VIBRATION OF A PLANE TRUSS    Revision B

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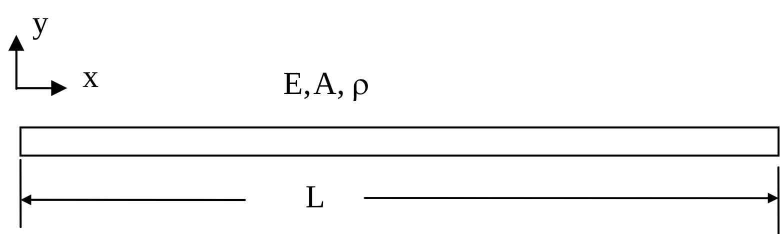
A truss is a structure which undergoes translation at its joints. The truss members are assumed to undergo no bending. This model is an idealization.

Each truss members is effectively assumed to undergo displacement in its respective longitudinal axis.

Consider an individual member.




E   is the modulus of elasticity
A   is the cross-section area
$\rho$   is the mass per volume

The one-dimensional element stiffness $K_j$ and mass $M_j$ matrices for element j are obtained from Reference 1.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{1}$$

$$K_j = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2}$$

$$M_j = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3}$$

Form two-dimensional matrices using the displacement vector:

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} \tag{4}$$

The corresponding two-dimensional matrices are

$$K_j = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

$$M_j = \frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{6}$$

Now consider the case where a coordinate transformation must be made to align a given element with one of the orthogonal axes. The following method is taken from Reference 2.

The rotational sub-matrix R is

$$R = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \tag{7}$$

where $\gamma$ is the angle from the global X-axis to the local X-axis.

The full transformation matrix for the two-dimensional problem is

$$R = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 & 0 \\ -\sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{bmatrix} \tag{8}$$

The element stiffness matrix is transformed to a global stiffness matrix $K_G$ as follows

$$K_{G,j} = R^T K_j R \tag{9}$$

The elemental mass matrix is transformed in a similar manner.

$$M_{G,j} = R^T M_j R \tag{10}$$

The transformed matrices are then assembled into final system matrices which contain all of the degrees-of-freedom. The natural frequencies are found by solving the corresponding generalized eigenvalue problem.

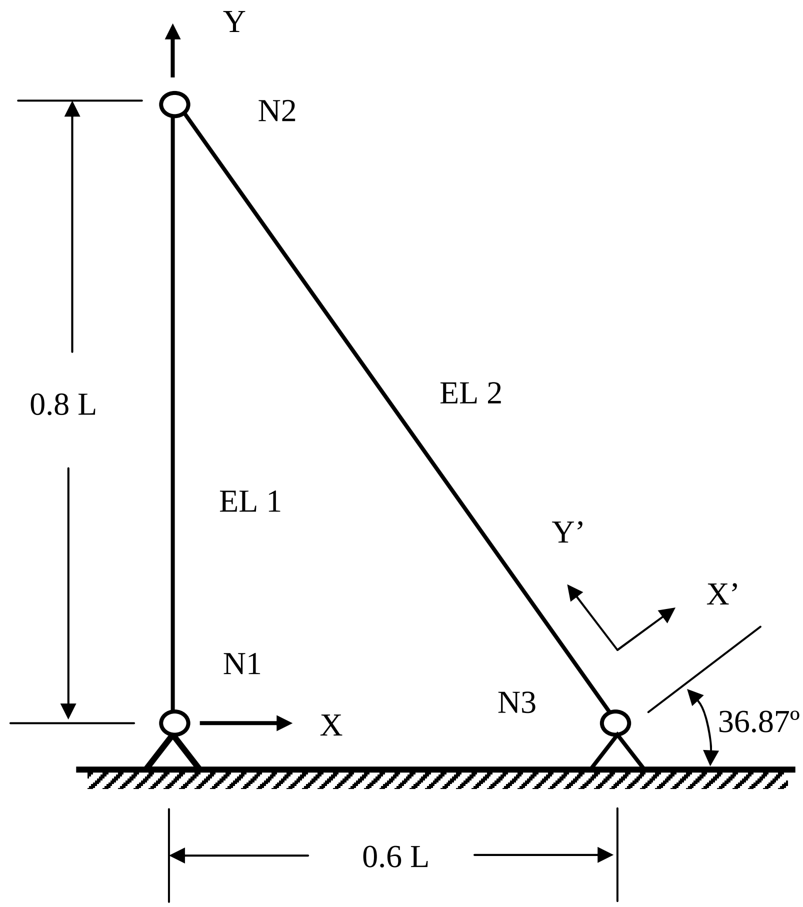
An example is given in Appendix A.

References

1. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Rev B, 2008.
2. W. Weaver and P. Johnson, Structural Dynamics by Finite Elements, Englewood Cliffs, New Jersey, 1987.

APPENDIX A

Example 1




EL indicates the element.  N is the node.

Furthermore, the cross-sectional areas are:

| | |
|---|---|
| EL1 | 0.8 A |
| EL2 | 1.0 A |

Element 1 Displacement Vector

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix} \tag{A-1}$$

The unconstrained system mass matrix is

MG = $\rho$AL *

$$\begin{matrix} 1.2800 & 0 & 0.6400 & 0 & 0 & 0 \\ 0 & 1.2800 & 0 & 0.6400 & 0 & 0 \\ 0.6400 & 0 & 3.2800 & 0 & 1.0000 & 0 \\ 0 & 0.6400 & 0 & 3.2800 & 0 & 1.0000 \\ 0 & 0 & 1.0000 & 0 & 2.0000 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 2.0000 \end{matrix} \tag{A-2}$$

The unconstrained system stiffness matrix is

KG = (EA/L)*

$$\begin{matrix} 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0 & 0 \\ 0.0000 & 1.0000 & -0.0000 & -1.0000 & 0 & 0 \\ -0.0000 & -0.0000 & 0.3600 & -0.4800 & -0.3600 & 0.4800 \\ -0.0000 & -1.0000 & -0.4800 & 1.6400 & 0.4800 & -0.6400 \\ 0 & 0 & -0.3600 & 0.4800 & 0.3600 & -0.4800 \\ 0 & 0 & 0.4800 & -0.6400 & -0.4800 & 0.6400 \end{matrix} \tag{A-3}$$

No displacement is allowed in either axis at nodes 1 and 3.

The generalized eigenvalue problem becomes.

$$\frac{EA}{L}\begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 1.64 \end{bmatrix} = \lambda \rho AL \begin{bmatrix} 0.5467 & 0 \\ 0 & 0.5467 \end{bmatrix}$$

(A-4)

The eigenvalues are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{E}{L^2 \rho} \begin{bmatrix} 0.3659 \\ 3.2927 \end{bmatrix}$$

(A-5)

Each natural frequency is the square root of the respective eigenvalue.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \frac{1}{L}\sqrt{\frac{E}{\rho}} \begin{bmatrix} 0.6049 \\ 1.8146 \end{bmatrix}$$

(A-6)

The eigenvectors are

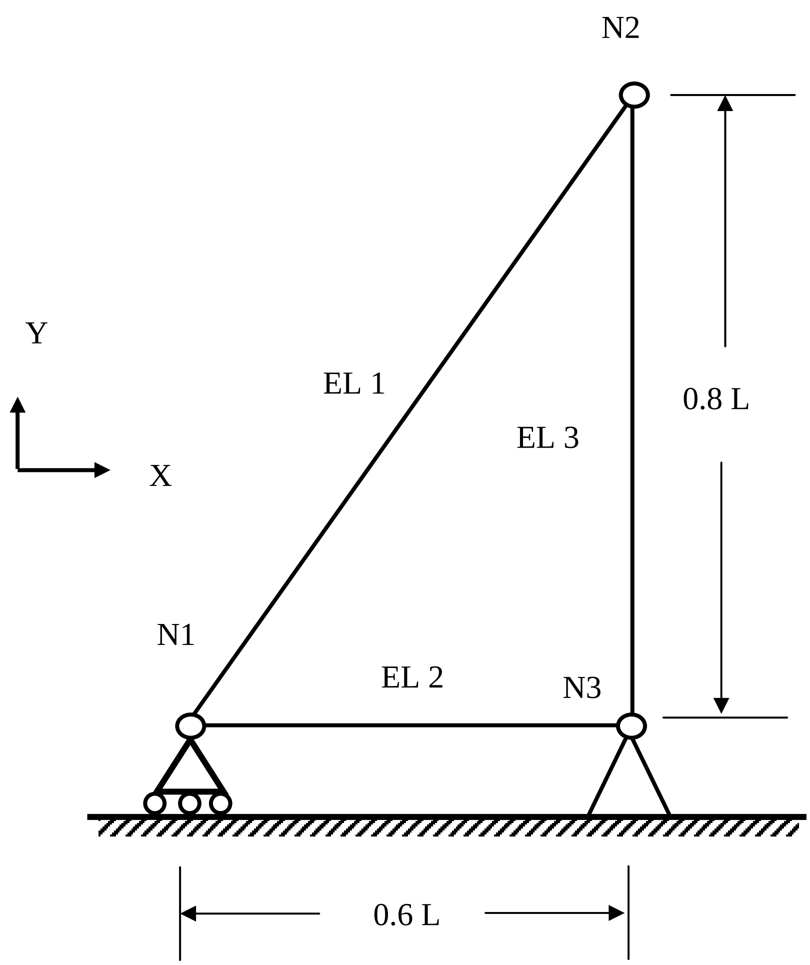
$$\begin{bmatrix} -1.283 \\ -0.4277 \end{bmatrix}, \quad \begin{bmatrix} -0.4277 \\ 1.283 \end{bmatrix}$$

(A-7)

The above eigenvectors are normalized with respect to the mass matrix.

APPENDIX B

Example 2




EL indicates the element.  N is the node.

Furthermore, the cross-sectional areas are:

| | |
|---|---|
| EL1 | 1.0 A |
| EL2 | 0.6 A |
| EL3 | 0.8A |

Element 1 Displacement Vector

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix} \tag{B-1}$$

The unconstrained system mass matrix is

MG = $\rho$AL *

| | | | | | |
|---|---|---|---|---|---|
| 0.4533 | 0 | 0.1667 | 0 | 0.0600 | 0 |
| 0 | 0.4533 | 0 | 0.1667 | 0 | 0.0600 |
| 0.1667 | 0 | 0.5467 | 0 | 0.1067 | 0 |
| 0 | 0.1667 | 0 | 0.5467 | 0 | 0.1067 |
| 0.0600 | 0 | 0.1067 | 0 | 0.3333 | 0 |
| 0 | 0.0600 | 0 | 0.1067 | 0 | 0.3333 |

(B-2)

The unconstrained system stiffness matrix is

KG = (EA/L)*

| | | | | | |
|---|---|---|---|---|---|
| 1.3600 | 0.4800 | -0.3600 | -0.4800 | -1.0000 | 0 |
| 0.4800 | 0.6400 | -0.4800 | -0.6400 | 0 | 0 |
| -0.3600 | -0.4800 | 0.3600 | 0.4800 | -0.0000 | 0.0000 |
| -0.4800 | -0.6400 | 0.4800 | 1.6400 | 0.0000 | -1.0000 |
| -1.0000 | 0 | -0.0000 | 0.0000 | 1.0000 | -0.0000 |
| 0 | 0 | 0.0000 | -1.0000 | -0.0000 | 1.0000 |

(B-3)

Only the x-axis displacement is allowed at node 1.

No displacement is allowed in either axis at node 3.

The generalized eigenvalue problem becomes.

$$\frac{EA}{L}\begin{bmatrix} 1.36 & -0.36 & -0.48 \\ -0.36 & 0.36 & 0.48 \\ -0.48 & 0.48 & 1.64 \end{bmatrix} = \lambda \rho AL \begin{bmatrix} 0.4533 & 0.1667 & 0 \\ 0.1667 & 0.5467 & 0 \\ 0 & 0 & 0.5467 \end{bmatrix} \tag{B-4}$$

The eigenvalues are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \frac{E}{L^2 \rho} \begin{bmatrix} 0.2701 \\ 2.0879 \\ 5.3078 \end{bmatrix} \tag{B-5}$$

Each natural frequency is the square root of the respective eigenvalue.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \frac{1}{L}\sqrt{\frac{E}{\rho}} \begin{bmatrix} 0.5197 \\ 1.445 \\ 2.304 \end{bmatrix} \tag{B-6}$$

The eigenvectors are

$$\begin{bmatrix} 0.2803 \\ 1.2114 \\ -0.2995 \end{bmatrix}, \quad \begin{bmatrix} -0.9384 \\ 0.1856 \\ -1.0820 \end{bmatrix}, \quad \begin{bmatrix} -1.2350 \\ 0.7472 \\ 0.7542 \end{bmatrix} \tag{B-7}$$

The above eigenvectors are normalized with respect to the mass matrix.