THE RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A WAVELET PULSE BASE EXCITATION

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

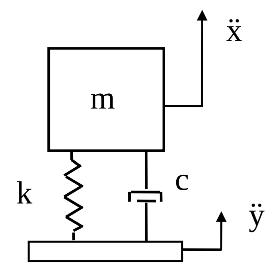

Figure 1.

where

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
x	=	absolute displacement of the mass
y	=	base input displacement

A free-body diagram is shown in Figure 2.

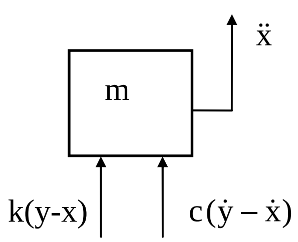

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

$$\begin{aligned} \text{Let } z &= x - y \quad \text{(relative displacement)} \\ \dot{z} &= \dot{x} - \dot{y} \\ \ddot{z} &= \ddot{x} - \ddot{y} \\ \ddot{x} &= \ddot{z} + \ddot{y} \end{aligned}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields

$$\ddot{z} + (c / m)\dot{z} + (k / m)z = -\ddot{y} \tag{5}$$

By convention,

$$\begin{aligned} (c / m) &= 2\xi\omega_n \\ (k / m) &= \omega_n{}^2 \end{aligned}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{6}$$

Response to Wavelet Base Excitation

The base excitation function is:

$$\ddot{y}(t) = \begin{cases} A\sin\left[\dfrac{2\pi\, f\, t}{N}\right]\sin\left[2\pi f\, t\right] , & 0 \le t \le T \\[2ex] 0 , & t > T \end{cases}$$

(7a)

where

A = wavelet acceleration amplitude

f = wavelet frequency

N = number of half-sines, odd integer $\geq$ 3

T = N / (2 f)

The base excitation may also be expressed as:

$$\ddot{y}(t) = \begin{cases} -\dfrac{A}{2}\cos\left[(N+1)\dfrac{2\pi\, f\, t}{N}\right] + \dfrac{A}{2}\cos\left[(N-1)\dfrac{2\pi\, f\, t}{N}\right] , & 0 \le t \le T \\[2ex] 0 , & t > T \end{cases}$$

(7b)

The equation of motion becomes

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z = \frac{A}{2}\cos\left[(N+1)\frac{2\pi\, f\, t}{N}\right] - \frac{A}{2}\cos\left[(N-1)\frac{2\pi\, f\, t}{N}\right], \qquad 0 \le t \le T$$

(8)

Let

$$\alpha = (N+1)\frac{2\pi f}{N} \tag{9a}$$

$$\beta = (N-1)\frac{2\pi f}{N} \tag{9b}$$

$$B = A / 2 \tag{9c}$$

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z = B\cos(\alpha t) - B\cos(\beta t) \quad , \quad 0 \le t \le T \tag{10}$$

Now take the Laplace transform.

$$L\left\{\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n^2 z\right\} = L\left\{B\cos(\alpha t) - B\cos(\beta t)\right\} \tag{11}$$

$$\begin{aligned} & s^2 Z(s) - s z(0) - \dot{z}(0) \\ & + 2\xi\omega_n s Z(s) - 2\xi\omega_n z(0) \\ & + \omega_n{}^2 Z(s) \quad = \frac{Bs}{s^2+\alpha^2} - \frac{Bs}{s^2+\beta^2} \end{aligned} \tag{12}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\} Z(s) + \{-1\}\dot{z}(0) + \{-s - 2\xi\omega_n\} z(0) = \frac{Bs}{s^2+\alpha^2} - \frac{Bs}{s^2+\beta^2} \tag{13}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}Z(s) = \dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0) + \frac{Bs}{s^2+\alpha^2} - \frac{Bs}{s^2+\beta^2} \tag{14}$$

$$Z(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} + \left\{\frac{Bs}{s^2+\alpha^2} - \frac{Bs}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{15}$$

Let

$$Z(s) = Z_n(s) + Z_f(s) \tag{16}$$

where

$$Z_n(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{17}$$

$$Z_f(s) = B\left\{\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{18}$$

Consider the denominator term,

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = \left(s + \xi\omega_n\right)^2 + \omega_n{}^2 - \left(\xi\omega_n\right)^2 \tag{19}$$

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = \left(s + \xi\omega_n\right)^2 + \omega_n{}^2\left(1 - \xi^2\right) \tag{20}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1-\xi^2} \tag{21}$$

Substitute equation (21) into (20),

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = \left(s + \xi\omega_n\right)^2 + \omega_d{}^2 \tag{22}$$

Substitute equation (22) into (18).

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\} z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{23}$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n) z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{24}$$

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n) z(0)}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \omega_d}{(s + \xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{25}$$

Take the inverse Laplace transform using Reference 1.

$$z_n(t) = z(0) \exp(-\xi\omega_n t) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \exp(-\xi\omega_n t) \sin(\omega_d t) \tag{26}$$

$$z_n(t) = \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \tag{27}$$

Take the first derivative to determine the relative velocity.

$$\begin{aligned} \dot{z}_n(t) = &-\xi\omega_n \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\ &+ \exp(-\xi\omega_n t) \{ -\omega_d z(0) \sin(\omega_d t) + \{ \dot{z}(0) + (\xi\omega_n) z(0) \} \cos(\omega_d t) \} \end{aligned} \tag{28}$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0)\cos(\omega_d t) - \xi\omega_n\left\{\frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0)+(\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

(29)

$$\dot{z}_n(t) =$$

$$\exp(-\xi\omega_n t)\{\{-\xi\omega_n z(0) + \dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\left\{-\omega_d z(0) - \xi\omega_n\left[\frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d}\right]\right\}\sin(\omega_d t)\right\}$$

(30)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d^2 z(0) - \xi\omega_n[\dot{z}(0)+(\xi\omega_n)z(0)]\right\}\sin(\omega_d t)\right\}$$

(31)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d^2 z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(32)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n^2(1-\xi^2)z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(33)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{\left(-\omega_n{}^2+\xi^2\omega_n{}^2\right)z(0)-\xi\omega_n\dot{z}(0)-(\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(34)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0)-\xi\omega_n\dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

(35)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t)+\frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0)-\xi\omega_n\dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

(36)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t)+\frac{\omega_n}{\omega_d}\{-\omega_n z(0)-\xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$

(37)

Take the second derivative to determine the acceleration.

$$\ddot{z}_n(t) = -\xi\omega_n\exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t)+\frac{\omega_n}{\omega_d}\{-\omega_n z(0)-\xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d\dot{z}(0)\sin(\omega_d t)+\omega_n\{-\omega_n z(0)-\xi\dot{z}(0)\}\cos(\omega_d t)\}$$

(38)

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\left\{-\xi\omega_n\dot{z}(0)\cos(\omega_d t)-\frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0)-\xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d\dot{z}(0)\sin(\omega_d t)+\omega_n\{-\omega_n z(0)-\xi\dot{z}(0)\}\cos(\omega_d t)\}$$

(39)

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\{-\xi\omega_n \dot{z}(0) + \omega_n\{-\omega_n z(0) - \xi\dot{z}(0)\}\}\cos(\omega_d t)$$
$$+\exp(-\xi\omega_n t)\left\{-\omega_d \dot{z}(0) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\right\}\sin(\omega_d t)$$
(40)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2\dot{z}(0) - \xi\omega_n{}^2\{-\omega_n z(0) - \xi\dot{z}(0)\}\right\}\sin(\omega_d t)$$
(41)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2\dot{z}(0)\right\}\sin(\omega_d t)$$
(42)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_n{}^2\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2\dot{z}(0)\right\}\sin(\omega_d t)$$
(43)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n z(0) + \xi^2\dot{z}(0)\right\}\sin(\omega_d t)$$
(44)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{\xi\omega_n z(0) + \left(-1+2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$
(45)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t)$$
$$-\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

(46)

$$\ddot{z}_n(t) =$$
$$-\exp(-\xi\omega_n t)\left\{\omega_n\left[\omega_n z(0) + 2\xi \dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n{}^2}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(47)

$$\ddot{z}_n(t) =$$
$$-\omega_n \exp(-\xi\omega_n t)\left\{\left[\omega_n z(0) + 2\xi \dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(48)

Recall equation (22).

$$Z_f(s) = B\left\{\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} \quad (49)$$

Expand into partial fractions using Reference 2.

$$B\left\{\frac{s}{s^2+\alpha^2}-\frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} \quad =$$

$$B\frac{-\left(\alpha^2-\omega_n^2\right)s+2\xi\alpha^2\omega_n}{\left[\left(\alpha^2-\omega_n^2\right)^2+(2\xi\alpha\omega_n)^2\right]\left[s^2+\alpha^2\right]}$$

$$+B\frac{\left(\alpha^2-\omega_n^2\right)s-2\xi\omega_n^3}{\left[\left(\alpha^2-\omega_n^2\right)^2+(2\xi\alpha\omega_n)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

$$-B\frac{-\left(\beta^2-\omega_n^2\right)s+2\xi\beta^2\omega_n}{\left[\left(\beta^2-\omega_n^2\right)^2+(2\xi\beta\omega_n)^2\right]\left[s^2+\beta^2\right]}$$

$$-B\frac{\left(\beta^2-\omega_n^2\right)s-2\xi\omega_n^3}{\left[\left(\beta^2-\omega_n^2\right)^2+(2\xi\beta\omega_n)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]} \tag{50}$$

Let

$$C_1 = B\frac{-\left(\alpha^2-\omega_n^2\right)}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]} \tag{51}$$

$$C_2 = B\frac{2\xi\alpha^2\omega_n}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]} \tag{52}$$

$$C_3 = B\frac{\left(\alpha^2-\omega_n^2\right)}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]} \tag{53}$$

$$C_4 = B\frac{-2\xi\omega_n^3}{\left[\left(\alpha^2-\omega_n^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]} \tag{54}$$

$$C_5 = -B\frac{-\left(\beta^2-\omega_n^2\right)}{\left[\left(\beta^2-\omega_n^2\right)^2+\left(2\xi\beta\omega_n\right)^2\right]} \tag{55}$$

$$C_6 = -B\frac{2\xi\beta^2\omega_n}{\left[\left(\beta^2-\omega_n^2\right)^2+\left(2\xi\beta\omega_n\right)^2\right]} \tag{56}$$

$$C_7 = -B \frac{\left(\beta^2 - \omega_n^2\right)}{\left[\left(\beta^2 - \omega_n^2\right)^2 + (2\xi\beta\omega_n)^2\right]} \tag{57}$$

$$C_8 = -B \frac{-2\xi\omega_n^3}{\left[\left(\beta^2 - \omega_n^2\right)^2 + (2\xi\beta\omega_n)^2\right]} \tag{58}$$

By substitution into equation (50),

$$B\left\{\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} =$$

$$\frac{C_1 s + C_2}{\left[s^2+\alpha^2\right]} + \frac{C_3 s + C_4}{\left[s^2+2\xi\omega_n s+\omega_n^2\right]} + \frac{C_5 s + C_6}{\left[s^2+\beta^2\right]} + \frac{C_7 s + C_8}{\left[s^2+2\xi\omega_n s+\omega_n^2\right]} \tag{59}$$

$$B\left\{\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} =$$

$$\frac{C_1 s + C_2}{\left[s^2+\alpha^2\right]} + \frac{\left[C_3 + C_7\right]s + \left[C_4 + C_8\right]}{\left[s^2+2\xi\omega_n s+\omega_n^2\right]} + \frac{C_5 s + C_6}{\left[s^2+\beta^2\right]} \tag{60}$$

Let

$$C_{10} = C_3 + C_7 \tag{61}$$

$$C_{11} = C_4 + C_8 \tag{62}$$

By substitution into equation ,

$$B\left\{\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} =$$

$$\frac{C_1 s + C_2}{\left[\ s^2+\alpha^2\right]} + \frac{C_{10}\ s + \ C_{11}}{\left[s^2+2\xi\omega_n s+\omega_n^2\right]} + \frac{C_5 s + C_6}{\left[\ s^2+\beta^2\right]}$$

(63)

$$B\left\{\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n{}^2}\right\} =$$

$$\frac{C_1 s + C_2}{\left[\ s^2+\alpha^2\right]} + \frac{C_{10}\ s + \ C_{11}}{\left[(s+\xi\omega_n)^2+\omega_d{}^2\right]} + \frac{C_5 s + C_6}{\left[\ s^2+\beta^2\right]}$$

(64)

Take the inverse Laplace transform using Reference 1.

$$z_f(t) = C_1 \cos(\alpha t) + \frac{C_2}{\alpha}\sin(\alpha t)$$
$$+ \exp(-\xi\omega_n t)\left[C_{10} \cos(\omega_d t) + \frac{1}{\omega_d}\left[C_{11} - \xi\omega_n C_{10}\right]\sin(\omega_d t)\right]$$
$$+ C_5 \cos(\beta t) + \frac{C_6}{\beta}\sin(\beta t) \tag{65}$$

Let

$$C_{20} = C_{11} - \xi\omega_n C_{10} \tag{66}$$

$$z_f(t) = C_1 \cos(\alpha t) + \frac{C_2}{\alpha}\sin(\alpha t)$$
$$+ \exp(-\xi\omega_n t)\left[C_{10} \cos(\omega_d t) + \frac{1}{\omega_d} C_{20} \sin(\omega_d t)\right]$$
$$+ C_5 \cos(\beta t) + \frac{C_6}{\beta}\sin(\beta t) \tag{67}$$

The total relative displacement for $0 \le t \le T$ is

$$z(t) = z_n(t) + z_f(t) \tag{68}$$

$$z(t) = \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n) z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\}$$

$$+ \; C_1 \cos(\alpha t) \; + \; \frac{C_2}{\alpha} \sin(\alpha t) \; + \; C_5 \cos(\beta t) \; + \; \frac{C_6}{\beta} \sin(\beta t)$$

$$+ \; \exp(-\xi\omega_n t)\left[C_{10} \cos(\omega_d t) + \frac{1}{\omega_d} C_{20} \sin(\omega_d t) \right]$$

(69)

Let

$$R_3(t) = \exp(-\xi\omega_n t)\left[C_{10} \cos(\omega_d t) + \frac{1}{\omega_d} C_{20} \sin(\omega_d t) \right] \tag{70}$$

The total relative velocity for $0 \leq t \leq T$ is

$$\dot{z}(t) = \exp(-\xi\omega_n t)\left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \{-\omega_n z(0) - \xi \dot{z}(0)\} \sin(\omega_d t) \right\}$$

$$- \; \alpha C_1 \sin(\alpha t) \; + \; C_2 \cos(\alpha t) \; - \; \beta C_5 \sin(\beta t) \; + \; C_6 \cos(\beta t)$$

$$- \xi\omega_n R_3(t)$$

$$+ \exp(-\xi\omega_n t)\left[-\omega_d C_{10} \sin(\omega_d t) + C_{20} \cos(\omega_d t) \right]$$

(71)

The total absolute acceleration for $0 \leq t \leq T$ calculated from the equation of motion.

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{72}$$

$$m\ddot{x} = -c\dot{z} - kz \tag{73}$$

$$\ddot{x} = -\frac{c}{m}\dot{z} - \frac{k}{m}z \tag{74}$$

$$\ddot{x} = -2\xi\omega_n \dot{z} - \omega_n^2 z \tag{75}$$

The solution is completed by substituting equations (69) through (71) into equation (75).

The relative displacement at t = T is

$$\begin{aligned} z(T) &= \exp(-\xi\omega_n T)\left\{ z(0)\cos(\omega_d T) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d T)\right\} \\ &+ C_1\cos(\alpha T) + \frac{C_2}{\alpha}\sin(\alpha T) + C_5\cos(\beta T) + \frac{C_6}{\beta}\sin(\beta T) \\ &+ \exp(-\xi\omega_n T)\left[C_{10}\cos(\omega_d T) + \frac{1}{\omega_d}C_{20}\sin(\omega_d T)\right] \end{aligned} \tag{76}$$

The relative velocity at t = T is

$$\begin{aligned} \dot{z}(T) &= \exp(-\xi\omega_n T)\left\{ \dot{z}(0)\cos(\omega_d T) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d T)\right\} \\ &- \alpha C_1\sin(\alpha T) + C_2\cos(\alpha T) - \beta C_5\sin(\beta T) + C_6\cos(\beta T) \\ &- \xi\omega_n R_3(T) \\ &+ \exp(-\xi\omega_n T)\left[-\omega_d C_{10}\sin(\omega_d T) + C_{20}\cos(\omega_d T)\right] \end{aligned} \tag{77}$$

The relative displacement for t > T is found by adding a delay into equation (27).

$$z(t) = \exp\left(-\xi\omega_n\left(t-T\right)\right)\left\{ z(T)\cos\left(\omega_d\left(t-T\right)\right) + \left\{\frac{\dot{z}(T) + \left(\xi\omega_n\right)z(T)}{\omega_d}\right\}\sin\left(\omega_d\left(t-T\right)\right)\right\} \tag{78}$$

Note that the absolute displacement is equal to the relative displacement for t > T.

The relative velocity for t > T is

$$\dot{z}(t) = \exp\left(-\xi\omega_n\left(t-T\right)\right)\left\{ \dot{z}(T)\ \cos\left(\omega_d\left(t-T\right)\right) + \frac{\omega_n}{\omega_d}\left\{-\omega_n z(T) - \xi\,\dot{z}(T)\right\}\sin\left(\omega_d\left(t-T\right)\right)\right\} \tag{79}$$

Note that the absolute acceleration t > T is found via the equation of motion.

$$\ddot{x} = -2\xi\omega_n\ \dot{z}\ -\omega_n^2 z \tag{80}$$

References

1. T. Irvine, Table of Laplace Transforms, Vibrationdata Publications, 1999.
2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata Publications, 1999.

APPENDIX A

<u>Example</u>

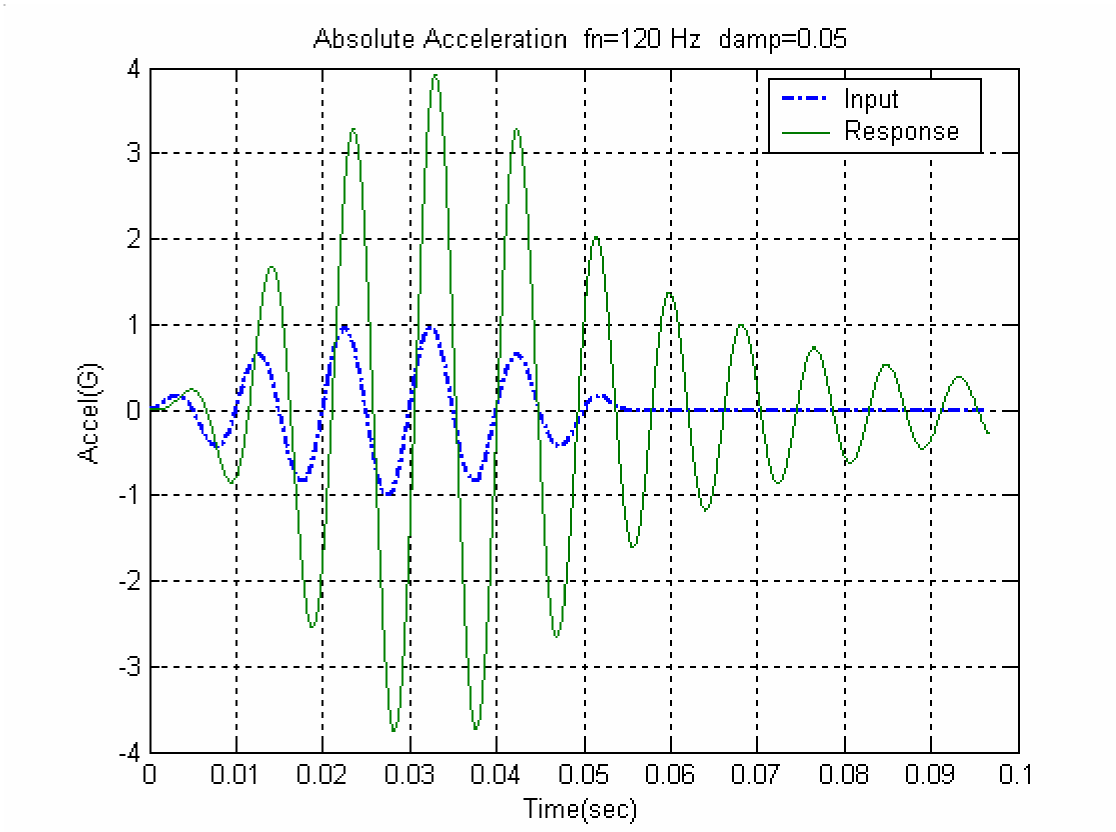

Figure 1.

A single-degree-of-freedom system has a natural frequency of 120 Hz with 5% damping.

The system is subjected to a wavelet base input, with an amplitude of 1 G, a frequency of 100 Hz, and 11 half-sine pulses. The acceleration response is shown in Figure 1.