Lateral Natural Frequencies of Rotor Shaft Systems via the Finite Element Method

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Abstract

Two analytical methods, the Transfer Matrix method (TMM) and the Finite Element method (FEM), are discussed for evaluating the lateral natural frequencies of a shaft rotor system. Hypothetical cases are considered for evaluating fundamental and higher frequencies. A MATLAB program is developed for the Finite Element method. It may be useful for evaluating the frequencies of the system's higher modes as well as the fundamental frequency.

Introduction

Consider a rotor shaft system in which the operating speed matches the frequencies of the higher mode. In this case, it is necessary to evaluate the higher frequencies of the system. It is necessary to know the higher frequencies while taking the rotor through a critical speed to an operational speed. To avoid failures of shafting, the general practice in the design of rotors is to determine the bending critical speeds as well as higher frequencies. For most rotors, it is the fundamental mode which falls in the running speed zone. There are several methods of calculation of the critical speed:

1. Dunkerley method
2. Rayleigh method
3. Transfer Matrix method
4. Finite Element method

The first two methods are suitable for estimating the fundamental frequency by hand calculation. The transfer matrix method and FEM would require the use of computers. The third and fourth methods are also useful for evaluating the higher frequencies. In this paper, higher frequency methods are discussed, specifically the Transfer Matrix method and Finite Element method.

Transfer Matrix Method

In a manner similar to Holzer method, Myklestad and Prohl [1] developed highly successful method of computation for the bending critical speeds of a shaft. Consider an n mass system, each mass representing a gear, a disk, or a flywheel etc. All these masses are taken as lumped with their gyroscopic inertia neglected. We adopt a root searching technique to determine the natural frequency w. The algorithm of the program for TMM is as follows:

TMM Algorithm

1. Take the trial value of frequency w.
2. Assign the element in the matrix with different variable.
3. Form the mass and field matrix.
4. Arrange the matrix in the order of multiplication and multiply it.
5. Apply boundary condition to overall transfer matrix.
6. Find the determinant value. If it is zero, or nearly zero, then this w will be natural frequency, otherwise repeat the same procedure for different values of w till the determinant approaches to zero.
7. After evaluating the first mode, repeat the same procedure by starting the frequency value higher than the evaluated one.

Another method for evaluating the fundamental frequency, as well as higher frequencies, is the Finite Element Method.

Finite Element Method

The main procedures in the finite element analysis are [2]

1. Read input data and allocate proper array sizes.
2. Calculate element matrices and vectors for every element.
3. Assemble matrices and vectors into the system matrix and vector.
4. Apply constraints to the system matrix and vector.
5. Solve the matrix equation for the eigenvalues.

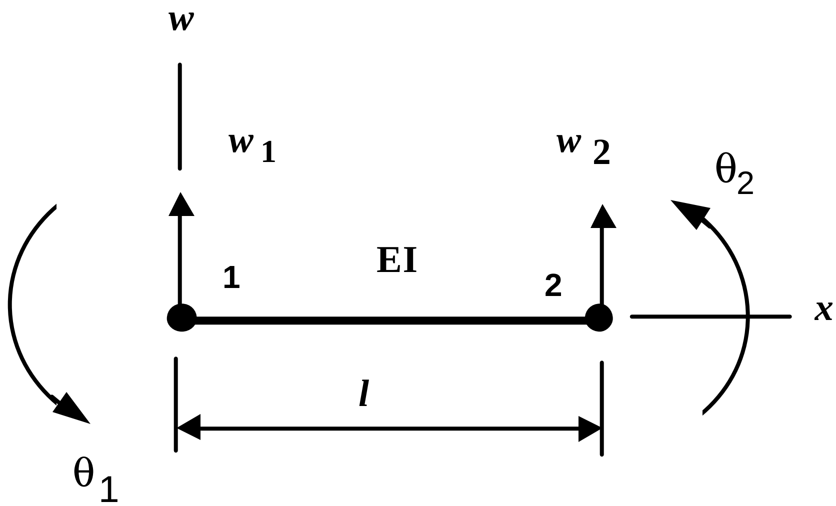


Figure 1: Beam Element

Input Data

The major input parameters needed for the finite element analysis program are:

- the number of total nodes in the system
- the number of total elements in the system
- coordinate values of every node in terms of the global coordinate system
- types of every element
- information for boundary condition
- element properties (modulus of elasticity, moment of inertia, length, area, poison's ratio, etc.)

Element Matrices for Stiffness and Mass

The beam element is used for analysis of the rotor shaft system. A typical beam element is shown in Figure 1. w_1 and w_2 are the vertical displacements at points 1 and 2, respectively. θ_1 and θ_2 are the slopes at point 1 and 2, respectively. l is length of beam element. The derivation of the stiffness and mass matrices is given in Appendix A.

The following are the stiffness matrix and mass matrices used for evaluating the natural frequencies.

The stiffness matrix $[K]$ is

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{1}$$

The consistent mass matrix $[M]_{cons}$ is

$$[M]_{cons} = \frac{m}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{2}$$

The lumped mass matrix $[M]_{Lumped}$ is

$$[M]_{Lumped} = \begin{bmatrix} m/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

The diagonal mass matrix $[M]_{diag}$ is

$$[M]_{diag} = \begin{bmatrix} m/2 & 0 & 0 & 0 \\ 0 & ml^2/78 & 0 & 0 \\ 0 & 0 & m/2 & 0 \\ 0 & 0 & 0 & ml^2/78 \end{bmatrix} \tag{4}$$

Assembly of Element Matrices

Once these matrices are computed, they are assembled into the global system stiffness and mass matrices.

Application of Constraints

The boundary condition of the simply supported shaft rotor system is applied to the global mass and stiffness matrices, i.e. the deflection at each simply supported node point is zero.

Solving for the eigenvalues

Once the system matrices are modified than simply using following command line in MATLAB software, we can evaluate the eigenvalue of the system.

$$w = sqrt(eig(K, M))$$

This two methods are demonstrated by solving the following three problems.

Problem - I

A uniform shaft with two disks and supported bearings is shown in Figure 2. Evaluate the natural frequencies of the system.

Problem - II

A stepped shaft with one disk and supported bearings is shown in Figure 3. Evaluate the natural frequencies of the system.

Problem - III

A uniform shaft with three disks and supported bearings is shown in Figure 4. Evaluate the natural frequencies of the system.

Solution

The Transfer Matrix Method and modified computer program are used to obtain the natural frequencies shown in Table 1. The table also shows the natural frequencies using FEM.

6, 4, and 4 beam elements are considered for evaluating the frequencies of problems I, II and III, respectively. Typical FEM models for the above three problems are shown in Figures 5, 6 and 7. In these figures, Roman letters denote the number of elements. The length of each element is also given. All problems are evaluated using consistent, lumped and diagonal mass matrix. The results are shown in Table 2. Table 1 shows a comparison of the Transfer matrix method and the Finite element method using a lumped mass matrix.

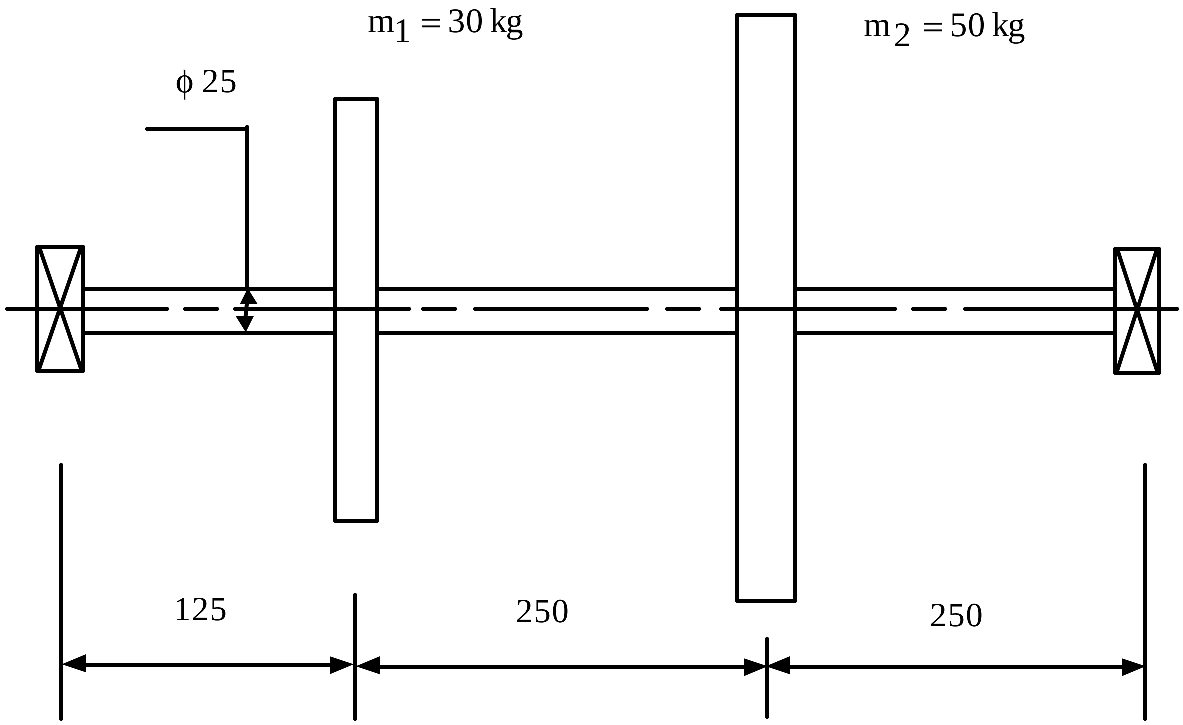


Figure 2: Example 1. Two disk and shaft system

All dimensions are in mm unless otherwise specified.

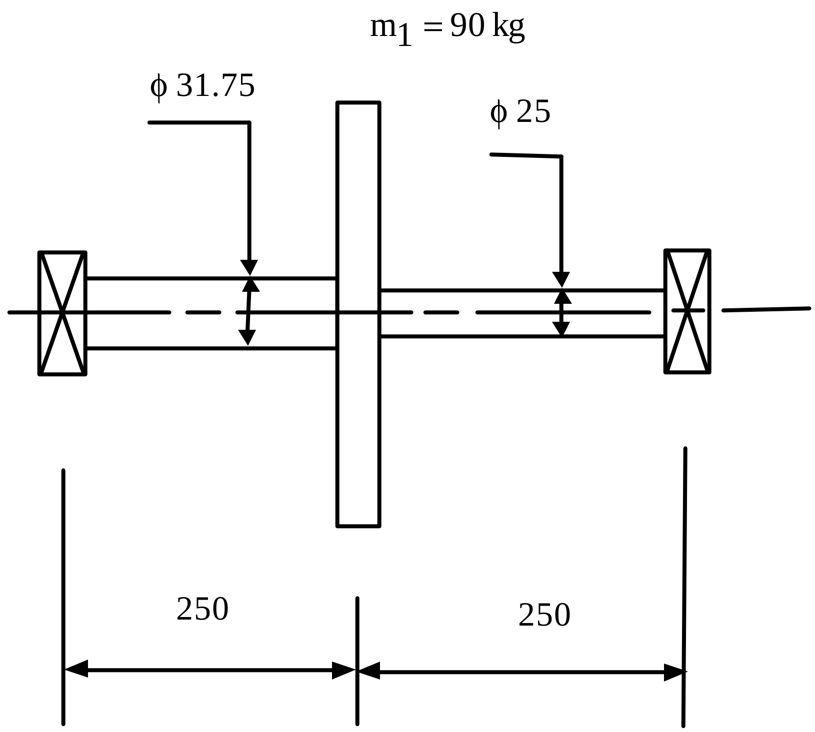


Figure 3: Example 2. Step shaft and disk system

All dimensions are in mm unless otherwise specified.

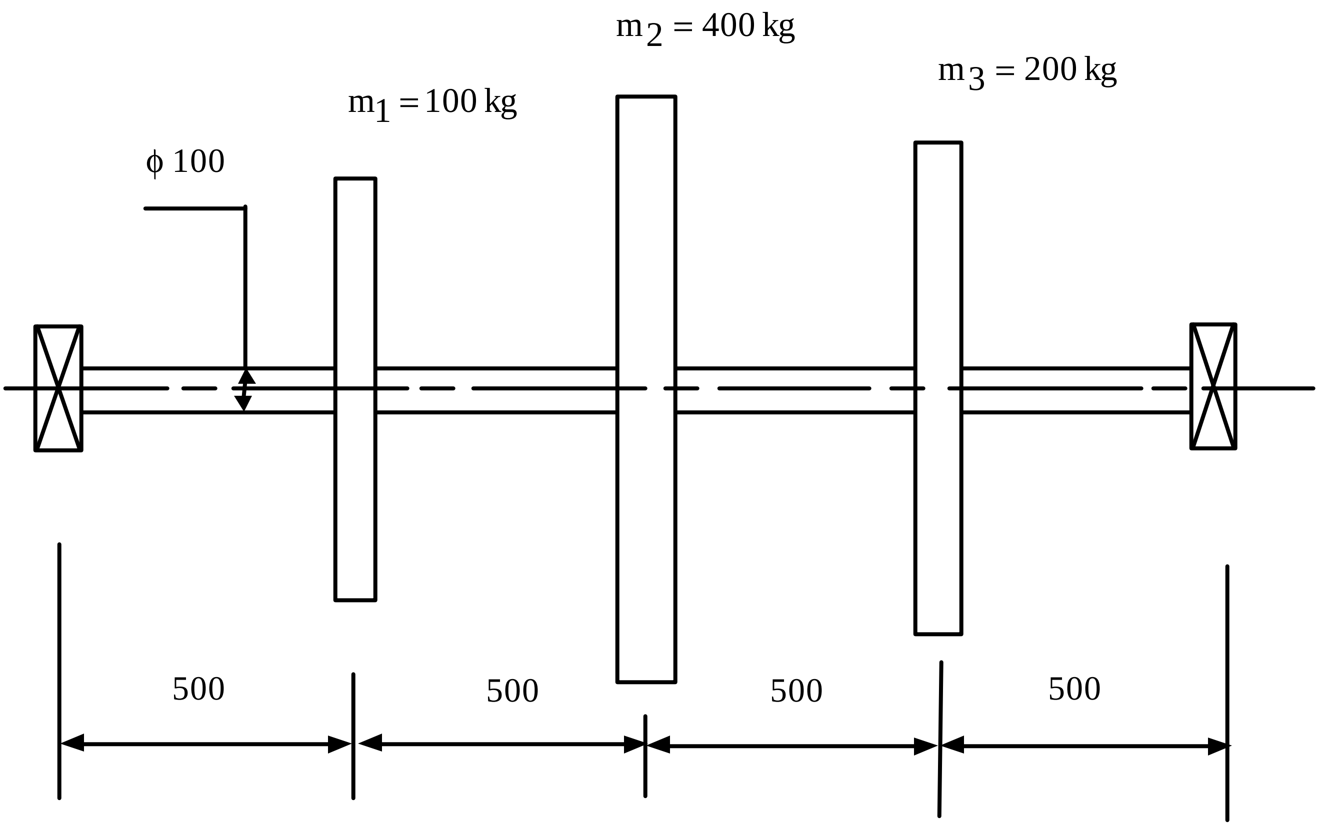
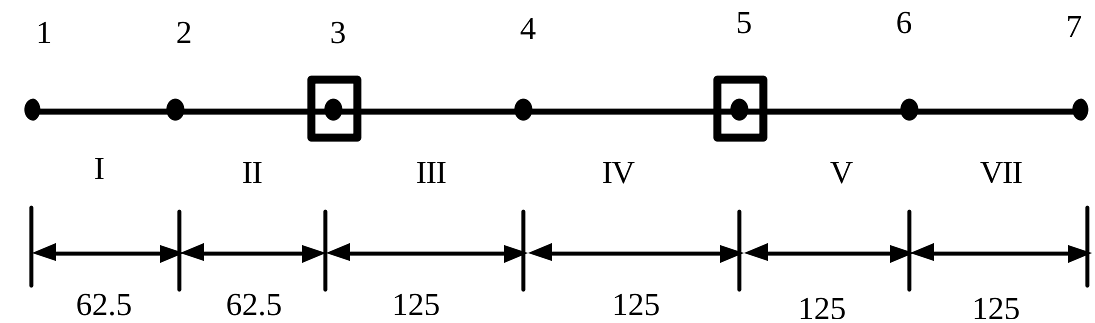


Figure 4: Example 3. Three disk and shaft system

All dimensions are in mm unless otherwise specified.



The dimensions are in units of mm.

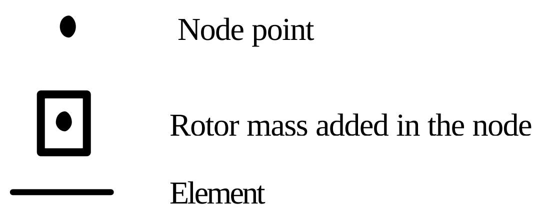
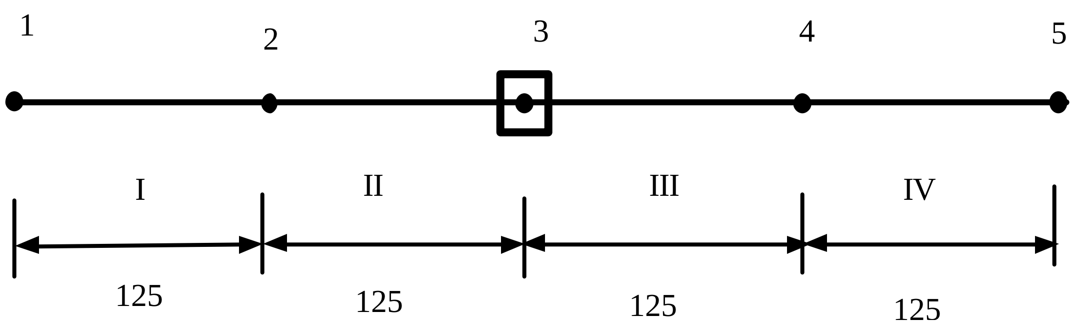


Figure 5: FEM model for example 1



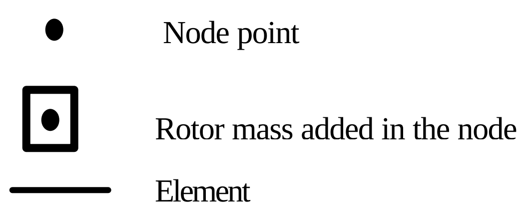


Figure 6: FEM model for example 2

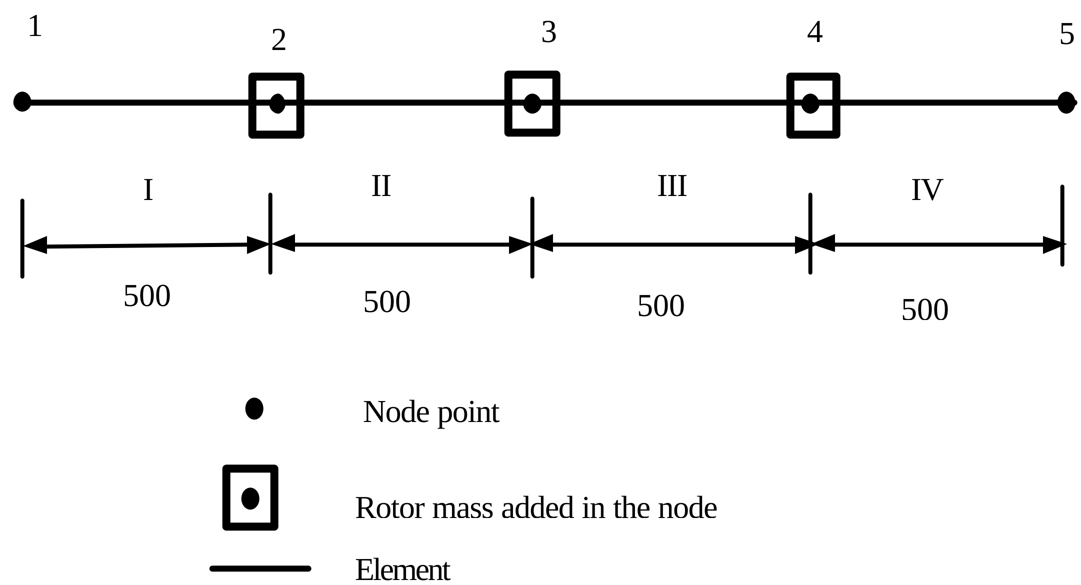


Figure 7: FEM model for example 3

Table 1: Comparison of natural frequencies using transfer matrix method with FEM						
Problem	Transfer Matrix Method			Finite Element Method with Lumped Mass Matrix		
	fn 1(Hz)	fn 2 (Hz)	fn 3 (Hz)	fn 1 (Hz)	fn 2 (Hz)	fn 3 (Hz)
I	18.34	78.06	*	18.42	77.24	873.36
II	24.26	*	*	24.31	906.09	1320.00
III	16.52	88.87	167.52	15.70	81.23	151.67
III	15.54**	79.24**	147.85	15.70	81.23	151.67

* - unable to evaluate using developed program due to divergence problem.
** - shaft mass is added into rotor mass for transfer matrix method.
1,2,3 - denotes first, second, and third mode frequencies respectively.

Table 2: Comparison of natural frequencies using FEM with different mass matrix									
Problem No.	Consistent Mass Matrix			Lumped Mass Matrix			Diagonal Mass Matrix		
	fn 1(Hz)	fn 2 (Hz)	fn 3 (Hz)	fn 1 (Hz)	fn 2 (Hz)	fn 3 (Hz)	fn 1 (Hz)	fn 2 (Hz)	fn 3 (Hz)
I	18.41	77.31	1009.03	18.42	77.24	873.36	18.41	77.18	1113.09
II	24.32	1026.18	1804.68	24.31	906.09	1320.00	24.29	1162.11	1526.99
III	15.70	81.42	153.77	15.70	81.23	151.67	15.69	80.84	150.88

1,2,3 - denotes first, second, and third mode frequencies respectively.

The corresponding mode shapes of the above examples are shown in Figures 8, 9, and 10. The mode shapes are plotted as relative amplitude versus length of beam.

Conclusion

Both the Transfer Matrix method and the Finite Element method are suitable for evaluating the fundamental as well as higher frequencies. The Transfer Matrix method gave very similar results to the Finite Element model with lumped mass for the fundamental frequency, as shown in Table 1.

The Transfer Matrix method for problem II was only able to obtain the fundamental frequency. It was unable to converge at higher frequencies. When the number of elements is increased from 2 to 4, the values of higher frequencies is obtained, as shown in Table 1. The Transfer Matrix method and FEM give nearly same values of the frequencies when the mass of the shaft is negligible.

Problem III gives a better idea of neglecting mass and adding mass on the system with TMM. The effect of mass matrix in FEM is also observed in Table 2. For higher modes, TMM takes more computation time due to iterations, but that is not case with FEM. If engineers take more elements in the same system, then it will take more time for computation otherwise it is faster than TMM. Sometimes more elements are required for increasing precision of eigenvalues to achieve exact solution and mode shapes.

The Rayleigh Ritz method also gives the higher frequencies, but it requires judgement of the mode shape. It is tedious and time consuming.

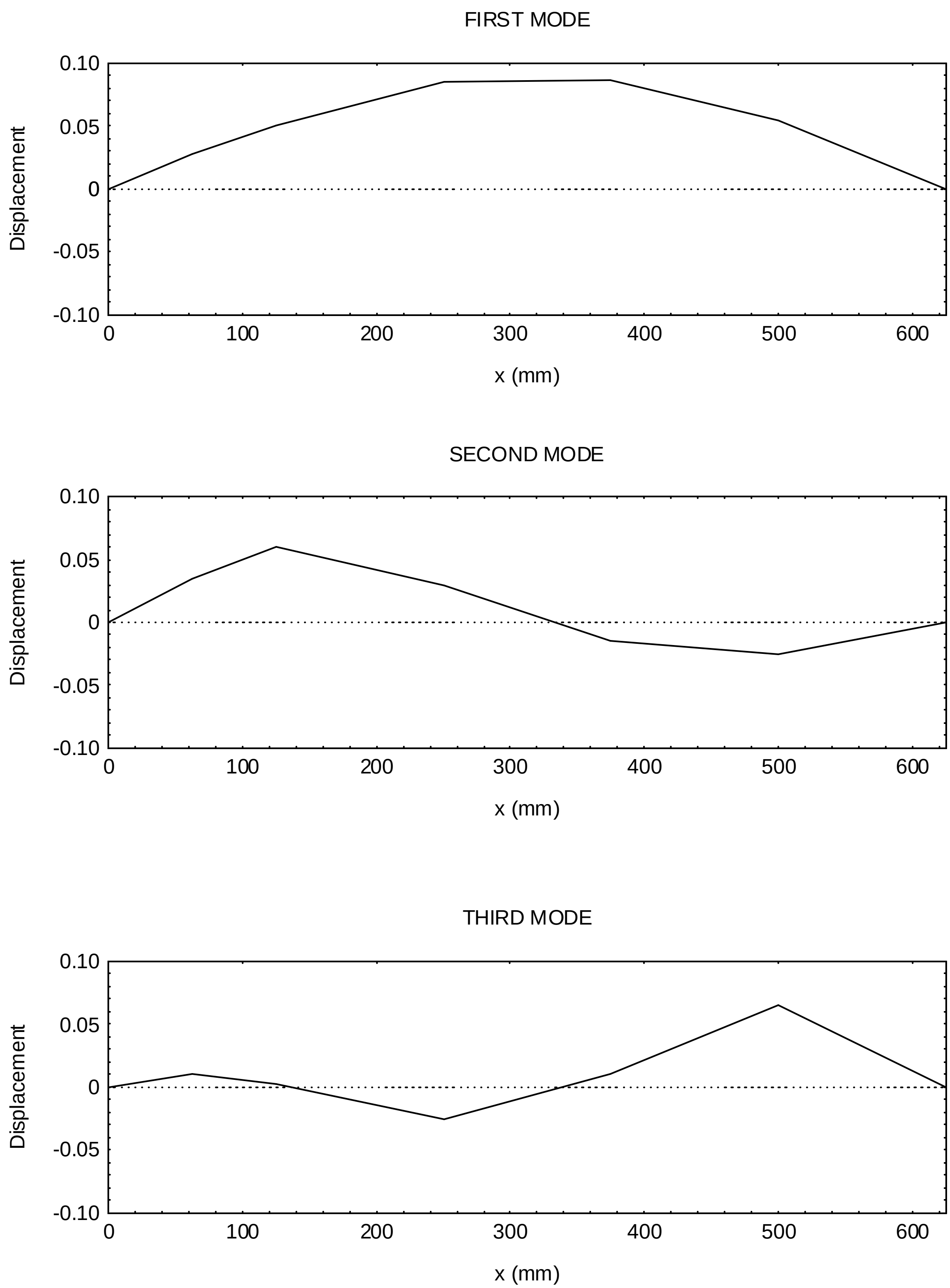


Figure 8. Mode Shape of Example 1

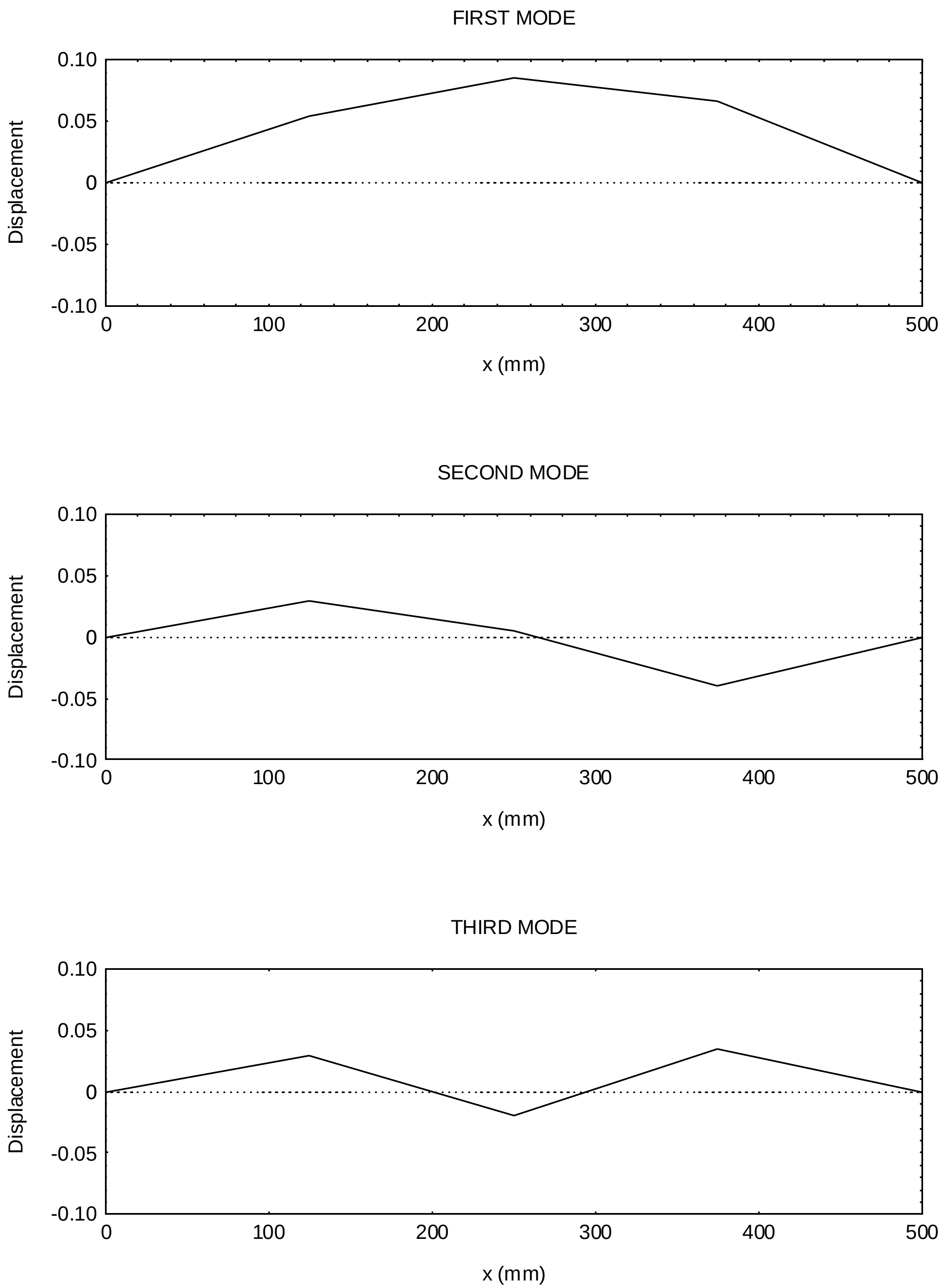


Figure 9. Mode Shape of Example 2

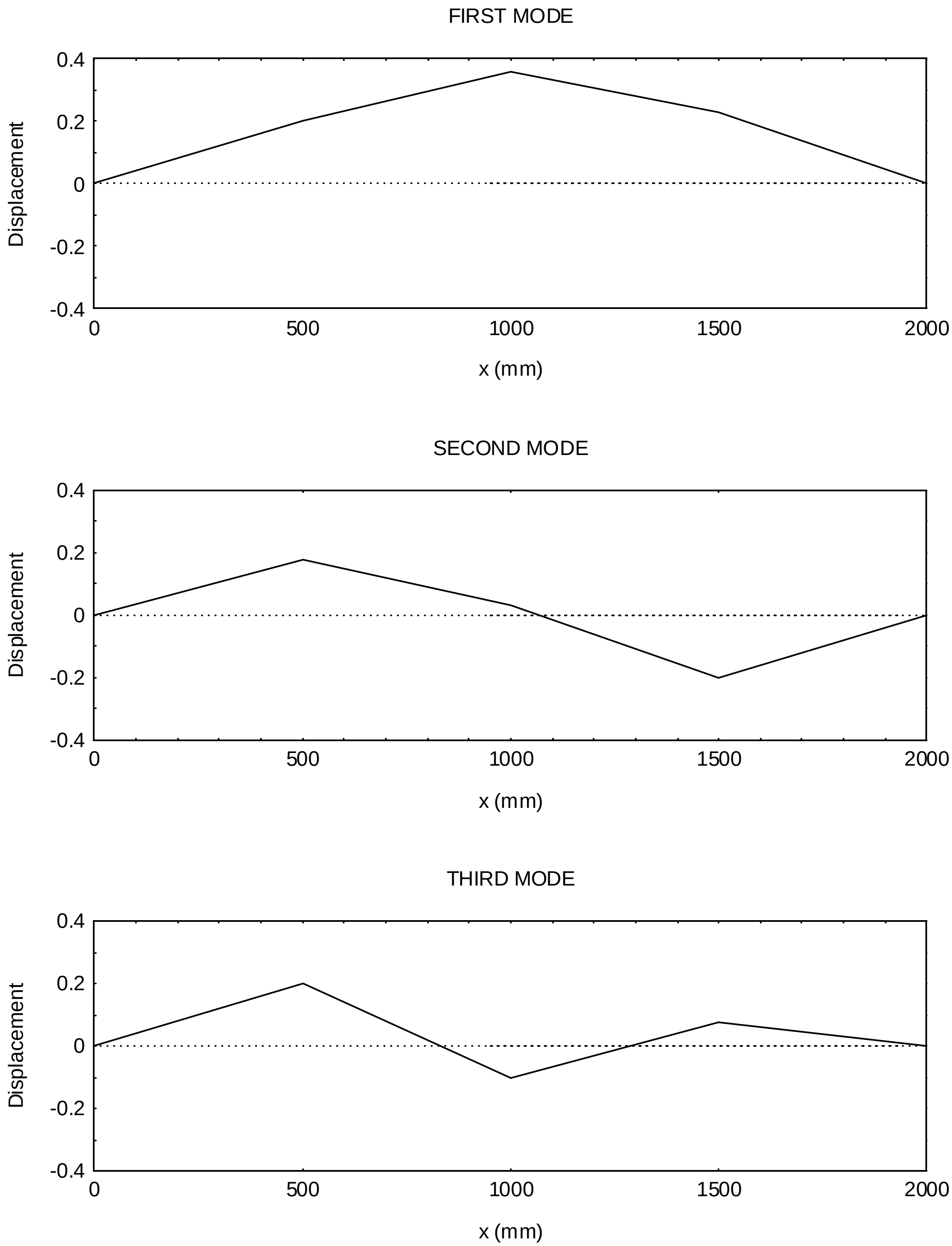


Figure 10. Mode Shape of Example 3

Appendix A

Stiffness Matrix

A two-node beam element having four degree of freedom is shown in Figure 1 [3]. Rotation θ is assumed to be small, so that $\theta = dw / dx$.

Four degree of freedom define a cubic lateral displacement field,

$$w = [\mathrm{N}][w_1\ \theta_1\ w_2\ \theta_2]^{\mathrm{T}} = [\mathrm{N}]\{\mathrm{d}\} \tag{5}$$

where,

[N] = assumed shape functions is corresponding to cubic lateral displacement and are given in Table 3,

θ_1 = rotation at node 1,
θ_2 = rotation at node 2,
w_1 = lateral displacement at node 1,
w_2 = lateral displacement at node 2,

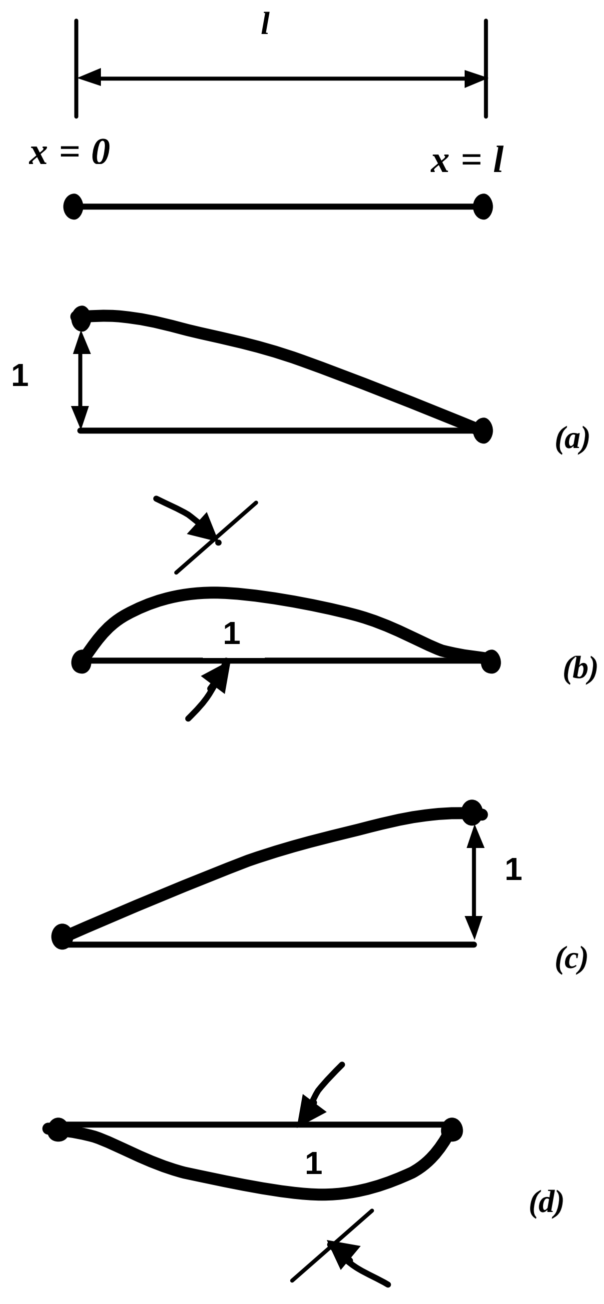


Figure 11:
Typical shape functions of a cubic fitted to ordinates and slopes at $x = 0$ and at $x = l$

Table 3: Typical shape functions of a cubic fitted to ordinates and slopes at $x=0$ and at $x=l$.					
Figure	Shape function	At $x = 0$		At $x = l$	
		N_i	$N_{i,x}$	N_i	$N_{i,x}$
11a	$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$	1	0	0	0
11b	$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$	0	1	0	0
11c	$N_3 = \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$	0	0	1	0
11d	$N_4 = -\frac{x^2}{l} + \frac{x^3}{l^3}$	0	0	0	1

For constant EI, the element stiffness matrix given by

$$[K] = \int [B]^T EI [B] dx = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{6}$$

$$[\mathrm{B}] = \frac{\mathrm{d}^2}{\mathrm{dx}^2}[\mathrm{N}] = \begin{bmatrix} -\frac{6}{l^2} + \frac{12}{l^3} & -\frac{4}{l} + \frac{6x}{l^2} & \frac{6}{l^2} - \frac{12x}{l^3} & -\frac{2}{l} + \frac{6x}{l^2} \end{bmatrix} \tag{7}$$

Mass Matrix

A mass matrix is a discrete representation of a continuous distribution of mass. A consistent mass matrix is defined by

$$[M]_{cons} = \int \rho [\mathrm{N}]^{\mathrm{T}} [\mathrm{N}] \mathrm{dV} = \frac{m}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{8}$$

It is termed "consistent" because [N] represents the same shape functions as are used to generate element stiffness matrix. A simpler and historically earlier formulation is the lumped mass matrix, which is obtained by placing particle masses M_i at nodes i of an element, such that $\sum m_i$ is the total element mass. Particle "lumps" have no rotary inertia unless rotary inertia is arbitrarily assigned. A lumped mass matrix is diagonal, but a consistent matrix is not. Historical lumped mass matrix is given by Lumped mass matrix,

$$[M]_{Lumped} = \begin{bmatrix} \mathrm{m}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{m}/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Another scheme is HRZ Lumping scheme. The HRZ scheme is an effective method for producing a diagonal mass matrix. It is recommenced for arbitrary elements. The idea is to use only the diagonal terms of the consistent mass matrix, but to scale them in such a way that the total mass of the element is preserved. The specific procedure steps are as follows.

1. Compute only diagonal coefficients of the consistent mass matrix.

2. Compute the total mass of the element, m

3. Compute a number s by adding the diagonal coefficients M_{ii} associated with translation d.o.f (but not rotational d.o.f, if any) that are mutually parallel and in the same direction.

4. Scale all the diagonal coefficients by multiplying them by the ratio *m/s*, thus preserving the total mass of the element.

As an example, consider a beam element, and follow the steps.

1. Diagonal element of consistent mass matrix,

$$[M] = \frac{m}{420}\begin{bmatrix} 156 & 0 & 0 & 0 \\ 0 & 4l^2 & 0 & 0 \\ 0 & 0 & 156 & 0 \\ 0 & 0 & 0 & 4l^2 \end{bmatrix} \tag{10}$$

2. Total mass element is element is *m*,

3. $s = M(1,1) + M(3,3) = \frac{m}{420}(156+156) = \frac{312}{420} m$. Hence $\frac{m}{s} = \frac{420}{312}$

4. Hence Diagonal mass matrix $[M]_{diag}$,

5.

$$[M]_{diag} = \frac{m}{s}[m] \tag{11}$$

$$[M]_{diag} = \begin{bmatrix} m/2 & 0 & 0 & 0 \\ 0 & ml^2/78 & 0 & 0 \\ 0 & 0 & m/2 & 0 \\ 0 & 0 & 0 & ml^2/78 \end{bmatrix} \tag{12}$$

References

[1] J. S. Rao. *Rotor Dynamics*. Wiley Eastern Ltd., 1983.

[2] Y. W. Kwon and H. Bang. *The Finite Element Method using MATLAB*. CRC press, 1997.

[3] R. D. Cook, D. S. Malkus, and M. E. Plesha. *Concepts and Application of Finite Element Analysis*. John Wiley and Sons, 1989.