Notes on Nonlinear Oscillators

These notes are a supplement to the textbook's Chapter 4. The main purpose is to introduce you to just a couple of the fascinating things that happen in nonlinear systems, and to develop a simple but effective tool known as "harmonic balance" to quantitatively understand some of what happens.

The Jump Phenomenon: Numerical Investigation

We saw for the linear driven oscillator that the amplitude of the response was a continuous function of the various system parameters (drive amplitude and frequency, for example). Something very different can happen when we consider driven nonlinear oscillators. One new possibility is the "jump phenomenon", where the response amplitude changes suddenly at some critical value of the control parameter (this is our first example of a *bifurcation*). This behavior is accompanied by another new feature, namely *hysteresis*.

Equation of motion

Consider a plane pendulum, in the presence of a driving torque; ignore the damping force for now. The equation of motion is:

$$I\ddot{\phi} = -mgL\sin\phi + A\cos\omega t \tag{1}$$

where I is the moment of inertia, m is the bob mass, L is the rod length, g is the acceleration due to gravity, and ϕ is the angle the rod makes with respect to the vertical. For small angle motion, one can Taylor expand the sine term:

$$\sin\phi = \phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \cdots$$

Keeping only the first term yields the harmonic oscillator equation. Here, we'll keep the second term too, and see what effect the nonlinearity has on the resulting motion. Our equation becomes, after dividing through by I, and including the usual damping term:

$$\ddot{\phi} + 2\beta\dot{\phi} + {\omega_0}^2(\phi - \frac{1}{6}\phi^3) = F\cos\omega t \tag{2}$$

where ${\omega_0}^2 = \frac{mgL}{I}$ and $F = A/I$.

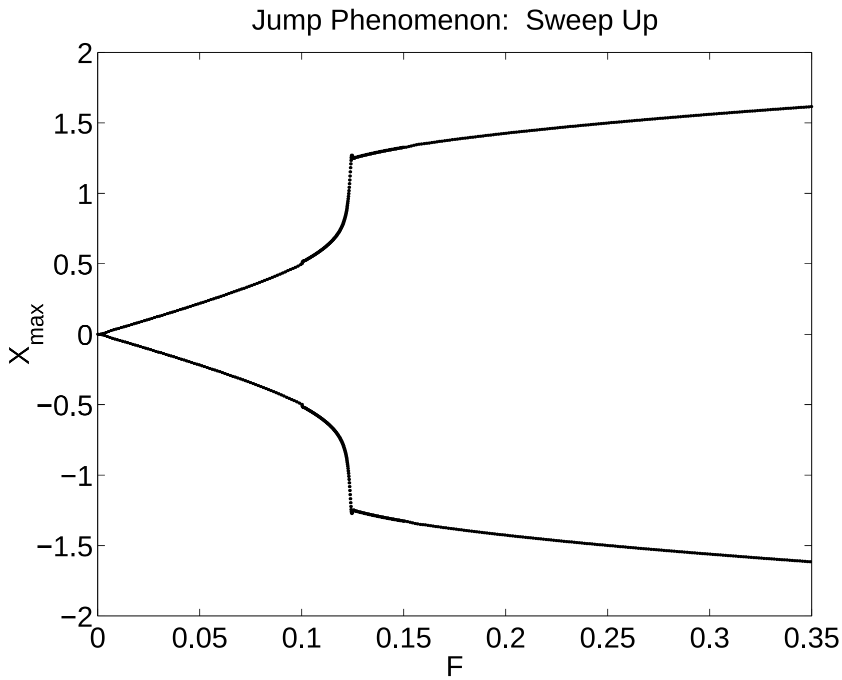

Figure 1:

Numerical Simulations

Here are the results of numerical simulations of this equation with parameter values $\gamma = 0.05$, $\omega_0 = 1.1$, $\omega = 1.0$, and $F \in [0, 0.35]$. Initially, with the system in equilibrium and at rest, the forcing amplitude is zero. As I very slowly ramp up F, the response amplitude grows. At first the response amplitude is linearly proportional to F, but as the nonlinearity becomes important, the growth is a bit faster than linear. Suddenly, at a critical value of F, the response jumps to a large value. As F continues to increase, the response increases, but now relatively slowly.

What happens if we lower the driving force? There is another "jump point", but it is somewhat lower than the first one. The result of the numerical simulation is shown on the next graph.

Finally, take a look at the two figures superimposed. There is a hysteresis loop: for values

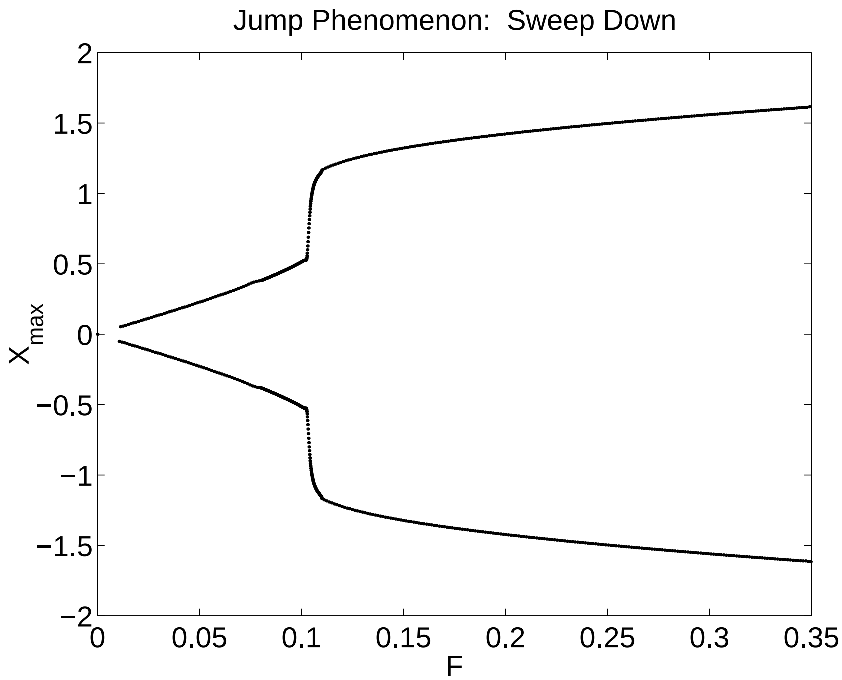

Figure 2:

of F between the two jump points, the system response can have two different amplitudes. Which one the system chooses depends on the initial conditions. This is an example of a rather common feature of nonlinear systems, namely *coexisting attractors*. This is impossible in a linear system.

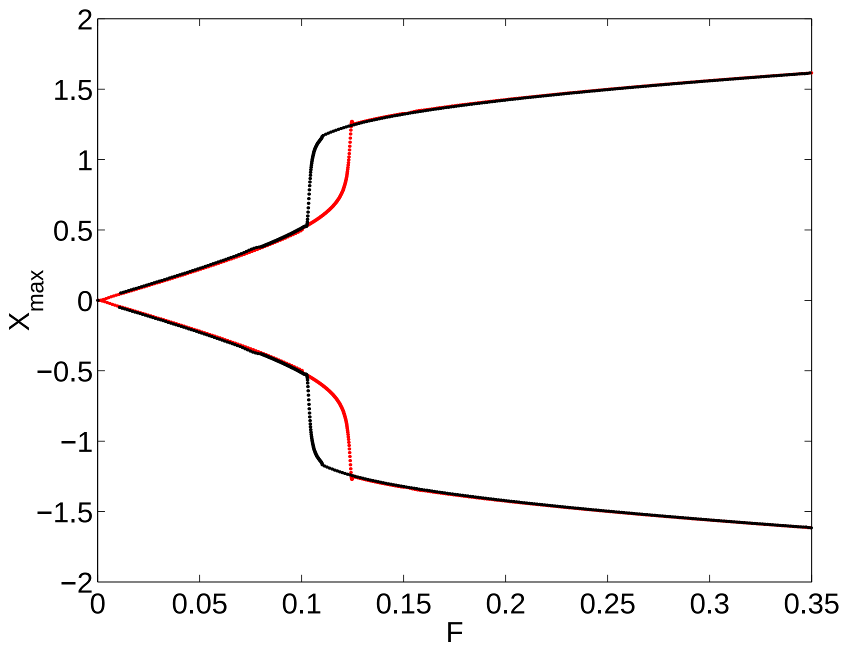

Figure 3:

The Method of Harmonic Balance

This is an approximate method which is similar to the Fourier series method we used for the linear oscillator. We can use it to investigate periodic solutions. The steps are:

1. Assume the solution can be represented as a truncated Fourier series.

2. Substitute the assumed solution into the equation of motion, and expand each term as a Fourier series.

3. Throw out any higher-frequency harmonics which are not included in the original assumed solution.

4. Balance coefficients for each Fourier term (harmonic). This leads to a set of algebraic equations.

5. Solve the algebraic equations.

Let's apply this method to the pendulum problem. For simplicity, ignore the damping term, so that

$$\ddot{\phi} + {\omega_0}^2(\phi - \frac{1}{6}\phi^3) = F \cos \omega t$$

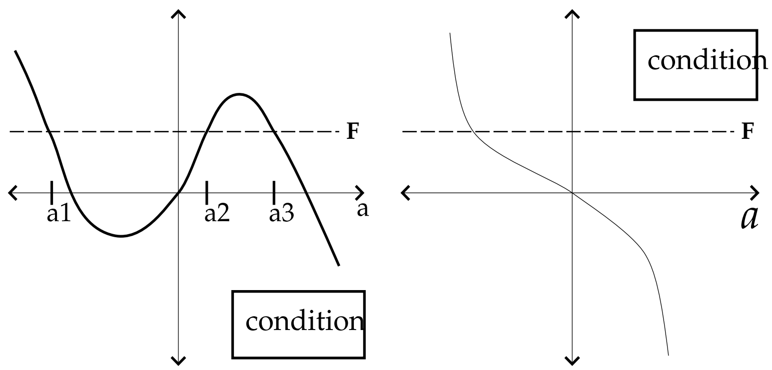

Figure 4: Solving the cubic graphically

Assume a solution of this form:

$$\phi(t) = a \cos \omega t$$

Our goal is to determine the allowed value(s) of a. We have:

$$\begin{aligned} \ddot{\phi} &= -a\omega^2 \cos \omega t \\ \phi^3 &= a \cos^3 \omega t = a^3 \left(\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \right) \end{aligned}$$

Substitute these into the differential equation and throw out the higher harmonic $\cos 3\omega t$. The remaining terms all oscillate as $\cos \omega t$. This common factor can be factored out with the result:

$$-a\omega^2 + {\omega_0}^2 (a - \frac{3}{(6)(4)} a^3) = F$$

$$({\omega_0}^2 - \omega^2) a - \frac{1}{8} {\omega_0}^2 a^3 = F \tag{3}$$

This is a cubic equation for the amplitude a, which may have either one or three real solutions. Now, it is possible to write down an exact formula for the solutions of this equation, though the expression is pretty cumbersome. Let's see what we can learn by trying to solve the cubic graphically, by plotting the lefthand and righthand sides of the Eq.(3) vs. a. There are two different cases, depending on the sign of $({\omega_0}^2 - \omega^2)$, shown in the figure.

We see that if $\omega^2 > {\omega_0}^2$, there is a unique solution, which is "just like" the linear oscillator problem. However, if $\omega^2 < {\omega_0}^2$, we can have either one or three solutions depending on the value of F!

Suppose we imagine slowly increasing the drive amplitude F, and measuring the response amplitude a. If $\omega^2 < {\omega_0}^2$ we get a surprising result. Assume the system starts in the low-amplitude state (a_2 in the figure 4), then as F increases from zero, the response increases, until F reaches a critical value F_C , beyond which the solution branch a_2 is missing! What happens? One possibility is that the system evolves to the solution branch with $a = a_1$, which is the only branch remaining. But there is more: if we now decrease F back past the value F_C, we don't expect the system to "jump back", since nothing special happens to the solution branch a_1 at that point. This phenomenon, where the output depends not just on the system parameters but on the previous history of the system is called *hysteresis*, and always accompanies this kind of "jump phenomenon".

We can calculate F_c by noting that it corresponds to the point where the cubic has a local maximum. In fact, F_c is equal to the local maximum of the cubic. We locate the critical value of a by setting $\partial F/\partial a = 0$ in Eq.(3:

$$\omega_0^2 - \omega^2 - \frac{3}{8}\omega_0^2 a^2 = 0$$

so that at the critical point

$$a = \sqrt{\frac{8}{3}\ (1 - \omega^2/\omega_0^2)}$$

and the corresponding value of F_c is obtained by plugging this back into Eq.(3).

The results of our thought experiment are summarized in figure **??**.

Harmonic Balance and Fourier Analysis

The approximation used above was a very simple application of the method called *harmonic balance*. This method is closely related to the method of Fourier Analysis: since we are looking for a periodic solution, we can write:

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

Fourier Analysis allows one to find an exact solution to linear equations with constant coefficients, by turning the differential equation into a set of *uncoupled linear* algebraic equations, one for each harmonic. The problem encountered for nonlinear equations is that

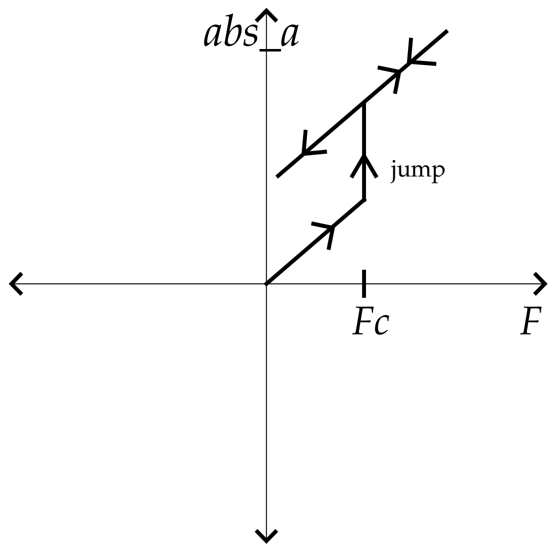

Figure 5: Results of our thought experiment

(as we have seen) the nonlinearities couple together these different harmonics, leading to an infinite number of *coupled nonlinear* algebraic equations. The method of harmonic balance simply truncates the Fourier series at some arbitrary (usually low) order, and throws out any unwanted harmonics generated by the nonlinearities. Although the approximation is thus "uncontrolled", it can be systematically improved by including more terms in the Fourier expansion.

The van der Pol Oscillator

The van der Pol oscillator was originally[1] studied as a model for an electronic device called a "triode", which can sustain voltage oscillations without benefit of any external driving input. The van der Pol equation is:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + {\omega_0}^2 x = 0 \tag{4}$$

You can also think of this equation as describing a mechanical oscillator with a nonlinear friction term which provides damping when $x^2 > 1$, and "anti-damping" when $x^2 < 1$. If you numerically integrate the van der Pol equation, you find that there is one unstable equilibrium point (at $x = \dot{x} = 0$). All other initial conditions settle down, after a transient,

[1]see: *Philosophical Magazine*, Volume 3, pages 65-80, (1927).

to a unique periodic oscillation. This periodic solution is called an *attractor*; a periodic attractor is also sometimes called a *limit cycle*.

Let's try to calculate the amplitude and frequency of the steady state periodic solution. In fact, no one has ever succeeded in writing down a formula for the exact solution to the van der Pol equation. So let's find an approximate solution using harmonic balance.

Begin by assuming $x(t) = a\cos\omega t$. Our goal is to find a and ω in terms of the system parameters ε and ω_0. Note that our assumed solution allows for the possibility that the output frequency ω is different from the "natural frequency" ω_0. Now,

$$\begin{aligned}
\dot{x} &= -a\omega\sin\omega t \\
\dot{x}x^2 &= -\frac{1}{4}a^3\omega(\sin\omega t + \sin 3\omega t) \\
\ddot{x} &= -a\omega^2\cos\omega t
\end{aligned}$$

Putting all this into Eq.(4) yields:

$$-a\omega^2\cos\omega t - \frac{\varepsilon a^3\omega}{4}(\sin\omega t + \sin 3\omega t) + \varepsilon a\omega\sin\omega t + a{\omega_0}^2\cos\omega t = 0$$

Next, we throw out the higher harmonic term $\sin 3\omega t$, so that

$$(a{\omega_0}^2 - a\omega^2)\cos\omega t + (\varepsilon a\omega - \frac{1}{4}\varepsilon a^3\omega)\sin\omega t = 0$$

The only way this can be true for all values of time t is for the coefficients to separately balance to zero, which implies that:

$$a = 2 \text{ and } \omega = \omega_0$$

(There is also a solution with $a = 0$, but this is just the fixed point solution $\dot{x} = x = 0$.) Note that, to this level of approximation, the oscillation amplitude is independent of the oscillation frequency.