NORMAL TOLERANCE FACTORS FOR UPPER TOLERANCE LIMITS

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The following tables and formulas are for one-sided intervals.

Introduction

This tutorial is an extension of Reference 1. The K factor calculation method presented in this tutorial is more accurate than the formula given in Reference 1.

The upper limit tolerance factor can be used to predict a maximum flight level from flight or test data. The sample mean and standard deviation are also required for this calculation.

The resulting level is sometimes referred to as the maximum predicted environment (MPE) or maximum expected flight level (MEFL). The level may represent a sound pressure level, a power spectral density, or a shock response spectrum.

The problem is to find a tolerance factor K that will yield an upper limit which covers at least 100β percent of the population with a confidence of 100γ percent as expressed by

$$P[\,F(\bar{x} + Ks) \geq \beta] = \gamma \tag{1}$$

where F is a cumulative distribution function.

The tolerance method assumes

1. The population has either a normal or a log-normal distribution.
2. The samples are selected at random.

The sample mean $\bar{x}$ is defined as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{2}$$

where

n is the number of samples

x_i individual sample values

The sample standard deviation s is defined as

$$s = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2} \tag{3}$$

The upper limit for the MPE or MEFL is thus

$$\text{Limit} = \bar{x} + Ks \tag{4}$$

Note that each of the parameters in equation (4) may vary with frequency for dynamic data.

A sample military reference which requires this approach is given in Appendix A.

<u>Tolerance Factor Equation</u>

A number of approximation formulas are given in various references. A drawback is that these formulas tend to give errors for cases where there are only a few samples.

The following approach gives a formula which is exact in theory, although series summations are required to evaluate some of the embedded terms.

Define a noncentrality parameter δ.

$$\delta = \sqrt{n}\, Z_p \tag{5}$$

where

Z_p is the Z-value abscissa corresponding to the cumulative probability area p under the normal distribution curve

The number of statistical degrees of freedom ν is

$$\nu = n - 1 \tag{6}$$

Let $T_{\nu,\delta}$ be the abscissa value corresponding to the cumulative area under the noncentral t-distribution curve with the area set equal to the confidence level γ.

The tolerance factor K for the upper limit calculation is

$$K = \frac{1}{\sqrt{n}} T_{\nu\,,\,\delta} \tag{7}$$

Equation (7) is taken from Reference 1.

The cumulative probability density function for the noncentral t-distribution is given in Appendix B.

Examples

The following calculations were made using Matlab script: k_factor.m

This script is available from the author.

It is particularly useful for cases where tabular reference data is unavailable.

Table 1. P95/50 for Three Samples, Tolerance Factor Calculation			
Description	Symbol	Value	Source
Samples	n	3	Given
Degrees-of-Freedom	ν	2	Equation (6)
Probability	P	95%	Given
Confidence	γ	50%	Given
Z limit corresponding to P area in the Normal Distribution curve	Z_p	1.645	Reference 5
Noncentrality Parameter	δ	2.849	Equation (5)
Noncentral T-distribution Abscissa	$T_{\nu\,,\,\delta}$	3.357	Appendix A
Tolerance Factor	K	1.938	Equation (7)

Table 2. P99/90 for Ten Samples, Tolerance Factor Calculation			
Description	Symbol	Value	Source
Samples	n	10	Given
Degrees-of-Freedom	ν	9	Equation (6)
Probability	P	99%	Given
Confidence	γ	90%	Given
Z limit corresponding to P area in the Normal Distribution curve	Z_p	2.326	Reference 5
Noncentrality Parameter	δ	7.357	Equation (5)
Noncentral T-distribution Abscissa	$T_{\nu,\delta}$	11.17	Appendix A
Tolerance Factor	K	3.532	Equation (7)

Table 3. P90/75 for Ten Samples, Tolerance Factor Calculation			
Description	Symbol	Value	Source
Samples	n	15	Given
Degrees-of-Freedom	ν	14	Equation (6)
Probability	P	90%	Given
Confidence	γ	75%	Given
Z limit corresponding to P area in the Normal Distribution curve	Z_p	1.282	Reference 5
Noncentrality Parameter	δ	4.963	Equation (5)
Noncentral T-distribution Abscissa	$T_{\nu,\delta}$	6.109	Appendix A
Tolerance Factor	K	1.577	Equation (7)

The K values in tables 1 through 3 agree with the reference data in the next section.

Reference Tables

Table 4. Normal Distribution Tolerance Factors for Upper Tolerance Limit, from Reference 3

	γ=0.50			γ=0.75			γ=0.90		
n	β=0.90	β=0.95	β=0.99	β=0.90	β=0.95	β=0.99	β=0.90	β=0.95	β=0.99
3	1.50	1.94	2.76	2.50	3.15	4.40	4.26	5.31	7.34
4	1.42	1.83	2.60	2.13	2.68	3.73	3.19	3.96	5.44
5	1.38	1.78	2.53	1.96	2.46	3.42	2.74	3.40	4.67
7	1.35	1.73	2.46	1.79	2.25	3.13	2.33	2.89	3.97
10	1.33	1.71	2.42	1.67	2.10	2.93	2.06	2.57	3.53
15	1.31	1.68	2.39	1.58	1.99	2.78	1.87	2.33	3.21
20	1.30	1.67	2.37	1.53	1.93	2.70	1.76	2.21	3.05
30	1.29	1.66	2.35	1.48	1.87	2.61	1.66	2.08	2.88
50	1.29	1.65	2.34	1.43	1.81	2.54	1.56	1.96	2.74
∞	1.28	1.64	2.33	1.28	1.64	2.33	1.28	1.64	2.33

Table 5. P95/50 Normal Distribution Tolerance Factors for Upper Tolerance Limit, from Reference 4

n	K-factor
2	2.339
3	1.939
4	1.830
5	1.779
6	1.750
7	1.732
8	1.719
9	1.709
10	1.702
11	1.696
12	1.691
13	1.687
14	1.684
15	1.681

n	K-factor
16	1.678
17	1.676
18	1.674
19	1.673
20	1.671
21	1.670
22	1.669
23	1.668
24	1.667
25	1.666
30	1.662
35	1.659
40	1.658
45	1.656
∞	1.64485

References

1. T. Irvine, P95/50 Rule – Theory and Application, Vibrationdata, 1996.
2. C. Link, An Equation for One-Sided Tolerance Limits for Normal Distributions, U.S. Department of Agriculture, Research Paper FPL 458, 1985. (Available from: http://www.fpl.fs.fed.us/documnts/fplrp/fplrp458.pdf).
3. Cyril Harris, editor; Shock and Vibration Handbook, 4th edition, McGraw-Hill, New York, 1995. See Chapter 20, Test Criteria and Specifications, Allan G. Piersol.
4. H. Himelblau, et al, "Development of Cassini Acoustic Criteria using Titan IV Flight Data," IES ATM, May 1992.
5. T. Irvine, Integration of the Normal Distribution Curve, Vibrationdata, 1999.
6. MIL-HDBK-340A (USAF), Department of Defense Handbook, Test Requirements for Launch, Upper Stage, and Space Vehicles, Washington, D.C., 2000.
7. NASA-HDBK-7005, Dynamic Environmental Criteria, 2001. See Chapter 6.

APPENDIX A

Excerpt from MIL-HDBK-340A, Vol. I.

3.3.2 Statistical Estimates of Vibration, Acoustic, and Shock Environments.

Qualification and acceptance tests for vibration, acoustic, and shock environments are based upon statistically expected spectral levels. The level of the extreme expected environment, used for qualification testing, is that not exceeded on at least 99% of flights, estimated with 90% confidence (P99/90 level). The level of the maximum expected environment, used for acceptance testing, is that not exceeded on at least 95% of flights, estimated with 50% confidence (P95/50 level).

These statistical estimates are made assuming a lognormal flight-to-flight variability having a standard deviation of 3 dB, unless a different assumption can be justified. As a result, the P95/50 level estimate is 5 dB above the estimated mean (namely, the average of the logarithmic values of the spectral levels of data from all available flights).

When data from N flights are used for the estimate, the P99/90 estimate in dB is

$$2.0 + 3.9 / \sqrt{N}$$

above the P95/50 estimate.

When data from only one flight are available, those data are assumed to represent the mean and so the P95/50 is 5 dB higher and the P99/90 level is 11 dB higher.

When ground testing produces the realistic flight environment (for example, engine operation or activation of explosive ordnance), the statistical distribution can be determined using the test data, providing data from a sufficient number of tests are available. The P99/90 and P95/50 levels are then determined from the derived distribution.

See also NASA-HDBK-7005.

APPENDIX B

Cumulative Distribution Function

The cumulative distribution function of the noncentral t-distribution with ν degrees of freedom and noncentrality parameter μ can be expressed as

$$F_{\nu,\mu}(x) = \begin{cases} \tilde{F}_{\nu,\mu}(x), & \text{if } x \geq 0 \\ 1-\tilde{F}_{\nu,-\mu}(-x), & \text{if } x < 0 \end{cases} \tag{B-1}$$

where

$$\tilde{F}_{\nu,\mu}(x) = \Phi(-\mu) + \frac{1}{2}\sum_{j=0}^{\infty}\left[p_j I_y\left(j+\frac{1}{2},\frac{\nu}{2}\right) + q_j I_y\left(j+1,\frac{\nu}{2}\right)\right] \tag{B-2}$$

where

Φ is the cumulative distribution function of the standard normal distribution

$I_y(a,b)$ is the regularized incomplete beta function.

$$y = \frac{x^2}{x^2+\nu^2} \tag{B-3}$$

$$p_j = \frac{1}{j!}\exp\left\{-\frac{\mu^2}{2}\right\}\left(\frac{\mu^2}{2}\right)^j \tag{B-4}$$

$$q_j = \frac{\mu}{\sqrt{2}\,\Gamma(j+3/2)}\exp\left\{-\frac{\mu^2}{2}\right\}\left(\frac{\mu^2}{2}\right)^j \tag{B-5}$$