LONGITUDINAL VIBRATION OF A ROD VIA THE FINITE ELEMENT METHOD
Revision B

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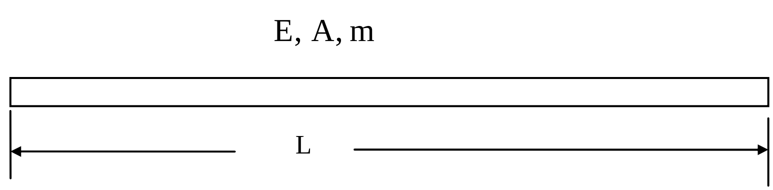
Note that the longitudinal vibration of a rod is analogous to the acoustic pressure oscillation in a pipe.

Introduction

Consider a thin rod.



E is the modulus of elasticity
A is the cross-section area
m is the mass per unit length

The longitudinal displacement u(x, t) is governed by the equation

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] = m(x)\frac{\partial^2 u}{\partial t^2} \tag{1}$$

This equation is taken from Reference 1.

For a uniform cross-section and mass density, the governing equation simplifies to

$$EA\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2} \tag{2}$$

Separate the variables. Let

$$u(x,t) = U(x)T(t) \tag{3}$$

By substitution

$$EA \frac{\partial^2}{\partial x^2} U(x)T(t) = m \frac{\partial^2}{\partial t^2} U(x)T(t) \tag{4}$$

Perform the partial differentiation,

$$EAU''(x)T(t) = mU(x)T''(t) \tag{5}$$

Divide through by U(x)T(t).

$$\frac{EAU''(x)}{mU(x)} = \frac{T''(t)}{T(t)} \tag{6}$$

Each side of equation (6) must equal a constant. Let ω be a constant.

$$\frac{EAU''(x)}{mU(x)} = \frac{T''(t)}{T(t)} = -\omega^2 \tag{7}$$

The spatial equation is

$$\frac{EAU''(x)}{mU(x)} = -\omega^2 \tag{8}$$

$$EAU''(x) = -\omega^2 mU(x) \tag{9}$$

$$EAU''(x) + \omega^2 mU(x) = 0 \tag{10}$$

Again, m is the mass per length.

As an aside, the time equation is

$$T''(t) + \omega^2 T(t) = 0 \tag{11}$$

Equation (10) is a homogeneous, second order, ordinary differential equation.

The weighted residual method is applied to equation (10). This method is suitable for boundary value problems. An alternative method would be the energy method.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (10a) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of p and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral is set equation to zero.

$$\int \phi(x) \left\{ \frac{d^2}{dx^2} U(x) + \left(\frac{\omega}{c}\right)^2 U(x) \right\} dx = 0 \tag{12}$$

The test function $\phi(x)$ can be regarded as a virtual displacement. The differential equation in the brackets represents an internal force. This term is also regarded as the residual. Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Define the domain over the limits from a to b. These limits represent the boundary points of the entire rod.

$$\int_a^b \phi(x) \left\{ EA \frac{d^2}{dx^2} U(x) + m\,\omega^2\, U(x) \right\} dx = 0 \tag{13}$$

$$\int_a^b \phi(x) \left\{ EA \frac{d^2}{dx^2} U(x) \right\} dx \;+\; \int_a^b \phi(x) \left\{ m\,\omega^2 U(x) \right\} dx = 0 \tag{14}$$

$$EA\int_a^b \phi(x)\left\{\frac{d^2}{dx^2}U(x)\right\}dx + m\omega^2\int_a^b \phi(x)\{U(x)\}dx = 0 \tag{15}$$

Integrate the first integral by parts.

$$EA\int_a^b \frac{d}{dx}\left\{\phi(x)\frac{d}{dx}U(x)\right\}dx - EA\int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{d}{dx}U(x)\right\}dx$$

$$+ m\omega^2\int_a^b \phi(x)\{U(x)\}dx = 0 \tag{16}$$

$$EA\left\{\phi(x)\frac{d}{dx}U(x)\right\}\Bigg|_a^b - EA\int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{d}{dx}U(x)\right\}dx$$

$$+ m\omega^2\int_a^b \phi(x)\{U(x)\}dx = 0 \tag{17}$$

Consider a free-free rod. The boundary conditions are

$$\left.\frac{dU}{dx}\right|_{x=a} = 0 \tag{18a}$$

$$U(b) = 0 \tag{18b}$$

Thus, the test functions must satisfy

$$\left.\frac{d\phi}{dx}\right|_{x=a} = 0 \tag{20a}$$

$$\phi(b) = 0 \tag{20b}$$

Equations (20a) and (20b) require

$$\left\{ \phi(x) \frac{d}{dx} U(x) \right\} \Big|_a^b = 0 \tag{21}$$

Apply the boundary conditions to equation (17). The result is

$$-EA\int_a^b \left\{ \frac{d}{dx}\phi(x) \right\} \left\{ \frac{d}{dx} U(x) \right\} dx + m\omega^2 \int_a^b \phi(x) \left\{ U(x) \right\} dx = 0 \tag{22}$$

$$EA\int_a^b \left\{ \frac{d}{dx}\phi(x) \right\} \left\{ \frac{d}{dx} U(x) \right\} dx - m\omega^2 \int_a^b \phi(x) \left\{ U(x) \right\} dx = 0 \tag{23}$$

Note that equation (23) would also be obtained for other simple boundary condition cases.

Now consider that the rod consists of number of segments, or elements. The elements are arranged geometrically in series form. Furthermore, the endpoints of each element are called nodes.

The following equation must be satisfied for each element.

$$EA\int \left\{ \frac{d}{dx}\phi(x) \right\} \left\{ \frac{d}{dx} U(x) \right\} dx - m\omega^2 \int \phi(x) \left\{ U(x) \right\} dx = 0 \tag{24}$$

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = U(x) \tag{25}$$

Thus

$$EA\int \left\{ \frac{d}{dx}U(x) \right\} \left\{ \frac{d}{dx} U(x) \right\} dx - m\omega^2 \int \left\{ U(x) \right\}^2 dx = 0 \tag{27}$$

Express the displacement function U(x) in terms of nodal displacement u_{j-1} and u_j .

$$U(x) = L_1\, u_{j-1} + L_2\, u_j \ , \qquad (j-1)h \le x \le jh \tag{28}$$

Note that h is the element length. In addition, each L coefficients is a function of x.

Now introduce a nondimensional natural coordinate ξ.

$$\xi = j - x/h \tag{29}$$

$$h\,\xi = h\,j - x \tag{30}$$

$$x = h\,j - h\,\xi \tag{31}$$

$$\left(\frac{x}{h}\right) = j - \xi \tag{32}$$

The derivative is

$$dx = -\,h\, d\xi \tag{33}$$

$$d\xi = -\frac{1}{h} dx \tag{34}$$

Note that h is the segment length.

Change the integration variable in equation (27) using equation (33). Also, apply the integration limits.

$$-\,EAh\int_0^1 \left\{\frac{d}{dx}U(x)\right\}\left\{\frac{d}{dx}U(x)\right\} d\xi + hm\omega^2 \int_0^1 \{U(x)\}^2\, d\xi = 0 \tag{35}$$

$$EAh\int_0^1 \left\{\frac{d}{dx}U(\xi)\right\}\left\{\frac{d}{dx}U(\xi)\right\} d\xi - hm\omega^2 \int_0^1 \{U(\xi)\}^2\, d\xi = 0 \tag{36}$$

The displacement function becomes.

$$U(\xi) = L_1\, u_{j-1} + L_2\, u_j \ , \qquad 0 \le \xi \le 1 \tag{37}$$

The slope equation is

$$U'(\xi) = L_1' \, y_{j-1} + L_2' \, h \, \theta_{j-1} \; , \qquad 0 \le \xi \le 1 \tag{38}$$

$$L_1 = 1 - \xi \tag{39}$$

$$L_1' = -1 \tag{40}$$

$$L_2 = \xi \tag{41}$$

$$L_2' = 1 \tag{42}$$

Now Let

$$U(x) = \underline{L}^T \; \bar{a} \, , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{43}$$

where

$$\underline{L} = [1 - \xi \qquad \xi \;]^T \tag{44}$$

$$\bar{a} = [\; p_{j-1} \qquad p_j \;]^T \tag{45}$$

The derivative terms are

$$\frac{d}{dx} U(x) = \frac{d}{dx} \underline{L}^T \; \bar{a} \, , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{46}$$

$$\frac{d}{dx} U(x) = \frac{d}{d\xi} \frac{d\xi}{dx} \underline{L}^T \; \bar{a} \, , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{47}$$

$$\frac{d}{dx}U(x) = \left(-\frac{1}{h}\right)\underline{L'}^{\,T}\ \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{48}$$

where

$$\underline{L}' = [-1 \qquad 1\ ]^T \tag{49}$$

Note that primes indicate derivatives with respect to ξ.

(50)

Equation (36) becomes

$$EAh\int_0^1 \left\{ \left[\left(\frac{1}{-h}\right)\underline{L'}^{\,T}\ \ \bar{a}\right]\left[\left(\frac{1}{-h}\right)\underline{L'}^{\,T}\ \ \bar{a}\right] \right\} d\xi$$

$$+hm\omega^2\int_0^1 \left[\ \underline{L}^T\ \ \bar{a}\right]\left[\ \underline{L}^T\ \ \bar{a}\right]\ d\xi = 0 \tag{51}$$

$$\frac{EA}{h}\int_0^1 \left\{ \left[\ \underline{L'}^{\,T}\ \ \bar{a}\right]\left[\ \underline{L'}^{\,T}\ \ \bar{a}\right] \right\} d\xi \ - h^2\, m\omega^2\int_0^1 \left[\ \underline{L}^T\ \ \bar{a}\right]\left[\ \underline{L}^T\ \ \bar{a}\right]\ d\xi = 0 \tag{52}$$

$$\frac{EA}{h}\int_0^1 \left\{ \bar{a}^T\,\underline{L'}\ \underline{L'}^{\,T}\ \ \bar{a} \right\} d\xi \ - \ h^2\, m\omega^2\int_0^1 \left[\ \bar{a}^T\,\underline{L}\,\underline{L}^T\ \ \bar{a}\right]\ d\xi = 0 \tag{53}$$

$$\bar{a}^T\,\frac{EA}{h}\left\{\int_0^1 \left\{ \underline{L'}\ \underline{L'}^{\,T} \right\} d\xi \ - \ h^2\, m\omega^2\int_0^1 \left[\ \underline{L}\,\underline{L}^T\right]\ d\xi \right\}\bar{a}\ \ = 0 \tag{54}$$

$$\frac{EA}{h}\int_0^1 \left\{ \underline{L'}\ \underline{L'}^{\,T} \right\} d\xi \ - \ h^2\, m\omega^2\int_0^1 \left[\ \underline{L}\,\underline{L}^T\right]\ d\xi = 0 \tag{55}$$

For a system of n elements,

$$K_j - \lambda M_j = 0 , \qquad j = 1, 2, \ldots, n \tag{56}$$

where

$$K_j = \frac{EA}{h} \int_0^1 \left\{ \underline{L}' \underline{L}'^T \right\} d\xi \tag{57}$$

$$M_j = h\, m \int_0^1 \left\{ \underline{L}\, \underline{L}^T \right\} d\xi \tag{58}$$

$$\lambda = \omega^2 \tag{59}$$

$$\underline{L}' \underline{L}'^T = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{60}$$

$$\underline{L}' \underline{L}'^T = \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{61}$$

Note that only the upper triangular components are shown due to symmetry.

$$K_j = \frac{EA}{h} \int_0^1 \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} d\xi \tag{62}$$

$$K_j = \frac{EA}{h} \begin{bmatrix} \xi & -\xi \\ & \xi \end{bmatrix} \Bigg|_0^1 \tag{63}$$

$$K_j = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{64}$$

$$\underline{L}\,\underline{L}^T = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \tag{65}$$

$$\underline{L}\,\underline{L}^T = \begin{bmatrix} 1-2\xi+\xi^2 & \xi-\xi^2 \\ & \xi^2 \end{bmatrix} \tag{66}$$

Recall

$$M_j = hm\int_0^1 \left\{ \underline{L}\,\underline{L}^T \right\} d\xi \tag{67}$$

$$M_j = hm\int_0^1 \begin{bmatrix} 1-2\xi+\xi^2 & \xi-\xi^2 \\ & \xi^2 \end{bmatrix} d\xi \tag{68}$$

$$M_j = hm\begin{bmatrix} \xi-\xi^2+\frac{1}{3}\xi^3 & \frac{1}{2}\xi^2-\frac{1}{3}\xi^3 \\ & \frac{1}{3}\xi^3 \end{bmatrix}\Bigg|_0^1 \tag{69}$$

$$M_j = hm\begin{bmatrix} 1-1+\frac{1}{3} & \frac{1}{2}-\frac{1}{3} \\ & \frac{1}{3} \end{bmatrix} \tag{70}$$

$$M_j = hm\begin{bmatrix} 1/3 & 1/6 \\ & \\ & 1/3 \end{bmatrix} \tag{71}$$

$$M_j = \frac{hm}{6}\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \tag{72}$$

Again, m is mass per length, and h is the element length.

A derivation of the mass and stiffness matrices via the energy method is given in Appendix A.

Examples are given in Appendices B and C.

References

1. L. Segerlind, Applied Finite Element Analysis, Wiley, New York, 1984.
2. T. Irvine, The Generalized Eigenvalue Problem, 1999.
3. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.
4. W. Weaver and P. Johnson, Structural Dynamics by Finite Elements, Englewood Cliffs, New Jersey, 1987.

APPENDIX A

Energy Method

The total strain energy P of a bar is

$$P = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx \tag{A-1}$$

The total kinetic energy T of a bar is

$$T = \frac{1}{2}\omega_n^2 \int_0^L m[u]^2 \, dx \tag{A-2}$$

Again let

$$U(x) = \underline{L}^{T} \quad \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{A-3}$$

$$\frac{d}{dx}U(x) = \left(-\frac{1}{h}\right)\underline{L'}^{T} \quad \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{A-4}$$

$$d\xi = -dx/h \tag{A-5}$$

Assume constant mass density and stiffness.

The strain energy is converted to a localized stiffness matrix as

$$K_j = \frac{EA}{h}\int_0^1 \left\{ \underline{L}' \, \underline{L}'^{T} \right\} d\xi \tag{A-6}$$

The kinetic energy is converted to a localized mass matrix as

$$M_j = hm\int_0^1 \left\{ \underline{L}\,\underline{L}^T \right\} d\xi \tag{A-7}$$

The total strain energy is set equal to the total kinetic energy per the Rayleigh method.

The result is a generalized eigenvalue problem. For a system of n elements,

$$K_j - \omega^2 M_j = 0 \,, \qquad j = 1, 2, 3, \ldots. \tag{A-8}$$

where

$$M_j = \frac{hm}{6}\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \tag{A-9}$$

$$K_j = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{A-10}$$

Again, m is mass per length.

APPENDIX B

Example 1:   Free-Free Rod, FE Model, Two Elements

The finite element model of the rod is shown in Figure B-1. It consists of two elements and three nodes. The rod has length L. Assume constant mass and stiffness Each element has an equal length.

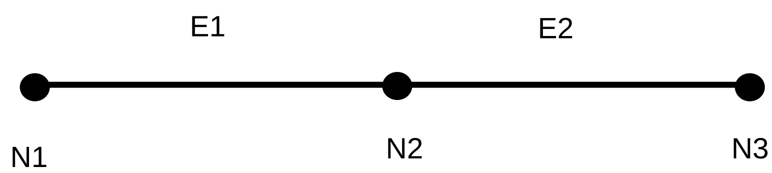


Figure B-1.

The boundary conditions are

$$\left.\frac{dU}{dx}\right|_{x=0} = 0 \tag{B-1}$$

$$\left.\frac{dU}{dx}\right|_{x=L} = 0 \tag{B-2}$$

The generalized eigenvalue problem is

$$K_j - \lambda\, M_j = 0\,, \qquad j = 1, 2, \ldots, n \tag{B-3}$$

The elemental mass matrix is

$$M_j = \frac{hm}{6}\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \tag{B-4}$$

The elemental stiffness matrix is

$$K_j = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{B-5}$$

The generalized eigenvalue problem with global mass and stiffness matrices is

$$\det\left\{ \frac{EA}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \left(\frac{h\, m\, \omega^2}{6}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right\} = 0 \tag{B-6}$$

$$\det\left\{ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \left(\frac{h^2 m\, \omega^2}{6EA}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right\} = 0 \tag{B-7}$$

$$\det\left\{ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \left(\frac{\lambda}{6}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right\} = 0 \tag{B-8}$$

Note that

$$\lambda = \left(\frac{h^2 m\, \omega^2}{EA}\right) \tag{B-9}$$

Let

$$c = \sqrt{\frac{EA}{m}} \tag{B-10}$$

$$h = L / 2 \tag{B-11}$$

The eigenvalues are obtained via Matlab as follows. (See also References 2 and 3.)

```
k =

     1    -1     0
    -1     2    -1
     0    -1     1

m =

    0.3333    0.1667         0
    0.1667    0.6667    0.1667
         0    0.1667    0.3333

>> [ModeShapes,Eigenvalues]=eig(k,m);
>> Eigenvalues

Eigenvalues =

   -0.0000         0         0
         0    3.0000         0
         0         0   12.0000
```

The resulting natural frequencies and mode shapes are shown in Table B-1 and B-2, respectively. The mode shapes for the second and third modes are plotted in Figures B-2 and B-3, respectively.

Table B-1. Free-Free Rod, Natural Frequencies

i	FEM λ_i	FEM ω_i (rad/sec)	FEM f_i (Hz)	Classical Solution f_i (Hz)
1	0	0	0	0
2	3.0	3.464 c / L	0.551 c / L	0.5 c / L
3	12.0	6.928 c / L	1.103 c / L	1.0 c / L

Table B-2. Free-Free Rod,
Displacement Eigenvectors with Arbitrary Scale

x / L	Mode 1	Mode 2	Mode 3
0	0.577	0.707	0.577
0.5	0.577	0.000	-0.577
1.0	0.577	-0.707	0.577

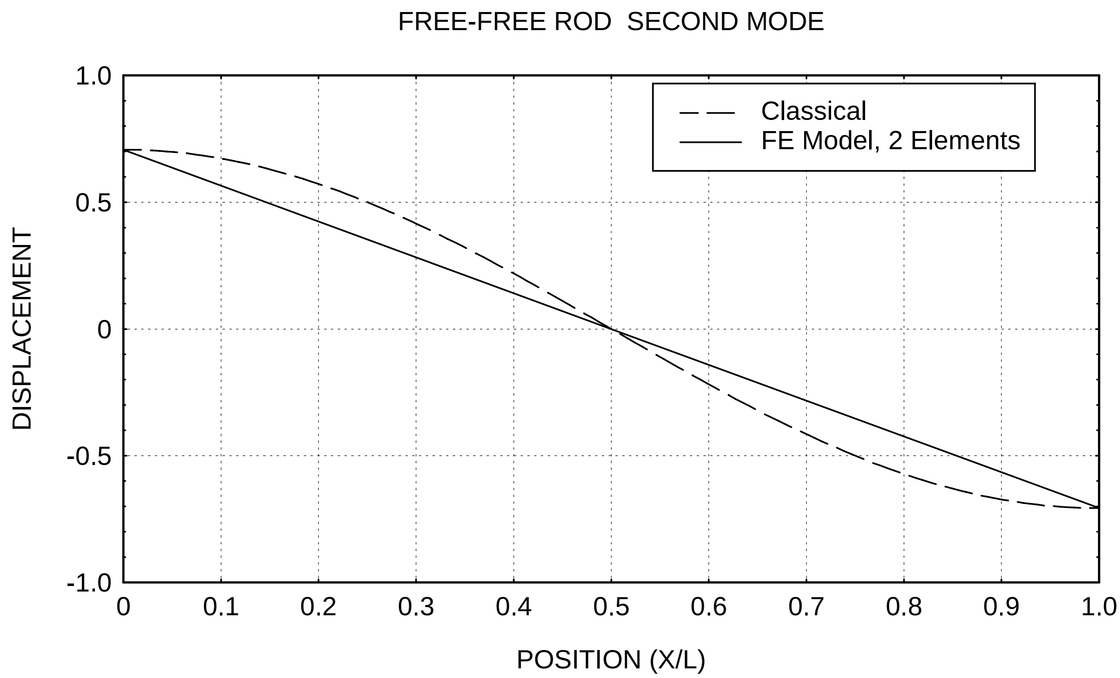


Figure B-2.

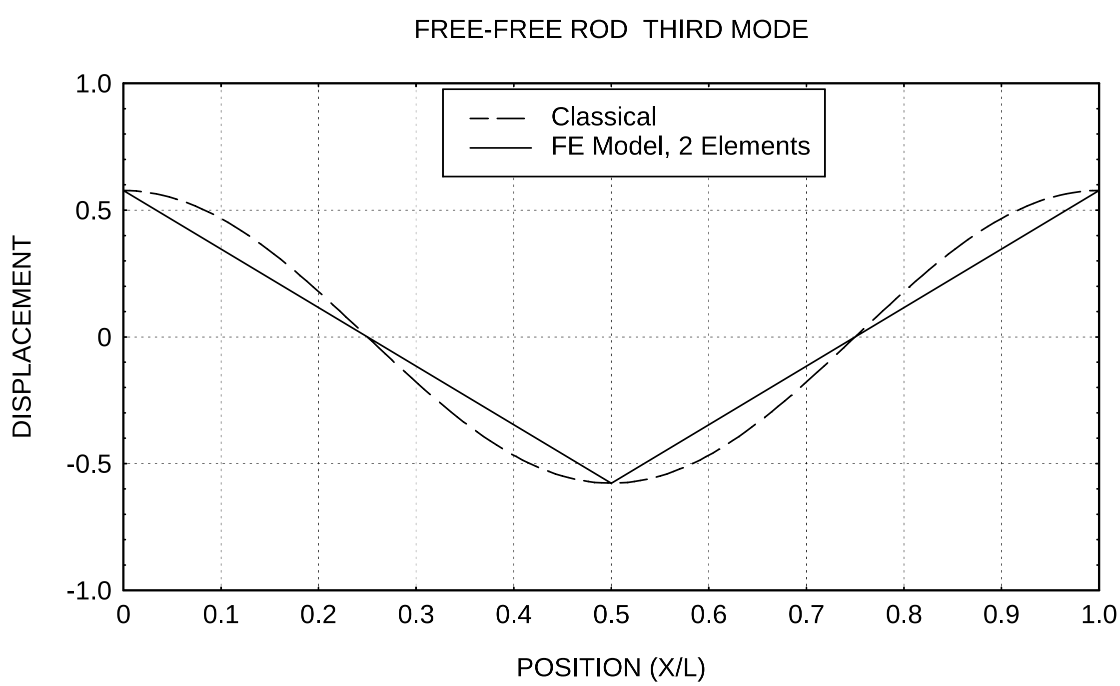


Figure B-3.

APPENDIX C

Example 1: Fixed-Free Rod, FE Model, Four Elements

The finite element model of the rod is shown in Figure C-1. It consists of four elements and five nodes. The rod has length L. Each element has an equal length.

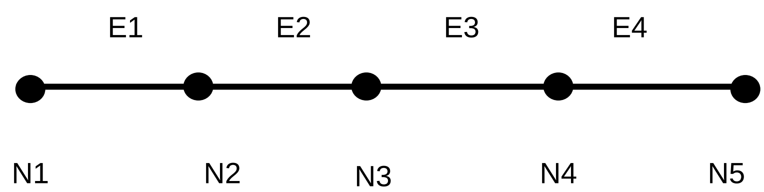


Figure C-1.

The boundary conditions are

$$U(0) = 0 \qquad \text{(Fixed end)} \tag{C-1}$$

$$\left.\frac{dU}{dx}\right|_{x=L} = 0 \qquad \text{(Free end)} \tag{C-2}$$

The generalized eigenvalue problem with global mass and stiffness matrices is assemble in the same manner as the example in Appendix B. The open boundary condition must be considered, however.

Application of the $U(0) = 0$ boundary condition causes each entry in the first column and first row of each matrix to equal zero. The last column and last row are thus removed from the problem. The resulting eigenvalue problem is

$$\det\left\{\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} - \left(\frac{\lambda}{6}\right)\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\right\} = 0 \tag{C-3}$$

Recall

$$\lambda = \left(\frac{h^2 m \omega^2}{EA} \right) \tag{C-4}$$

$$c = \sqrt{\frac{EA}{m}} \tag{C-5}$$

$$h = L / 4 \tag{C-6}$$

The eigenvalues are obtained via Matlab as follows.

```
m =

    0.6667    0.1667         0         0
    0.1667    0.6667    0.1667         0
         0    0.1667    0.6667    0.1667
         0         0    0.1667    0.3333


k =

     2    -1     0     0
    -1     2    -1     0
     0    -1     2    -1
     0     0    -1     1

>> [ModeShapes,Eigenvalues]=eig(k,m);
>> Eigenvalues

Eigenvalues =

    0.1562         0         0         0
         0    1.5545         0         0
         0         0    5.1295         0
         0         0         0   10.7268
```

The resulting natural frequencies are given in Table C-1. The first and second mode shapes are plotted in Figures C-2 and C-3, respectively. Each mode shape has an arbitrary scale factor.

Table C-1. Fixed-Free Rod, Natural Frequencies				
i	FEM λ_i	FEM ω_i (rad/sec)	FEM f_i (Hz)	Classical Solution f_i (Hz)
1	0.1562	1.581 c / L	0.25 c / L	0.25 c / L
2	1.5545	4.987 c / L	0.79 c / L	0.75 c / L
3	5.1295	9.060 c / L	1.44 c / L	1.25 c / L
4	10.7268	13.101 c / L	2.09 c / L	1.75 c / L

Table C-2. Fixed-Free Rod, Displacement Eigenvectors with Arbitrary Scale			
x / L	Mode 1	Mode 2	Mode 3
0	0	0	0
0.25	-0.274	0.733	0.890
0.50	-0.507	0.561	-0.681
0.75	-0.662	-0.304	-0.369
1.00	-0.716	-0.793	0.963

FIXED-FREE ROD FIRST MODE

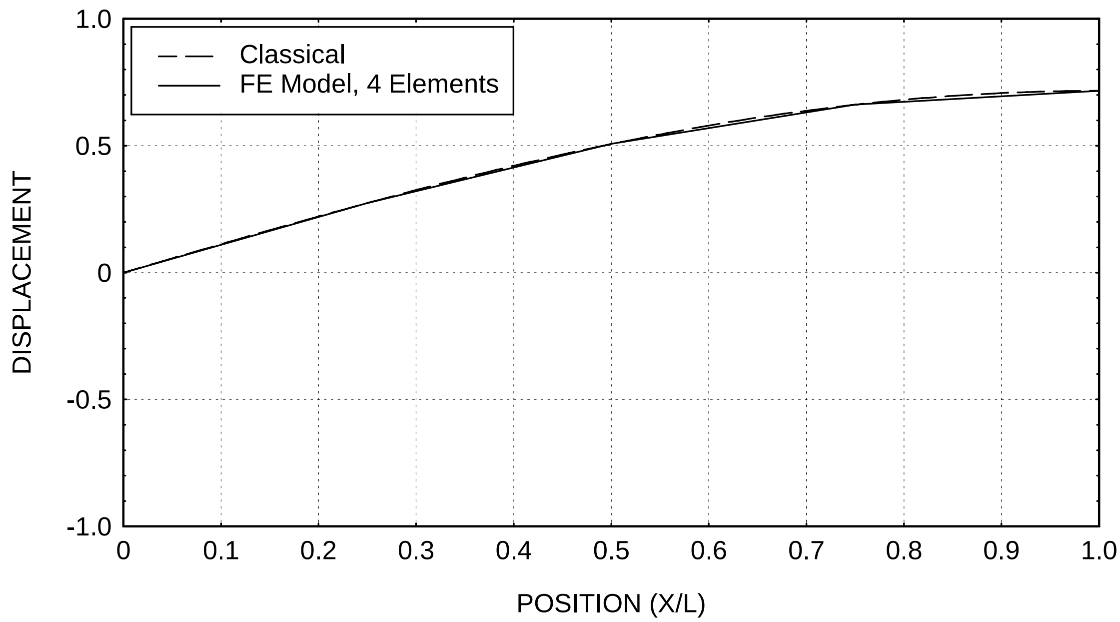


Figure C-2.

FIXED-FREE ROD SECOND MODE

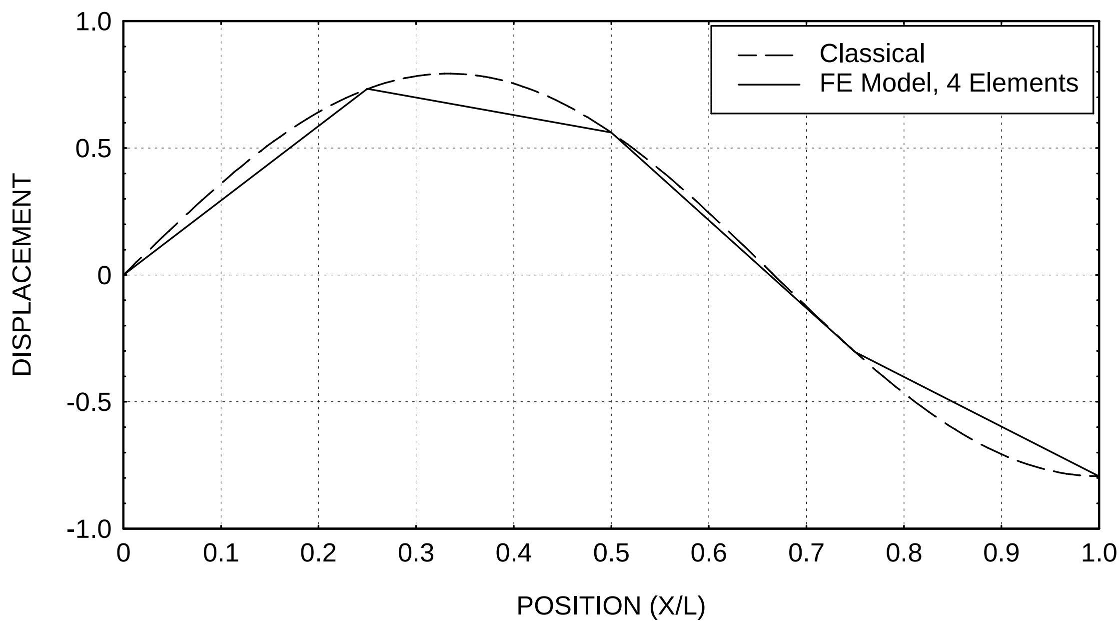


Figure C-3.