NATURAL FREQUENCIES OF RECTANGULAR PLATE BENDING MODES
Revision B

By Tom Irvine
Email: tomirvine@aol.com

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This tutorial presents both the Rayleigh method and the direct Eigenvalue solution.

Introduction

Rayleigh Method

The Rayleigh method is used to determine the fundamental bending frequency. A displacement function is assumed which satisfies the geometric boundary conditions. The geometric conditions are the displacement and slope conditions at the boundaries.

The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limited of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the rectangular plate in Figure 1.

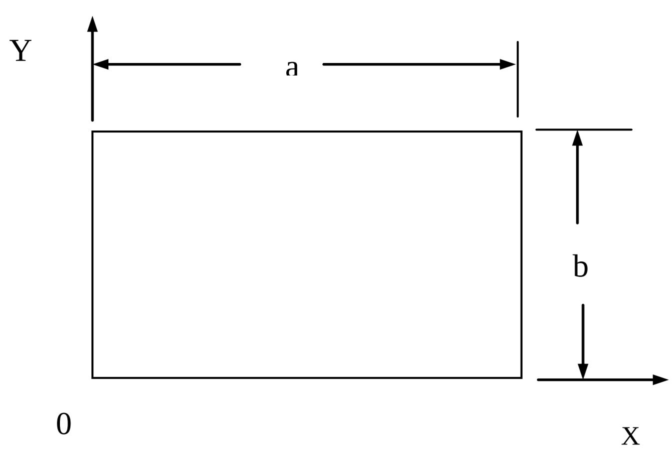

Figure 1.

Let Z represent the out-of-plane displacement. The total strain energy V of the plate is

$$V = \frac{D}{2}\int_0^b \int_0^a \left[\left(\frac{\partial^2 Z}{\partial X^2} \right)^2 + \left(\frac{\partial^2 Z}{\partial Y^2} \right)^2 + 2\mu \left(\frac{\partial^2 Z}{\partial X^2} \right)\left(\frac{\partial^2 Z}{\partial Y^2} \right) + 2(1-\mu) \left(\frac{\partial^2 Z}{\partial X \partial Y} \right)^2 \right] dXdY$$

(1)

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{2}$$

where

E = elastic modulus
h = plate thickness
μ = Poisson's ratio

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^b \int_0^a Z^2 \, dX \, dY \tag{3}$$

where

ρ = mass per volume
Ω = angular natural frequency

Direct Eigenvalue Solution of the Partial Differential Equation

An example showing the direct solution of the governing equation of motion is given in Appendix B.

APPENDIX A

Fixed-Free-Free-Free Plate, Rayleigh Method

Consider the plate in Figure A-1.

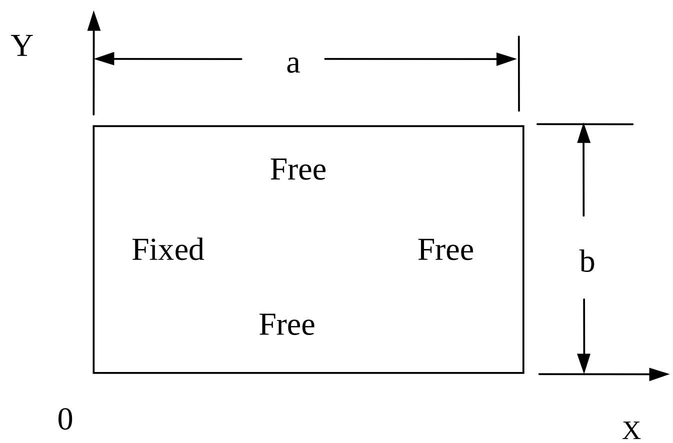

Figure A-1.

Seek a displacement function that is similar to a cantilever beam bending function. The geometric boundary conditions are

$$Z(0, y) = 0 \tag{A-1}$$

$$Z(a, y) = Z_o \tag{A-2}$$

$$\left.\frac{\partial Z}{\partial x}\right|_{x=0} = 0 \tag{A-3}$$

An additional geometric condition is

$$\left.\frac{\partial^2 Z}{\partial y^2}\right|_{0 \le y \le b} = 0 \tag{A-4}$$

The following quarter-cosine wave function satisfies the geometric boundary conditions.

$$Z(x,y) = Z_o\left[1-\cos\left(\frac{\pi x}{2a}\right)\right] \tag{A-5}$$

The partial derivatives are

$$\frac{\partial Z}{\partial x} = Z_o\left(\frac{\pi}{2a}\right)\sin\left(\frac{\pi x}{2a}\right) \tag{A-6}$$

$$\frac{\partial^2 Z}{\partial x^2} = Z_o\left(\frac{\pi}{2a}\right)^2 \cos\left(\frac{\pi x}{2a}\right) \tag{A-7}$$

$$\frac{\partial Z}{\partial y} = 0 \tag{A-8}$$

$$\frac{\partial^2 Z}{\partial y^2} = 0 \tag{A-9}$$

$$\frac{\partial^2 Z}{\partial X\,\partial Y} = 0 \tag{A-10}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h\,\Omega^2}{2}\int_0^b\int_0^a Z_o\left[1-\cos\left(\frac{\pi x}{2a}\right)\right]^2 dx\,dy \tag{A-11}$$

$$T = \frac{\rho h\,\Omega^2 Z_o{}^2}{2}\int_0^b\int_0^a\left[1-2\cos\left(\frac{\pi x}{2a}\right)+\cos^2\left(\frac{\pi x}{2a}\right)\right]dx\,dy \tag{A-12}$$

$$T = \frac{\rho h\,\Omega^2 Z_o{}^2}{2}\int_0^b\int_0^a\left[1-2\cos\left(\frac{\pi x}{2a}\right)+\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi x}{a}\right)\right]dx\,dy \tag{A-13}$$

$$T = \frac{\rho h\,\Omega^2 Z_o{}^2}{2}\int_0^b\int_0^a\left[\frac{3}{2}-2\cos\left(\frac{\pi x}{2a}\right)+\frac{1}{2}\cos\left(\frac{\pi x}{a}\right)\right]dx\,dy \tag{A-14}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{2} \int_0^b \left[\frac{3}{2} x - 2\left(\frac{2a}{\pi}\right) \sin\left(\frac{\pi x}{2a}\right) + \frac{1}{2}\left(\frac{a}{\pi}\right) \sin\left(\frac{\pi x}{a}\right) \right] \Bigg|_0^a dy \qquad \text{(A-15)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{2} \int_0^b \left[\frac{3}{2} a - 2\left(\frac{2a}{\pi}\right) \right] dy \qquad \text{(A-16)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a}{2} \left[\frac{3}{2} - \frac{4}{\pi} \right] \int_0^b dy \qquad \text{(A-17)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 ab}{2} \left[\frac{3}{2} - \frac{4}{\pi} \right] \qquad \text{(A-18)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 ab [3\pi - 8]}{4\pi} \qquad \text{(A-19)}$$

The total strain energy is given by

$$V = \frac{D}{2} \int_0^b \int_0^a \left[\left(Z_o \left(\frac{\pi}{2a}\right)^2 \cos\left(\frac{\pi x}{2a}\right) \right)^2 \right] dxdy \qquad \text{(A-20)}$$

$$V = \frac{D}{2} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 \int_0^b \int_0^a \left[\cos^2\left(\frac{\pi x}{2a}\right) \right] dxdy \qquad \text{(A-21)}$$

$$V = \frac{D}{2} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 \int_0^b \int_0^a \left[\frac{1}{2} + \cos\left(\frac{\pi x}{a}\right) \right] dxdy \qquad \text{(A-22)}$$

$$V = \frac{D}{2} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 \int_0^b \left[\frac{1}{2} x + \left(\frac{a}{\pi}\right) \sin\left(\frac{\pi x}{a}\right) \right] \Bigg|_0^a dy \qquad \text{(A-23)}$$

$$V = \frac{D}{2} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 \int_0^b \frac{a}{2}\, dy \tag{A-24}$$

$$V = \frac{D}{2} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 \frac{a\,b}{2} \tag{A-25}$$

$$V = \frac{D}{4} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 a\,b \tag{A-26}$$

Now equate the total kinetic energy with the total strain energy per Rayleigh's method.

$$\frac{\rho h\, \Omega^2 Z_o{}^2\, ab\,[3\pi - 8]}{4\pi} = \frac{D}{4} Z_o{}^2 \left(\frac{\pi}{2a}\right)^4 a\,b \tag{A-27}$$

$$\frac{\rho h\, \Omega^2\ [3\pi - 8]}{\pi} = D\left(\frac{\pi}{2a}\right)^4 \tag{A-28}$$

$$\Omega^2 = D\left(\frac{\pi}{2a}\right)^4 \left(\frac{\pi}{\rho h[3\pi - 8]}\right) \tag{A-29}$$

$$\Omega = \sqrt{\frac{D}{\rho h}\left(\frac{\pi}{2a}\right)^4 \left(\frac{\pi}{[3\pi - 8]}\right)} \tag{A-30}$$

$$\Omega = \left(\frac{\pi}{2a}\right)^2 \sqrt{\frac{D}{\rho h}\left(\frac{\pi}{[3\pi - 8]}\right)} \tag{A-31}$$

The natural frequency f_n is

$$f_n = \frac{1}{2\pi}\Omega \tag{A-32}$$

$$f_n = \left(\frac{1}{2\pi}\right)\left(\frac{\pi}{2a}\right)^2 \sqrt{\frac{D}{\rho h}\left(\frac{\pi}{[3\pi - 8]}\right)} \tag{A-33}$$

$$f_n = \left(\frac{\pi}{8a^2}\right)\sqrt{\frac{D}{\rho h}\left(\frac{\pi}{[3\pi - 8]}\right)} \tag{A-34}$$

$$f_n = \left(\frac{0.583}{a^2}\right)\sqrt{\frac{D}{\rho h}} \tag{A-35}$$

A more proper equation is

$$f_n \le \left(\frac{0.583}{a^2}\right)\sqrt{\frac{D}{\rho h}} \tag{A-36}$$

APPENDIX B

Plate Simply-Supported on All Sides, Eigenvalue Solution

The governing equation of motion is

$$D\left(\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2 \partial y^2}+\frac{\partial^4 z}{\partial y^4}\right)+\rho h\frac{\partial^2 z}{\partial t^2}=0 \qquad \text{(B-1)}$$

Assume a harmonic response.

$$z(x,y,t)=Z(x,y)\exp(j\omega t) \qquad \text{(B-2)}$$

$$D\left(\frac{\partial^4 Z}{\partial x^4}+2\frac{\partial^4 Z}{\partial x^2 \partial y^2}+\frac{\partial^4 Z}{\partial y^4}\right)\exp(j\omega t)-\rho h\omega^2 Z\exp(j\omega t)=0 \qquad \text{(B-3)}$$

$$D\left(\frac{\partial^4 Z}{\partial x^4}+2\frac{\partial^4 Z}{\partial x^2 \partial y^2}+\frac{\partial^4 Z}{\partial y^4}\right)-\rho h\omega^2 Z=0 \qquad \text{(B-4)}$$

The boundary conditions are

$$Z(x,y)=0, \quad M_x(x,y)=0 \qquad \text{for} \quad x = 0, a \qquad \text{(B-5)}$$

$$Z(x,y)=0, \quad M_y(x,y)=0 \qquad \text{for} \quad y = 0, b \qquad \text{(B-6)}$$

Assume the following displacement function which satisfies the boundary conditions, where A_{mn} is an amplitude coefficient determined from the initial conditions and m an n are integers.

$$Z_{mn} = A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-7}$$

The partial derivates are

$$\frac{\partial}{\partial x} Z_{mn} = \left(\frac{m\pi}{a}\right) A_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-8}$$

$$\frac{\partial^2}{\partial x^2} Z_{mn} = -\left(\frac{m\pi}{a}\right)^2 A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-9}$$

$$\frac{\partial^3}{\partial x^3} Z_{mn} = -\left(\frac{m\pi}{a}\right)^3 A_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-10}$$

$$\frac{\partial^4}{\partial x^4} Z_{mn} = \left(\frac{m\pi}{a}\right)^4 A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-11}$$

Similarly,

$$\frac{\partial^4}{\partial y^4} Z_{mn} = \left(\frac{n\pi}{n}\right)^4 A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-12}$$

Also,

$$\frac{\partial^2}{\partial x^2 \partial y^2} Z_{mn} = \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{a}\right)^2 A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{B-13}$$

$$D\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$
$$- \rho h \omega^2 A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = 0 \tag{B-14}$$

$$D\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) - \rho h\omega^2 = 0 \tag{B-15}$$

$$\omega^2 = \frac{D}{\rho h}\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) \tag{B-16}$$

$$\omega = \sqrt{\frac{D}{\rho h}\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right)} \tag{B-17}$$

$$\omega = \sqrt{\frac{D}{\rho h}}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right) \tag{B-18}$$

Note that the wave numbers are

$$k_m = m\pi / a \tag{B-19}$$

$$k_n = n\pi / b \tag{B-20}$$

Mass-normalize the mode shapes. The mass density is constant.

$$\rho h \int_0^b \int_0^a \left[Z_{mn}(x,y)\right]^2 dxdy = 1 \tag{B-21}$$

$$\rho h \int_0^b \int_0^a \left[A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\right]^2 dxdy = 1 \tag{B-22}$$

$$\rho h A_{mn}{}^2 \int_0^b \left[\sin\left(\frac{n\pi y}{b}\right)\right]^2 \left\{\int_0^a \left[\sin\left(\frac{m\pi x}{a}\right)\right]^2 dx\right\} dy = 1 \tag{B-23}$$

$$\rho h\, A_{mn}{}^{2} \int_0^b \left[\sin\left(\frac{n\pi\, y}{b}\right)\right]^2 \left\{\frac{1}{2}\int_0^a \left[1-\cos\left(\frac{2m\pi\, x}{a}\right)\right] dx\right\} dy = 1 \qquad \text{(B-24)}$$

$$\frac{1}{2}\rho h\, A_{mn}{}^{2} \int_0^b \left[\sin\left(\frac{n\pi\, y}{b}\right)\right]^2 \left\{ x - \frac{a}{2m\pi}\sin\left(\frac{2m\pi\, x}{a}\right)\Bigg|_0^a \right\} dy = 1 \qquad \text{(B-25)}$$

$$\frac{1}{2}\rho a\ h\, A_{mn}{}^{2} \int_0^b \left[\sin\left(\frac{n\pi\, y}{b}\right)\right]^2 dy = 1 \qquad \text{(B-26)}$$

$$\frac{1}{4}\rho a\, b\, h\, A_{mn}{}^{2} = 1 \qquad \text{(B-27)}$$

$$A_{mn}{}^{2} = \frac{4}{\rho a\, b\, h} \qquad \text{(B-28)}$$

$$A_{mn} = \sqrt{\frac{4}{\rho a\, b\, h}} \qquad \text{(B-29)}$$

$$A_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \qquad \text{(B-30)}$$

Calculate the participation factors.

$$\Gamma_{mn} = \rho h \int_0^b \int_0^a Z_{mn}(x, y)\, dx dy \qquad \text{(B-31)}$$

$$\Gamma_{mn} = \rho h \frac{2}{\sqrt{\rho a\, b\, h}} \int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy \tag{B-32}$$

$$\Gamma_{mn} = 2\sqrt{\frac{\rho h}{a\, b}} \int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy \tag{B-33}$$

$$\Gamma_{mn} = 2\sqrt{\frac{\rho h}{a\, b}} \int_0^b \left[\sin\left(\frac{n\pi y}{b}\right) \right] \left\{ \int_0^a \sin\left(\frac{m\pi x}{a}\right) dx \right\} dy \tag{B-34}$$

$$\Gamma_{mn} = -2\sqrt{\frac{\rho h}{a\, b}} \left(\frac{a}{m\pi}\right) \int_0^b \left[\sin\left(\frac{n\pi y}{b}\right) \right] \left\{ \cos\left(\frac{m\pi x}{a}\right) \Big|_0^a \right\} dy \tag{B-35}$$

$$\Gamma_{mn} = -2\sqrt{\frac{\rho h}{a\, b}} \left(\frac{a}{m\pi}\right) \int_0^b \left[\sin\left(\frac{n\pi y}{b}\right) \right] dy \; [\cos(m\pi) - 1] \tag{B-36}$$

$$\Gamma_{mn} = 2\sqrt{\frac{\rho h}{a\, b}} \left(\frac{a\, b}{m\, n\, \pi^2}\right) [\cos(n\pi) - 1][\cos(m\pi) - 1] \tag{B-37}$$

$$\Gamma_{mn} = \left(\frac{2\sqrt{\rho\, a\, b\, h}}{m\, n\, \pi^2}\right) [\cos(n\pi) - 1][\cos(m\pi) - 1] \tag{B-38}$$

Calculate the effective modal mass.

$$m_{eff,mn} = \frac{\left[\rho h \int_0^b \int_0^a Z_{mn}(x, y)\, dxdy \right]^2}{\rho h \int_0^b \int_0^a [Z_{mn}(x, y)]^2\, dxdy} \tag{B-39}$$

The eigenvectors are already normalized such that

$$\rho h \int_0^b \int_0^a \left[Z_{mn}(x,y)\right]^2 dxdy = 1 \tag{B-40}$$

Thus,

$$m_{eff,mn} = \left[\Gamma_n\right]^2 = \left[\rho h \int_0^b \int_0^a Z_{mn}(x,y)\, dxdy\right]^2 \tag{B-41}$$

$$m_{eff,mn} = \left\{\left(\frac{2\sqrt{\rho\, a\, b\, h}}{m\, n\, \pi^2}\right)\left[\cos(n\pi)-1\right]\left[\cos(m\pi)-1\right]\right\}^2 \tag{B-42}$$

$$m_{eff,mn} = \left\{\left(\frac{2\sqrt{\rho\, a\, b\, h}}{m\, n\, \pi^2}\right)\left[\cos(n\pi)-1\right]\left[\cos(m\pi)-1\right]\right\}^2 \tag{B-43}$$

$$m_{eff,mn} = \frac{4\rho\, a\, b\, h}{\left(m\, n\, \pi^2\right)^2}\left\{\left[\cos(n\pi)-1\right]\left[\cos(m\pi)-1\right]\right\}^2 \tag{B-44}$$

Some sample values are

$$m_{eff,11} = 0.6570\ \rho\, a\, b\, h \tag{B-45}$$

$$m_{eff,mn} = 0 \quad \text{if either m or n is even} \tag{B-46}$$

$$m_{eff,13} = m_{eff,31} = 0.0730\ \rho\, a\, b\, h \qquad \text{(B-45)}$$