THE RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A VERSED SINE BASE EXCITATION Revision A

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

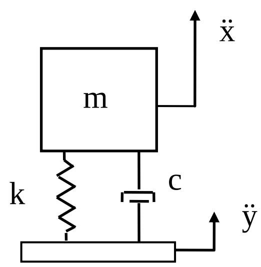

Figure 1.

where

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
x	=	absolute displacement of the mass
y	=	base input displacement

A free-body diagram is shown in Figure 2.

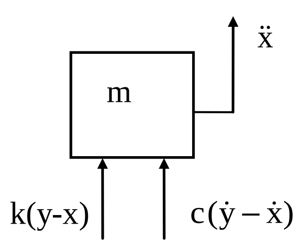

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

$$\begin{aligned} \text{Let } z &= x - y \quad \text{(relative displacement)} \\ \dot{z} &= \dot{x} - \dot{y} \\ \ddot{z} &= \ddot{x} - \ddot{y} \\ \ddot{x} &= \ddot{z} + \ddot{y} \end{aligned}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{5}$$

By convention,

$$\begin{aligned} (c/m) &= 2\xi\omega_n \\ (k/m) &= \omega_n^{\,2} \end{aligned}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^{\,2}z = -\ddot{y} \tag{6}$$

Response to Versed Sine

The base excitation function is:

$$\ddot{y}(t) = \begin{cases} \dfrac{A}{2}\left[1-\cos\left(\dfrac{2\pi t}{T}\right)\right], & 0 \le t \le T \\ \\ 0, & t > T \end{cases} \tag{7}$$

where A = the acceleration amplitude.

The equation of motion becomes

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z = -\frac{A}{2}\left[1-\cos\left(\frac{2\pi t}{T}\right)\right], \qquad 0 \le t \le T \tag{8}$$

Now take the Laplace transform.

$$L\left\{\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n^2 z\right\} = L\left\{-\frac{A}{2}\left[1-\cos\left(\frac{2\pi t}{T}\right)\right]\right\} \tag{9}$$

Let $\alpha = \left(\dfrac{2\pi}{T}\right)$

$$L\left\{\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n^2 z\right\} = L\left\{-\frac{A}{2}\left[1-\cos(\alpha t)\right]\right\} \tag{10}$$

$$s^2 Z(s) - sz(0) - \dot{z}(0) + 2\xi\omega_n s Z(s) - 2\xi\omega_n z(0) + \omega_n{}^2 Z(s) = -\frac{A}{2}\left\{\frac{1}{s} - \frac{s}{s^2 + \alpha^2}\right\} \tag{11}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\} Z(s) = \dot{z}(0) + \left\{s + 2\xi\omega_n\right\} z(0) - \frac{A}{2}\left\{\frac{1}{s} - \frac{s}{s^2 + \alpha^2}\right\} = 0 \tag{12}$$

$$Z(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\} z(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} - \frac{A}{2}\left\{\frac{1}{s} - \frac{s}{s^2 + \alpha^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{13}$$

Let

$$Z(s) = Z_n(s) + Z_f(s) \tag{14}$$

where

$$Z_n(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\} z(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{15}$$

$$Z_f(s) = -\frac{A}{2}\left\{\frac{1}{s} - \frac{s}{s^2 + \alpha^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{16}$$

Consider the denominator term,

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = \left(s + \xi\omega_n\right)^2 + \omega_n{}^2 - \left(\xi\omega_n\right)^2 \tag{17}$$

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = \left(s + \xi\omega_n\right)^2 + \omega_n{}^2\left(1 - \xi^2\right) \tag{18}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{19}$$

Substitute equation (19) into (18),

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = (s + \xi\omega_n)^2 + \omega_d{}^2 \tag{20}$$

Substitute equation (20) into (16).

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\} z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{21}$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n) z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{22}$$

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n) z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \omega_d}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{23}$$

Take the inverse Laplace transform using Reference 1.

$$z_n(t) = z(0) \exp(-\xi\omega_n t) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \exp(-\xi\omega_n t) \sin(\omega_d t) \tag{24}$$

$$z_n(t) = \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \tag{25}$$

Take the first derivative to determine the relative velocity.

$$\dot{z}_n(t) = -\xi\omega_n \exp(-\xi\omega_n t)\left\{z(0)\cos(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$
$$+ \exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

(26)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0)\cos(\omega_d t) - \xi\omega_n\left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$
$$+ \exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

(27)

$$\dot{z}_n(t) =$$

$$\exp(-\xi\omega_n t)\{\{-\xi\omega_n z(0) + \dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

$$+ \exp(-\xi\omega_n t)\left\{\left\{-\omega_d z(0) - \xi\omega_n\left[\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right]\right\}\sin(\omega_d t)\right\}$$

(28)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+ \exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d{}^2 z(0) - \xi\omega_n[\dot{z}(0) + (\xi\omega_n)z(0)]\right\}\sin(\omega_d t)\right\}$$

(29)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+ \exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d{}^2 z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(30)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2\left(1-\xi^2\right)z(0)-\xi\omega_n\dot{z}(0)-(\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(31)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{\left(-\omega_n{}^2+\xi^2\omega_n{}^2\right)z(0)-\xi\omega_n\dot{z}(0)-(\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(32)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0)-\xi\omega_n\dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

(33)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t)+\frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0)-\xi\omega_n\dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

(34)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t)+\frac{\omega_n}{\omega_d}\{-\omega_n z(0)-\xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$

(35)

Take the second derivative to determine the acceleration.

$$\ddot{z}_n(t) = -\xi\omega_n\exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t)+\frac{\omega_n}{\omega_d}\{-\omega_n z(0)-\xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$

$$+\exp(-\xi\omega_n t)\{-\omega_d\dot{z}(0)\sin(\omega_d t)+\omega_n\{-\omega_n z(0)-\xi\dot{z}(0)\}\cos(\omega_d t)\}$$

(36)

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\left\{-\xi\omega_n \dot{z}(0)\cos(\omega_d t) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d \dot{z}(0)\sin(\omega_d t) + \omega_n\{-\omega_n z(0) - \xi\dot{z}(0)\}\cos(\omega_d t)\}$$

(37)

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\{-\xi\omega_n \dot{z}(0) + \omega_n\{-\omega_n z(0) - \xi\dot{z}(0)\}\}\cos(\omega_d t)$$
$$+\exp(-\xi\omega_n t)\left\{-\omega_d \dot{z}(0) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\right\}\sin(\omega_d t)$$

(38)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2\dot{z}(0) - \xi\omega_n{}^2\{-\omega_n z(0) - \xi\dot{z}(0)\}\right\}\sin(\omega_d t)$$

(39)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2\dot{z}(0)\right\}\sin(\omega_d t)$$

(40)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_n{}^2\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2\dot{z}(0)\right\}\sin(\omega_d t)$$

(41)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n z(0) + \xi^2\dot{z}(0)\right\}\sin(\omega_d t)$$

(42)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\,\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n^{\,2}}{\omega_d}\exp(-\xi\omega_n t)\left\{\xi\omega_n z(0) + \left(-1+2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

(43)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\,\dot{z}(0)\}\cos(\omega_d t)$$
$$-\frac{\omega_n^{\,2}}{\omega_d}\exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

(44)

$$\ddot{z}_n(t) =$$
$$-\exp(-\xi\omega_n t)\left\{\omega_n\left[\omega_n z(0) + 2\xi\,\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n^{\,2}}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(45)

$$\ddot{z}_n(t) =$$
$$-\omega_n \exp(-\xi\omega_n t)\left\{\left[\omega_n z(0) + 2\xi\,\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(46)

Recall equation (16).

$$Z_f(s) = -\frac{A}{2}\left\{\frac{1}{s} - \frac{s}{s^2+\alpha^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right\} \tag{47}$$

$$Z_f(s) = \frac{A}{2}\left\{\frac{s}{s^2+\alpha^2} - \frac{1}{s}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right\} \tag{48}$$

Expand into partial fractions using Reference 2.

$$\left\{\frac{s}{s^2+\alpha^2}-\frac{1}{s}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} =$$

$$\frac{-\left(\alpha^2-\omega_n^2\right)s+2\xi\alpha^2\omega_n}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]\left[s^2+\alpha^2\right]}$$

$$+\left(\alpha^2-\omega_n^2\right)\frac{s-\dfrac{2\xi\omega_n^3}{\left(\alpha^2-\omega_n^2\right)}}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

$$-\frac{1}{\omega_n^2 s}+\frac{s+2\xi\omega_n}{\omega_n^2\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

(49)

The inverse Laplace transform of $Z_f(s)$ is

$$\frac{A/2}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]}\left[-\left(\alpha^2-\omega_n^2\right)\cos(\alpha t)+2\xi\alpha\omega_n\sin(\alpha t)\right]$$

$$+\frac{A/2}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]}\exp(-\xi\omega_n t)\left[\left(\alpha^2-\omega_n^2\right)\cos\left(\omega_d t\right)\right]$$

$$-\frac{A/2}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]}\exp(-\xi\omega_n t)\left[\left(\frac{\xi\omega_n}{\omega_d}\right)\left(\alpha^2+\omega_n^2\right)\sin\left(\omega_d t\right)\right]$$

$$\frac{A/2}{\omega_n^2}\left\{-1+\exp(-\omega_n t)\left[\cos\left(\omega_d t\right)+\left(\frac{\xi\omega_n}{\omega_d}\right)\sin\left(\omega_d t\right)\right]\right\}$$

(50)

Assemble the inverse Laplace transforms. The resulting relative displacement for $0 \le t \le T$ is

$$z(t) = \exp(-\xi\omega_n t)\left\{z(0)\cos(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+ \frac{A/2}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\omega_n)^2\right]}\left[-\left(\alpha^2 - \omega_n^2\right)\cos(\alpha t) + 2\xi\alpha\omega_n \sin(\alpha t)\right]$$

$$+ \frac{A/2}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\omega_n)^2\right]}\exp(-\xi\omega_n t)\left[\left(\alpha^2 - \omega_n^2\right)\cos\left(\omega_d t\right)\right]$$

$$- \frac{A/2}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\omega_n)^2\right]}\exp(-\xi\omega_n t)\left[\left(\frac{\xi\omega_n}{\omega_d}\right)\left(\alpha^2 + \omega_n^2\right)\sin\left(\omega_d t\right)\right]$$

$$+ \frac{A/2}{\omega_n^2}\left\{-1 + \exp(-\omega_n t)\left[\cos\left(\omega_d t\right) + \left(\frac{\xi\omega_n}{\omega_d}\right)\sin\left(\omega_d t\right)\right]\right\}$$

(51)

The resulting relative velocity for $0 \le t \le T$ is

$$\dot{z}(t) = -\xi\omega_n \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+\omega_d \exp(-\xi\omega_n t)\left\{ -z(0)\sin(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\cos(\omega_d t)\right\}$$

$$+\frac{\alpha A/2}{\left[\left(\alpha^2-\omega_n^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\left[\left(\alpha^2-\omega_n^2\right)\sin(\alpha t)+2\xi\alpha\omega_n\cos(\alpha t)\right]$$

$$+\frac{\left(\alpha^2-\omega_n^2\right)A/2}{\left[\left(\alpha^2-\omega_n^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\exp(-\xi\omega_n t)\left[-\xi\omega_n\cos\left(\omega_d t\right)-\omega_d\sin\left(\omega_d t\right)\right]$$

$$-\left(\frac{\xi\omega_n}{\omega_d}\right)\frac{\left(\alpha^2+\omega_n^2\right)A/2}{\left[\left(\alpha^2-\omega_n^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\exp(-\xi\omega_n t)\left[-\xi\omega_n\sin\left(\omega_d t\right)+\omega_d\cos\left(\omega_d t\right)\right]$$

$$+\frac{-A/2}{\omega_n}\left\{\exp(-\omega_n t)\left[\cos\left(\omega_d t\right)+\left(\frac{\xi\omega_n}{\omega_d}\right)\sin\left(\omega_d t\right)\right]\right\}$$

$$+\frac{\omega_d A/2}{\omega_n^2}\left\{\exp(-\omega_n t)\left[-\sin\left(\omega_d t\right)+\left(\frac{\xi\omega_n}{\omega_d}\right)\cos\left(\omega_d t\right)\right]\right\}$$

(52)

The resulting absolute acceleration for $0 \le t \le T$ is calculated via

$$\ddot{x}(t) = -\omega_n^2 z(t) - 2\xi\omega_n \dot{z}(t) \tag{53}$$

APPENDIX A

Versed-Sine Example

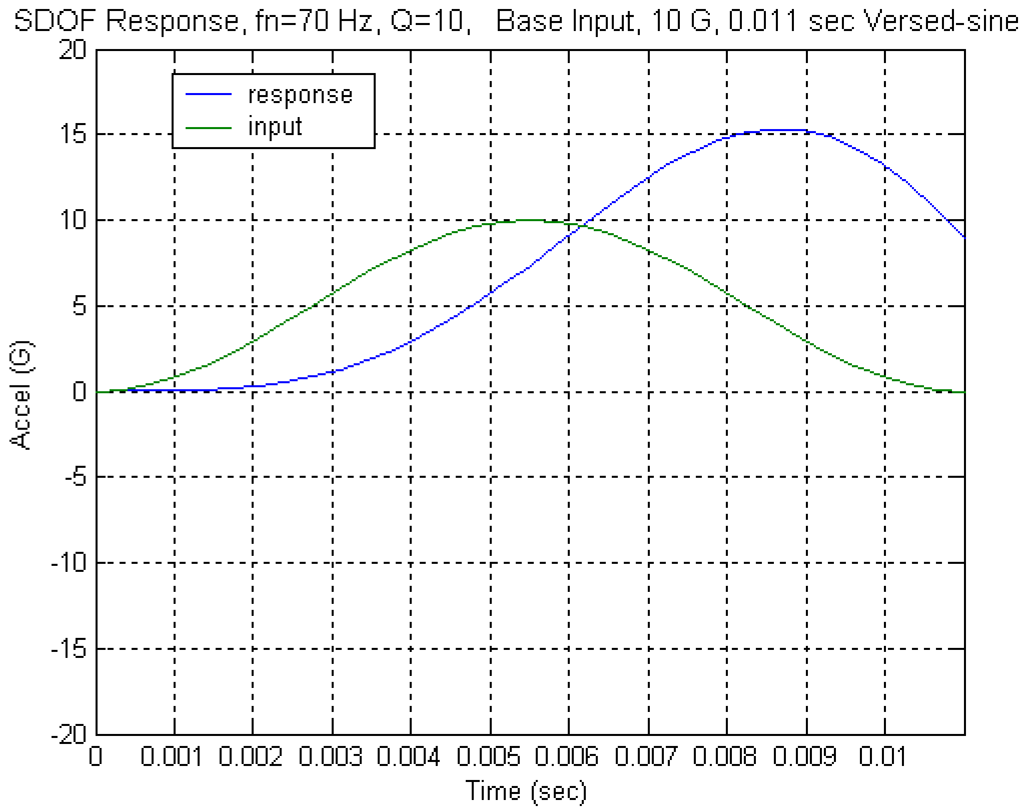

Figure A-1.

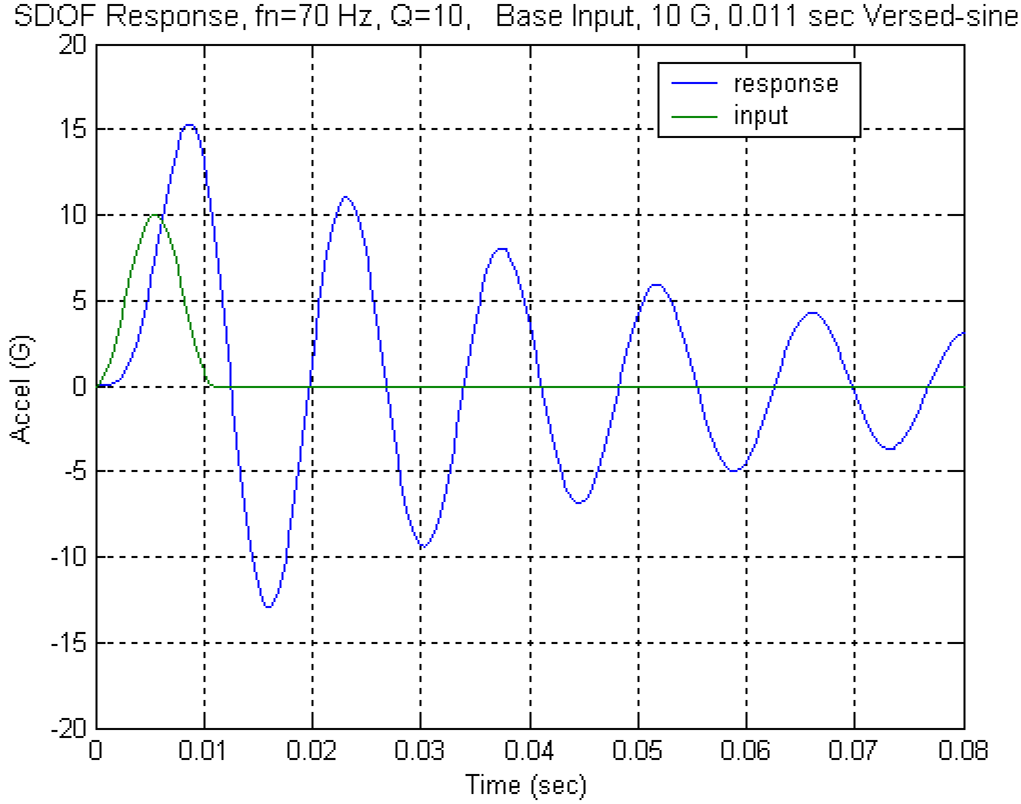

Figure A-2.

This is an extended view of the data in Figure A-1.