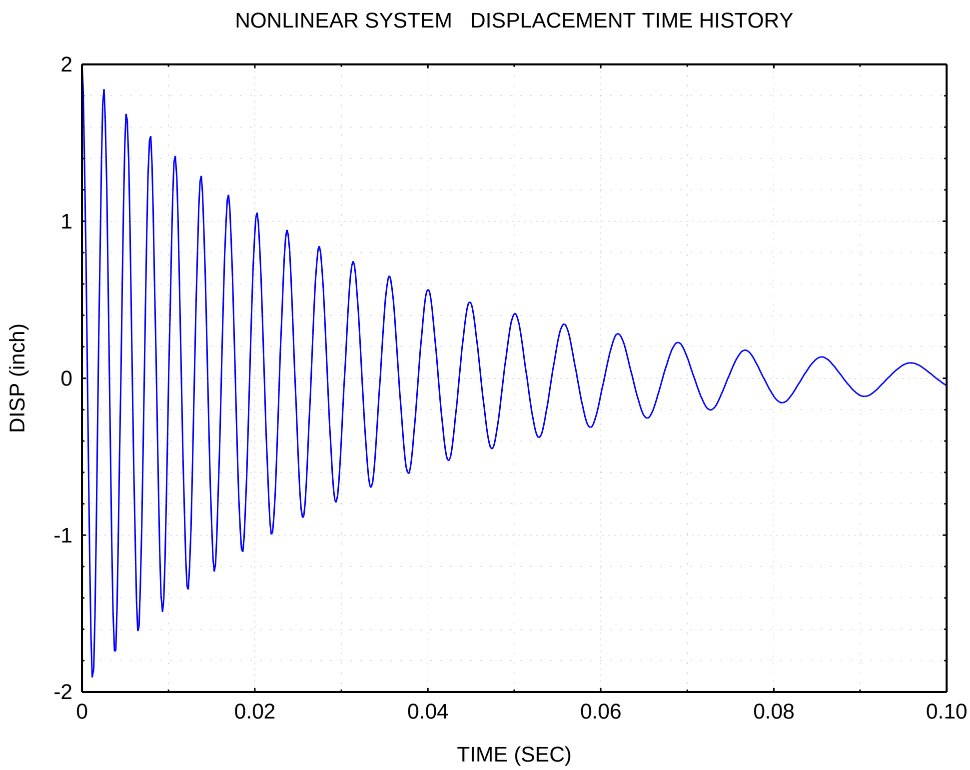


Figure 1.

Additional Systems

An additional case is analyzed in Appendix A.

Reference

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.

APPENDIX A

Consider a system governed by the following equation.

$$\ddot{x} + \omega^2 \sqrt{x\,\mathrm{sign}(x)} = 0 \tag{A-1}$$

Recall the equation for the period τ of a fully cycle.

$$\ddot{x} + \omega^2 f(x) = 0 \tag{A-2}$$

$$\tau = \frac{4}{\omega}\int_0^{u_m} \frac{du}{\sqrt{2\int_u^{u_m} f(u')du'}} \tag{A-3}$$

The maximum displacement must be found numerically solving equation (A-1).

For the sample problem,

$$f(u') = \sqrt{(u')} \tag{A-4}$$

The sign term is omitted from equation (A-4) because the function will be positive over the integration limits considered.

$$\int_u^{u_m} f(u')du' = \int_u^{u_m} \sqrt{(u')}\,du' \tag{A-5}$$

$$\int_u^{u_m} f(u')du' = \frac{2}{3}(u')^{3/2}\Big|_u^{u_m} \tag{A-6}$$

$$\int_{u}^{u_m} f(u')du' = \frac{2}{3}\left[(u_m)^{3/2} - (u)^{3/2}\right] \qquad \text{(A-7)}$$

$$\tau = \frac{4}{\omega}\int_{0}^{u_m} \frac{du}{\sqrt{\frac{4}{3}\left[(u_m)^{3/2} - (u)^{3/2}\right]}} \qquad \text{(A-8)}$$

$$\tau = \frac{4}{\omega}\sqrt{\frac{3}{4}}\int_{0}^{u_m} \left\{\frac{du}{\sqrt{\left[(u_m)^{3/2} - (u)^{3/2}\right]}}\right\} \qquad \text{(A-9)}$$

$$\tau = \frac{2}{\omega}\sqrt{3}\int_{0}^{u_m} \left\{\frac{du}{(u_m)^{3/4}\sqrt{1-(u/u_m)^{3/2}}}\right\} \qquad \text{(A-10)}$$

$$\tau = \frac{2}{\omega}\sqrt{3}\,(u_m)^{-3/4}\int_{0}^{u_m} \left\{\frac{du}{\sqrt{1-(u/u_m)^{3/2}}}\right\} \qquad \text{(A-11)}$$

Change the integration variable.

$$\xi = \left(\frac{u}{u_m}\right) \qquad \text{(A-12)}$$

$$d\xi = \left(\frac{du}{u_m}\right) \qquad \text{(A-13)}$$

$$u_m\, d\xi = du \qquad \text{(A-14)}$$

$$\tau = \frac{2}{\omega}\sqrt{3}(u_m)^{-3/4} \int_0^1 \left\{ \frac{u_m d\xi}{\sqrt{1-\xi^{3/2}}} \right\} \tag{A-15}$$

$$\tau = \frac{2}{\omega}\sqrt{3}\,(u_m)^{1/4} \int_0^1 \left\{ \frac{d\xi}{\sqrt{1-\xi^{3/2}}} \right\} \tag{A-16}$$

The integral is evaluated numerically,

$$\int_0^1 \left\{ \frac{d\xi}{\sqrt{1-\xi^{3/2}}} \right\} \approx 1.725 \tag{A-17}$$

By substitution,

$$\tau = \frac{2}{\omega}\sqrt{3}(u_m)^{1/4} \ \ (1.725) \tag{A-18}$$

The natural period for the sample problem is thus

$$\tau \approx 5.975 \ \frac{1}{\omega}(u_m)^{1/4} \tag{A-19}$$

A larger peak displacement yields a longer period.

The period approaches zero as the peak displacement approaches zero.