STEADY-STATE RELATIVE DISPLACEMENT RESPONSE TO BASE EXCITATION

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

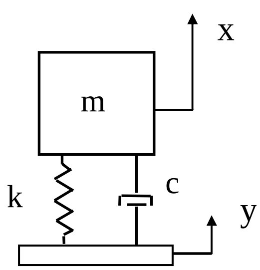

Figure 1.

where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass
y	is the base input displacement

Newton’s law can be applied to a free-body diagram of an individual system, as shown in Figure 2.

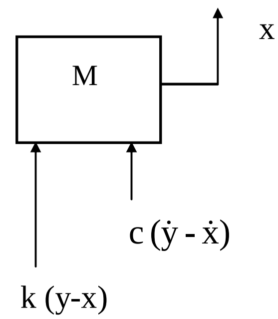

Figure 2. Free-body Diagram

A summation of forces yields the following governing differential equation of motion:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{1}$$

A relative displacement can be defined as $z = x - y$.

The following equation is obtained by substituting this expression into equation (1):

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{2}$$

Additional substitutions can be made as follows,

$$\omega_n{}^2 = \frac{k}{m} \tag{3}$$

$$2\xi\omega_n = \frac{c}{m} \tag{4}$$

Note that ξ is the damping ratio, and that ω_n is the natural frequency in radians per second.

Furthermore, ξ is often represented by the amplification factor Q, where $Q = 1/(2\,\xi)$.

Substitution of these terms into equation (2) yields an equation of motion for the relative response

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y}(t) \tag{5}$$

Take the Fourier transform of both sides of equation (5).

$$\int_{-\infty}^{\infty}\left\{\ddot{z}(t) + 2\xi\omega_n\,\dot{z}(t) + \omega_n^2 z(t)\right\}\exp[-j\omega t]dt = -\int_{-\infty}^{\infty}\ddot{y}(t)\,\exp[-j\omega t]dt \tag{6}$$

$$\int_{-\infty}^{\infty}\ddot{z}(t)\exp[-j\omega t]dt + 2\xi\omega_n\int_{-\infty}^{\infty}\dot{z}(t)\exp[-j\omega t]dt + \omega_n^2\int_{-\infty}^{\infty}z(t)\exp[-j\omega t]dt$$
$$= -\int_{-\infty}^{\infty}\ddot{y}(t)\exp[-j\omega t]dt \tag{7}$$

$$\int_{-\infty}^{\infty}\dot{z}(t)\exp[-j\omega t]dt = j\omega\int_{-\infty}^{\infty}z(t)\exp[-j\omega t]dt \tag{8}$$

$$\int_{-\infty}^{\infty}\ddot{z}(t)\exp[-j\omega t]dt = -\omega^2\int_{-\infty}^{\infty}z(t)\exp[-j\omega t]dt \tag{9}$$

Substitute equations (8) and (9) into (7).

$$-\omega^2\int_{-\infty}^{\infty} z(t)\exp[-j\omega t]dt + j2\xi\omega\omega_n\int_{-\infty}^{\infty} z(t)\exp[-j\omega t]dt + \omega_n^2\int_{-\infty}^{\infty} z(t)\exp[-j\omega t]dt$$

$$= -\int_{-\infty}^{\infty} \ddot{y}(t)\exp[-j\omega t]dt \tag{10}$$

$$\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}\int_{-\infty}^{\infty} z(t)\exp[-j\omega t]dt = -\int_{-\infty}^{\infty} \ddot{y}(t)\exp[-j\omega t]dt \tag{11}$$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\exp[-j\omega t]dt \tag{12}$$

$$\ddot{Y}(\omega) = \int_{-\infty}^{\infty} \ddot{y}(t)\exp[-j\omega t]dt \tag{13}$$

Substitute equations (12) and (13) into (11).

$$\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}\, Z(\omega) = -\ddot{Y}(\omega) \tag{14}$$

$$\frac{Z(\omega)}{\ddot{Y}(\omega)} = \frac{-1}{\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}} \tag{15}$$

The transfer function $H(\omega)$ is

$$H(\omega) = \frac{Z(\omega)}{\ddot{Y}(\omega)} \tag{16}$$

$$H(\omega) = \frac{-1}{\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}} \tag{17}$$

The transfer function magnitude is

$$|\,H(\omega)\,| = \frac{1}{\sqrt{(\omega_n{}^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2}} \quad (18)$$

$$|\,H(f)\,| = \frac{1}{4\pi^2\,\sqrt{(f_n{}^2 - f^2)^2 + (2\,\xi\, f\, f_n)^2}} \quad (19)$$

Recall

$$H(\omega) = \frac{-1}{\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}} \quad (20)$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$H(\omega) = -\frac{(\omega_n^2 - \omega^2) - j2\xi\omega\omega_n}{\{(\omega_n^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2\}} \quad (21)$$

$$H(\omega) = \frac{(\omega^2 - \omega_n{}^2) + j2\xi\omega\omega_n}{\{(\omega_n^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2\}} \quad (22)$$

The phase angle is

$$\varphi = \arctan\left[\frac{2\xi\omega\omega_n}{\omega^2 - \omega_n{}^2}\right] \quad (23)$$

$$\varphi = \arctan\left[\frac{2\,\xi\, f\, f_n}{f^2 - f_n^{\,2}}\right] \tag{24}$$

Recall

$$\left|\, H(\omega) \,\right| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2}} \tag{25}$$

$$\left|\, H(\omega) \,\right| = \frac{1}{\omega_n^2 \sqrt{(1-\rho^2)^2 + (2\xi\,\rho)^2}}$$

where $\rho = \omega / \omega_n$

(26)

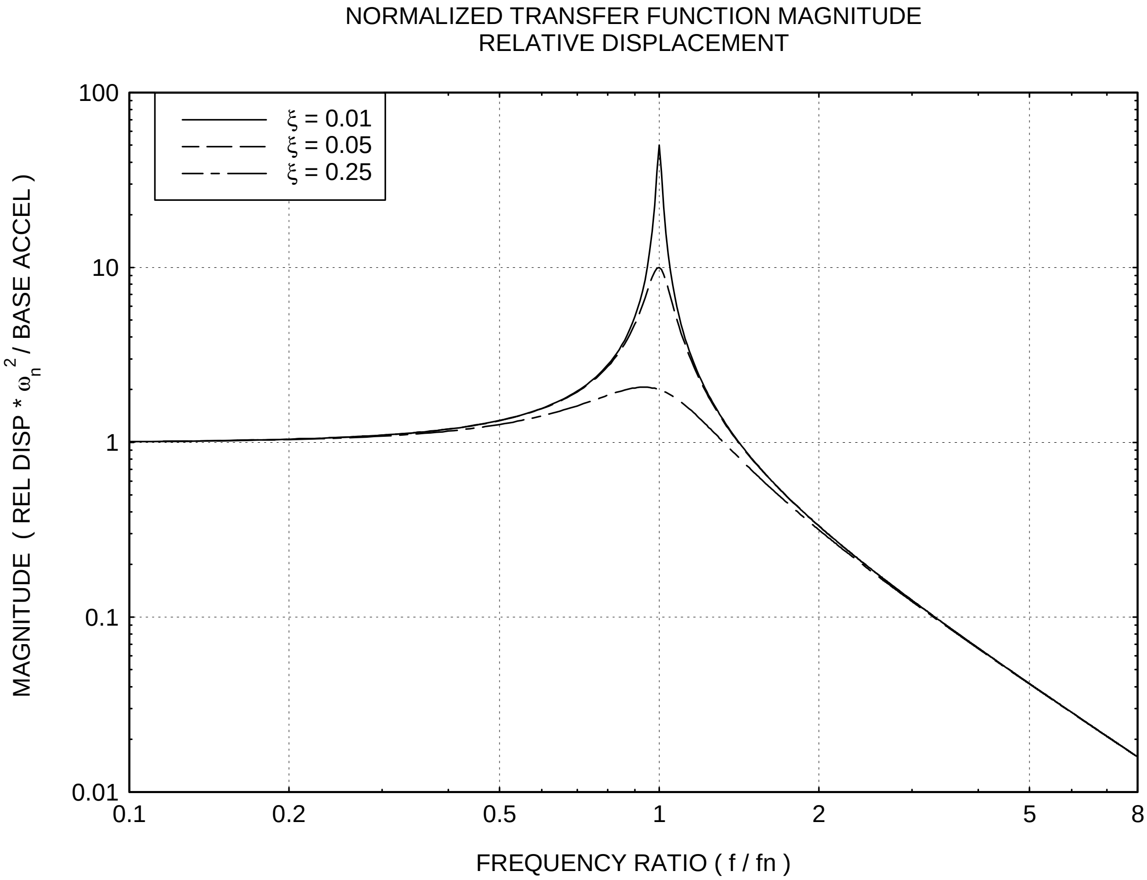

Figure 3.

Note that the units must be consistent. For example, acceleration must be in terms of in^2/sec if relative displacement is in terms of inches.

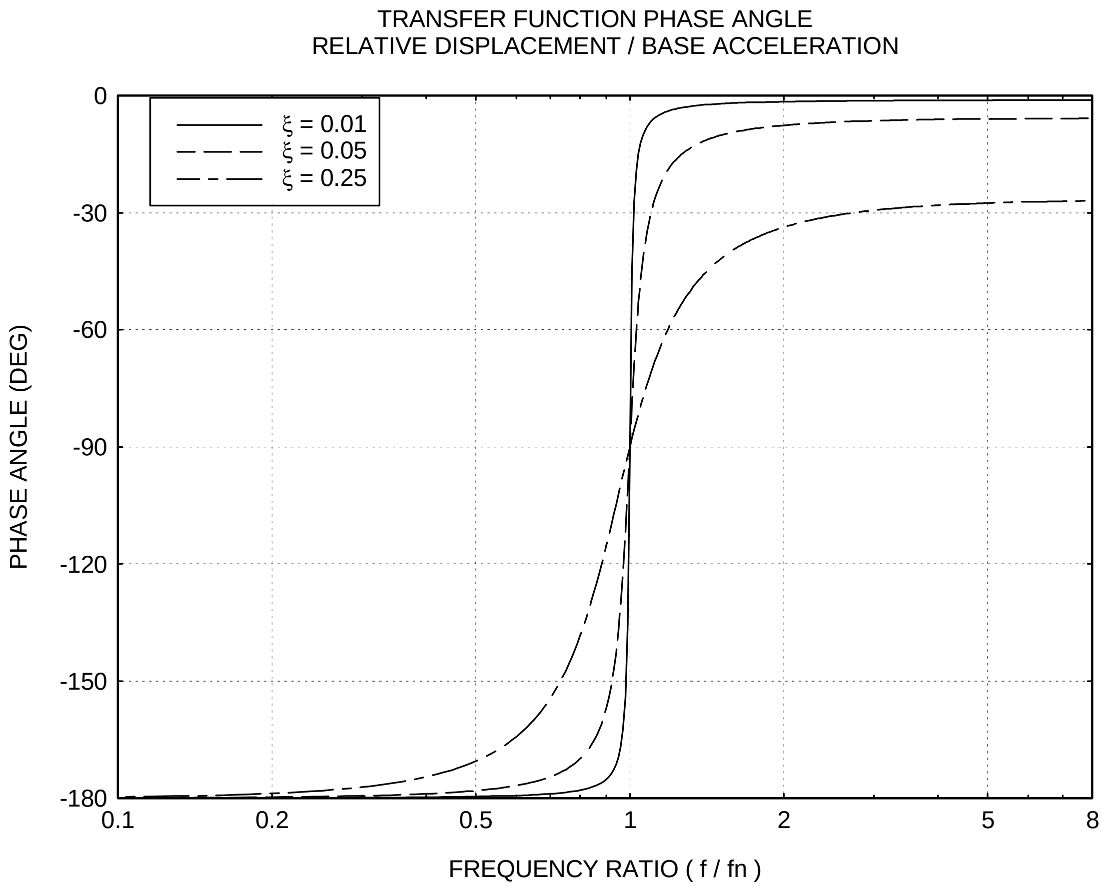
TRANSFER FUNCTION PHASE ANGLE
RELATIVE DISPLACEMENT / BASE ACCELERATION
PHASE ANGLE (DEG)
FREQUENCY RATIO (f / fn)
ξ = 0.01
ξ = 0.05
ξ = 0.25
0
-30
-60
-90
-120
-150
-180
0.1
0.2
0.5
1
2
5
8

Figure 4.