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March 30, 2006

## Introduction

This tutorial is based somewhat on a section in Reference 1. The tutorial makes an improvement by using a general solution to solve for the response.

Consider a vehicle system traveling along a road at a constant velocity as show in Figure 1. The vehicle is modeled as a single-degree-of-freedom system.


Figure 1.

The variables are:
m is the mass
c is the viscous damping coefficient
k is the stiffness
x is the absolute displacement of the mass
y is the base input displacement
V is the rigid-body velocity of the mass

A free-body diagram is shown in Figure 2.


Figure 2.

Summation of forces in the vertical direction

$$
\begin{align*}
& \sum F=m \ddot{x}  \tag{1}\\
& m \ddot{x}=c(\dot{y}-\dot{x})+k(y-x)  \tag{2}\\
& m \ddot{x}+c \dot{x}+k x=c \dot{y}+k y  \tag{3}\\
& \ddot{x}+(c / m) \dot{x}+(k / m) x=(c / m) \dot{y}+(k / m) y \tag{4}
\end{align*}
$$

By convention,

$$
\begin{aligned}
& (\mathrm{c} / \mathrm{m})=2 \xi \omega_{\mathrm{n}} \\
& (\mathrm{k} / \mathrm{m})=\omega_{\mathrm{n}}{ }^{2}
\end{aligned}
$$

where $\omega_{\mathrm{n}}$ is the natural frequency in (radians/sec), and $\xi$ is the damping ratio.
Substitute the convention terms into equation (4).

$$
\begin{equation*}
\ddot{\mathrm{x}}+2 \xi \omega_{\mathrm{n}} \dot{\mathrm{x}}+\omega_{\mathrm{n}}^{2} \mathrm{x}=2 \xi \omega_{\mathrm{n}} \dot{\mathrm{y}}+\omega_{\mathrm{n}}^{2} \mathrm{y} \tag{5}
\end{equation*}
$$

Take the Fourier transform.

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left\{\ddot{\mathrm{x}}+2 \xi \omega_{\mathrm{n}} \dot{\mathrm{x}}+\omega_{\mathrm{n}}^{2} \mathrm{x}\right\} \exp (-j \omega t) \mathrm{dt}=\int_{-\infty}^{\infty}\left\{2 \xi \omega_{\mathrm{n}} \dot{\mathrm{y}}+\omega_{\mathrm{n}}^{2} \mathrm{y}\right\} \exp (-j \omega t) \mathrm{dt} \tag{6}
\end{equation*}
$$

$$
\begin{gather*}
\int_{-\infty}^{\infty} \ddot{\mathrm{x}} \exp (-j \omega t) d t+2 \xi \omega_{n} \int_{-\infty}^{\infty} \dot{x} \exp (-j \omega t) d t+\omega_{n}^{2} \int_{-\infty}^{\infty} x \exp (-j \omega t) d t \\
=2 \xi \omega_{n} \int_{-\infty}^{\infty}\{\dot{y}\} \exp (-j \omega t) d t+\omega_{n}^{2} \int_{-\infty}^{\infty} y \exp (-j \omega t) d t \tag{7}
\end{gather*}
$$

Note that

$$
\begin{align*}
& \int_{-\infty}^{\infty} \dot{x}(t) \exp [-j \omega t] d t=j \omega \int_{-\infty}^{\infty} x(t) \exp [-j \omega t] d t  \tag{8}\\
& \int_{-\infty}^{\infty} \ddot{\mathrm{x}}(\mathrm{t}) \exp [-j \omega t] d t=-\omega^{2} \int_{-\infty}^{\infty} x(t) \exp [-j \omega t] d t \tag{9}
\end{align*}
$$

Thus,

$$
\begin{gather*}
-\omega^{2} \int_{-\infty}^{\infty} x(t) \exp [-j \omega t] d t+j 2 \xi \omega_{n} \omega \int_{-\infty}^{\infty} x(t) \exp [-j \omega t] d t+\omega_{n}^{2} \int_{-\infty}^{\infty}\{x\} \exp (-j \omega t) d t \\
=j 2 \xi \omega_{n} \omega \int_{-\infty}^{\infty}\{y(t)\} \exp (-j \omega t) d t+\omega_{n}^{2} \int_{-\infty}^{\infty}\{y(t)\} \exp (-j \omega t) d t \tag{10}
\end{gather*}
$$

$$
\begin{equation*}
\left[\left(\omega_{n}^{2}-\omega^{2}\right)+j 2 \xi \omega_{n} \omega\right] \int_{-\infty}^{\infty} x(t) \exp [-j \omega t] d t=\left[\omega_{n}^{2}+j 2 \xi \omega_{n} \omega\right] \int_{-\infty}^{\infty} y(t) \exp (-j \omega t) d t \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
\left[\left(\omega_{n}^{2}-\omega^{2}\right)+j 2 \xi \omega_{n} \omega\right] X(\omega)=\left[\omega_{n}^{2}+j 2 \xi \omega_{n} \omega\right] Y(\omega) \tag{12}
\end{equation*}
$$

Define a transfer function.

$$
\begin{equation*}
H(\omega)=\left[\frac{X(\omega)}{Y(\omega)}\right]=\frac{\left[\omega_{n}^{2}+j 2 \xi \omega_{n} \omega\right]}{\left[\left(\omega_{n}^{2}-\omega^{2}\right)+j 2 \xi \omega_{n} \omega\right]} \tag{13}
\end{equation*}
$$

$$
\begin{align*}
H(\omega) & =\frac{\left[1+\mathrm{j} 2 \xi\left(\omega / \omega_{\mathrm{n}}\right)\right]}{\left[\left(1-\left(\omega / \omega_{\mathrm{n}}\right)^{2}\right)+\mathrm{j} 2 \xi\left(\omega / \omega_{\mathrm{n}}\right)\right]}  \tag{14}\\
H(\rho) & =\frac{[1+\mathrm{j} 2 \xi \rho]}{\left[\left(1-\rho^{2}\right)+\mathrm{j} 2 \xi \rho\right]}, \quad \rho=\left(\omega / \omega_{\mathrm{n}}\right) \tag{15}
\end{align*}
$$

## General Approach

Consider the system is subjected to a base input power spectral density (PSD). The resulting response equation is

$$
\begin{equation*}
\mathrm{x}_{\mathrm{RMS}}(\mathrm{fn}, \xi)=\sqrt{\sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{\left[1+\left(2 \xi \rho_{\mathrm{i}}\right)^{2}\right]}{\left[\left(1-\rho_{\mathrm{i}}^{2}\right)+\left(2 \xi \rho_{\mathrm{i}}\right)^{2}\right]}} \hat{\mathrm{Y}}_{\mathrm{DPSD}}\left(\mathrm{f}_{\mathrm{i}}\right) \Delta \mathrm{f}_{\mathrm{i}} \quad, \quad \rho=\left(\mathrm{f}_{\mathrm{i}} / \mathrm{f}_{\mathrm{n}}\right) \tag{16}
\end{equation*}
$$

where

$$
\begin{array}{cl}
\mathrm{f}_{\mathrm{n}} & \text { is the natural frequency } \\
\hat{\mathrm{Y}}_{\mathrm{DPSD}}\left(\mathrm{f}_{\mathrm{i}}\right) & \text { is the base input displacement power spectral density }
\end{array}
$$

Equation (16) is taken from Reference 2.


Figure 3.

| Table 1. Spatial Displacement PSD |  |
| :---: | :---: |
| Scaled <br> Wavenumber <br> (cycles / meter) | Displacement <br> $\left(\mathrm{mm}^{2}\right.$ / (cycles/meter)) |
| 0.1 | 10 |
| 2 | 10 |
| 10 | 0.1 |

Assume that the PSD level is constant from 0 to 0.1 cycles/meter. Also assume that it its zero beyond 10 cycles/meter.

Now assume that the vehicle travels down a road characterized by the spatial power spectral density in Figure 3. The speed of the vehicle is an independent variable.

The wavenumber expresses the rate of change with respect to distance.
The wavenumber is normally expressed as (rad / meter). The wavenumber in Figure 3 is thus scaled so that its unit is (cycles / meter ).

TIME DISPLACEMENT POWER SPECTRAL DENSITY CURVES


## Figure 4.

The Spatial PSD is converted to a Time PSD as follows:

$$
\operatorname{TimePSD}\left(\mathrm{f}=\frac{\mathrm{V}}{\lambda}\right)=\left[\frac{1}{\mathrm{~V}}\right] \text { Spatial PSD }\left(\frac{1}{\lambda}=\frac{\mathrm{f}}{\mathrm{~V}}\right)
$$

Both the X and Y -axes must be scaled for the conversion.
The resulting Time PSD curves for two speed cases are shown in Figure 4. The overall displacement of 5.4 mm is maintained in each case. Again, the PSD plateau is assumed to be constant down to zero Hz for each curve.

The breakpoints for the $10 \mathrm{~km} / \mathrm{hr}$ case are given in Table 2.

| Table 2. Equivalent Time Displacement <br> PSD for $10 \mathrm{~km} / \mathrm{hr}$ |  |
| :---: | :---: |
| Frequency <br> $(\mathrm{Hz})$ | Displacement <br> $\left(\mathrm{mm}^{2} / \mathrm{Hz}\right)$ |
| 0.3 | 3.6 |
| 5.6 | 3.6 |
| 27.8 | 0.036 |

Assume that the vehicle has a natural frequency of 1.5 Hz with $10 \%$ damping. Also assume that the vehicle's "wheel" always remains in contact with the ground.

The resulting response for each of five speed cases is given in the following table, as calculated using equation (16).

| Table 1. Response Results |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Speed (km/h) | Speed <br> $(\mathrm{m} / \mathrm{sec})$ | Displacement <br> ( mm RMS ) | Acceleration <br> $\left(\mathrm{m} / \mathrm{sec}^{\wedge 2)}\right.$ RMS | Acceleration <br> (GRMS) |
| 10 | 2.78 | 6.6 | 0.87 | 0.09 |
| 20 | 5.56 | 4.7 | 1.08 | 0.11 |
| 30 | 8.33 | 3.8 | 1.43 | 0.15 |
| 50 | 13.9 | 3.0 | 2.23 | 0.23 |
| 100 | 27.8 | 2.1 | 4.34 | 0.44 |

Both the displacement and the acceleration are absolute.
The input Displacement Time PSD can be converted to an Acceleration Time PSD by multiplying it by $\omega^{4}$. Equation (16) can then be used to calculate the acceleration response. The acceleration results are also given in Table 1.

The displacement decreases with speed. The acceleration has the opposite trend.
The displacement response is driven primarily by the resonant response near the natural frequency for each speed case.

The acceleration response, however, is driven by both the resonant response and a broadband response. The broadband acceleration becomes more significant as the speed increases.

## References

1. D. E. Newland, An Introduction to Random Vibrations, Spectral \& Wavelet Analysis, Third Edition, Dover, New York, 1993.
2. T. Irvine, An Introduction to the Vibration Response Spectrum Rev C, Vibrationdata, 2000.
