THE MODAL TRANSIENT RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO HARMONIC BASE EXCITATION

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EQUATION OF MOTION

Consider a single degree-of-freedom system subjected to base excitation.

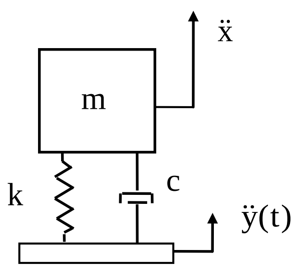

Figure 1.

where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass
y	is the base input displacement

The double-dot denotes acceleration.

The free-body diagram is

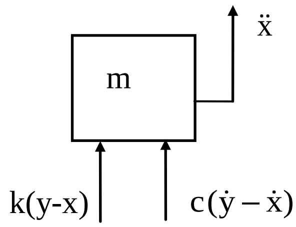

Figure 2.

Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \qquad (1)$$

$$m\ddot{x} = f(t) + c(\dot{y} - \dot{x}) + k(y - x) \qquad (2)$$

Define a relative displacement

$$z = x - y \qquad (3)$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \qquad (4)$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad (5)$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \qquad (6)$$

By convention,

$$(c/m) = 2\xi\omega_n \qquad (7)$$

$$(k/m) = \omega_n{}^2 \qquad (8)$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (6) yields

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{9}$$

Equation (9) can be solved using the methods in Reference 1 for the case where the base acceleration varies arbitrarily with time.

Now assume that the force and base excitation are harmonic functions.

$$\ddot{y}(t) = A \sin(\alpha t) \tag{10}$$

By substitution,

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z = -A \sin(\alpha t) \tag{11}$$

Take the Laplace transform,

$$L\left\{\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z\right\} = L\{-A \sin(\alpha t)\} \tag{12}$$

$$\begin{aligned} & s^2 Z(s) - s\,z(0) - \dot{z}(0) \\ & + 2\xi\omega_n s\,Z(s) - 2\xi\omega_n z(0) \\ & + \omega_n{}^2 Z(s) \;=\; \frac{-A\alpha}{s^2 + \alpha^2} \end{aligned} \tag{13}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\} Z(s) = \dot{z}(0) + \{s + 2\xi\omega_n\} - \frac{A\alpha}{s^2 + \alpha^2} \tag{14}$$

$$Z(s) = \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}}{\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}} + \left\{\frac{-A\alpha}{s^2 + \alpha^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{15}$$

Consider the denominator term.

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = (s + \xi\omega_n)^2 + \omega_n{}^2 - (\xi\omega_n)^2 \tag{16}$$

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = (s + \xi\omega_n)^2 + \omega_n{}^2\left(1 - \xi^2\right) \tag{17}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{18}$$

By substitution,

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = (s + \xi\omega_n)^2 + \omega_d{}^2 \tag{19}$$

$$Z(s) = \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}}{(s + \xi\omega_n)^2 + \omega_d{}^2} + \left\{\frac{-A\alpha}{s^2 + \alpha^2}\right\}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d{}^2}\right\} \tag{20}$$

The solution for the relative displacement is found using Reference 2.

$$z(t) = \exp(-\xi\omega_n t)\left\{z(0)\cos(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+ \frac{A}{\left[\left(\alpha^2 - \omega_n{}^2\right)^2 + (2\xi\alpha\omega_n)^2\right]}\left[(2\xi\alpha\omega_n)\cos(\alpha t) + \left(\alpha^2 - \omega_n{}^2\right)\sin(\alpha t)\right]$$

$$- \frac{A\dfrac{\alpha}{\omega_d}\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\alpha^2 - \omega_n{}^2\right)^2 + (2\xi\alpha\omega_n)^2\right]}\left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\alpha^2 - \omega_n{}^2\left(1 - 2\xi^2\right)\right)\sin(\omega_d t)\right]$$

(21)

The total relative velocity is

$$\dot{z}(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$

$$+\frac{A\alpha}{\left[\left(\alpha^2-\omega_n{}^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\left[\left(\alpha^2-\omega_n{}^2\right)\cos(\alpha t)-(2\xi\alpha\omega_n)\sin(\alpha t)\right]$$

$$+\frac{A\alpha[\exp(-\xi\omega_n t)]}{\left[\left(\alpha^2-\omega_n{}^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\left[\left(\omega_n{}^2-\alpha^2\right)\cos(\omega_d t)+\frac{\xi\omega_n}{\omega_d}\left(\omega_n{}^2+\alpha^2\right)\sin(\omega_d t)\right]$$

(22)

The total relative acceleration is

$$\ddot{z}(t) = -\omega_n \exp(-\xi\omega_n t)\left\{[\omega_n z(0) + 2\xi\dot{z}(0)]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

$$+\frac{A\alpha^2}{\left[\left(\alpha^2-\omega_n{}^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\left[-\left(\alpha^2-\omega_n{}^2\right)\sin(\alpha t)-(2\xi\alpha\omega_n)\cos(\alpha t)\right]$$

$$+\frac{A\alpha\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\alpha^2-\omega_n{}^2\right)^2+(2\xi\alpha\omega_n)^2\right]}\left[\left(2\xi\alpha^2\right)\cos(\omega_d t)+\left\{\frac{\omega_n}{\omega_d}\right\}\left\{-\omega_n{}^2+\alpha^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)\right]$$

(23)

The total absolute acceleration is

$$\ddot{x}(t) = -\omega_n \exp(-\xi\omega_n t)\left\{\left[\omega_n z(0) + 2\xi\,\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

$$+\frac{A\alpha^2}{\left[\left(\alpha^2-\omega_n{}^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]}\left[-\left(\alpha^2-\omega_n{}^2\right)\sin(\alpha t)-\left(2\xi\alpha\omega_n\right)\cos(\alpha t)\right]$$

$$+\frac{A\alpha\omega_n\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\alpha^2-\omega_n{}^2\right)^2+\left(2\xi\alpha\omega_n\right)^2\right]}\left[\left(2\xi\alpha^2\right)\cos(\omega_d t)+\left\{\frac{\omega_n}{\omega_d}\right\}\left\{-\omega_n{}^2+\alpha^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)\right]$$

$$+A\sin(\alpha t) \tag{24}$$

<u>References</u>

1. T. Irvine, An Introduction to the Shock Response Spectrum, Rev Q, Vibrationdata, 2009.
2. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Rev A, Vibrationdata, 1999.

APPENDIX A

Example using Matlab Script

```
>> sine_base

 sine_base.m  version 1.0    January 12, 2012
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 This program calculates the response of
 a single-degree-of-freedom system subjected
 to a sinusoidal base input.

 Assume zero initial displacement and zero initial velocity.

  Enter the base amplitude (G) 0.2
  Enter the base frequency (Hz) 10

  Enter the natural frequency (Hz) 10
  Enter amplification factor Q 10

  Enter duration (sec) 2


 maximum acceleration =        1.997 G
 minimum acceleration =       -1.997 G


 maximum relative disp =       0.1952 inch
 minimum relative disp =      -0.1951 inch
```

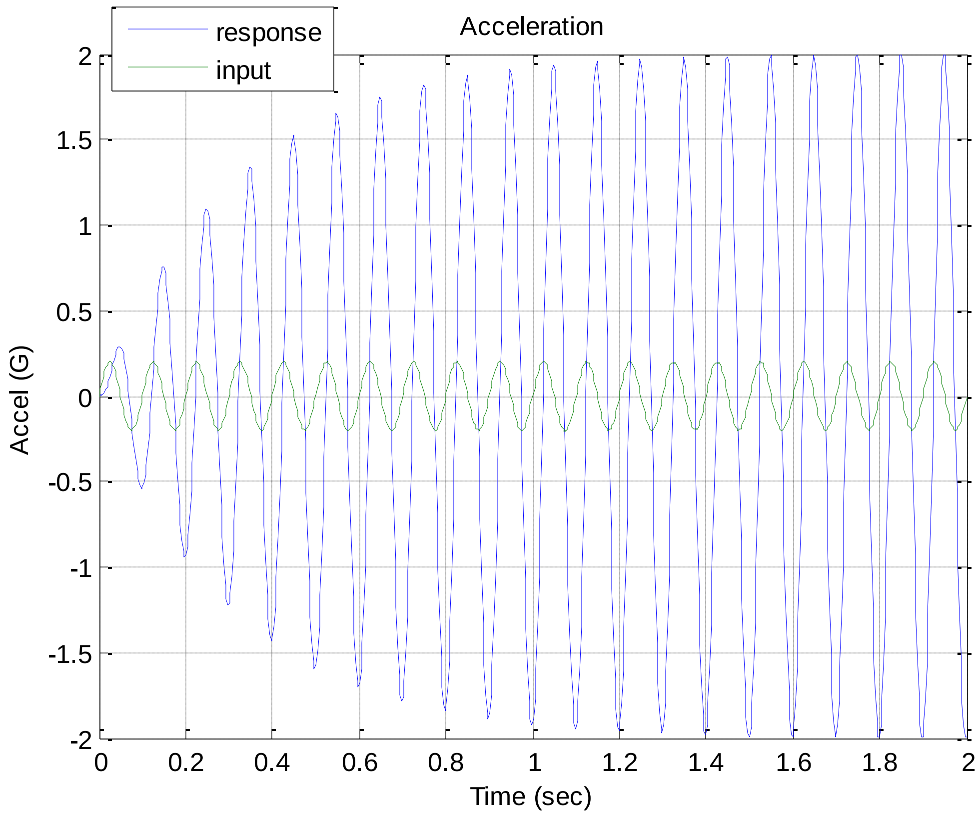
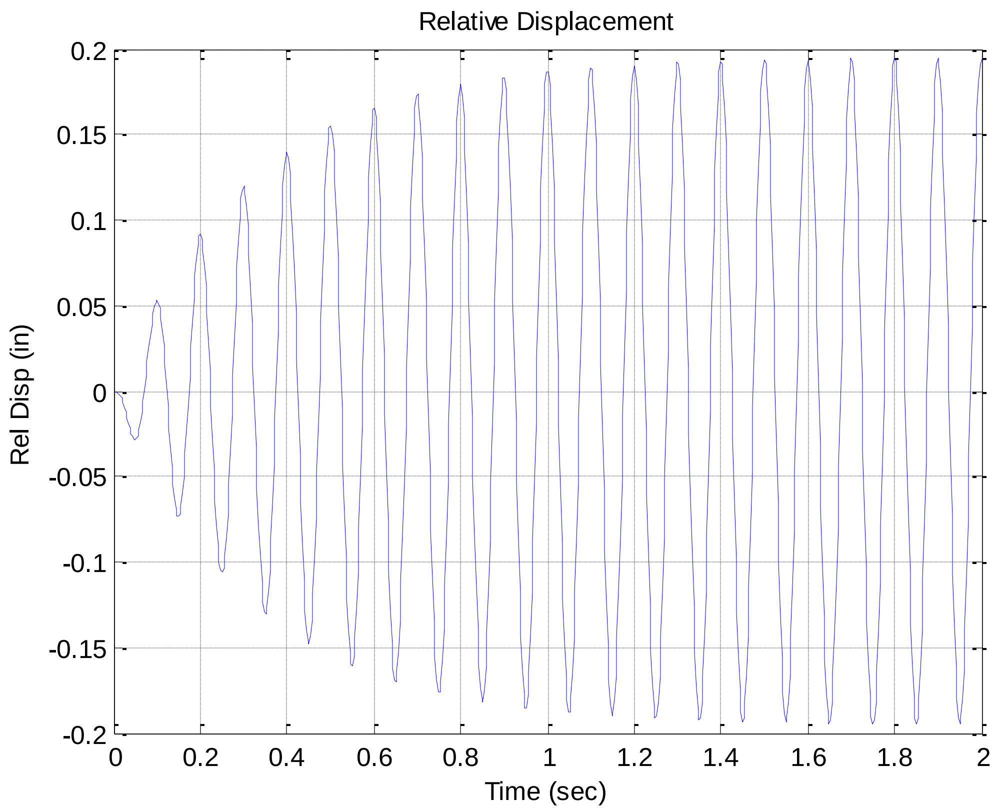

Acceleration
response
input
Accel (G)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time (sec)

Figure A-1.



Relative Displacement
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
Rel Disp (in)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time (sec)

Figure A-2.