

THE RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM
SYSTEM SUBJECTED TO COMBINED FORCE AND BASE EXCITATION
Revision A

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EQUATION OF MOTION

Consider a single degree-of-freedom system. Assume that the base excitation and applied force are completely independent.

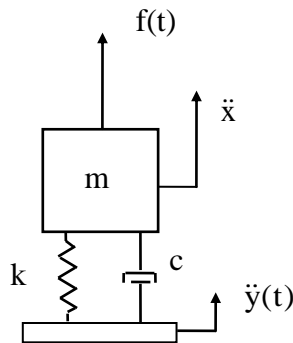


Figure 1.

where

- m is the mass
- c is the viscous damping coefficient
- k is the stiffness
- x is the absolute displacement of the mass
- y is the base input displacement
- f is the applied force

The double-dot denotes acceleration.

The free-body diagram is

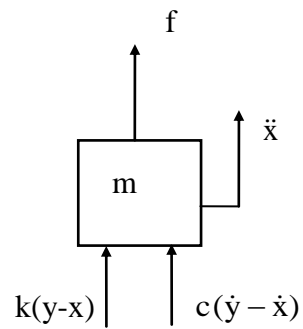


Figure 2.

Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \quad (1)$$

$$m\ddot{x} = f(t) + c(\dot{y} - \dot{x}) + k(y - x) \quad (2)$$

Define a relative displacement

$$z = x - y \quad (3)$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = f(t) - c\dot{z} - kz \quad (4)$$

$$m\ddot{z} + c\dot{z} + kz = f(t) - m\ddot{y} \quad (5)$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = [f(t)/m] - \ddot{y} \quad (6)$$

By convention,

$$(c/m) = 2\zeta\omega_n \quad (7)$$

$$(k/m) = \omega_n^2 \quad (8)$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (6) yields

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = [f(t)/m] - \ddot{y} \quad (9)$$

Equation (9) can be solved using the methods in Reference 1 for the case where either the force or the base acceleration varies arbitrarily with time.

Now assume that the force and base excitation are harmonic functions.

$$\ddot{y}(t) = A \sin(\alpha t) \quad (10)$$

$$[f(t)/m] = F \sin(\beta t) \quad (11)$$

By substitution,

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -A \sin(\alpha t) + F \sin(\beta t) \quad (12)$$

Take the Laplace transform,

$$L\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z\} = L\{-A \sin(\alpha t) + F \sin(\beta t)\} \quad (13)$$

$$\begin{aligned} & s^2 Z(s) - s z(0) - \dot{z}(0) \\ & + 2\xi\omega_n s Z(s) - 2\xi\omega_n z(0) \\ & + \omega_n^2 Z(s) = \frac{-A \alpha}{s^2 + \alpha^2} + \frac{F \beta}{s^2 + \beta^2} \end{aligned} \quad (14)$$

$$\{s^2 + 2\xi\omega_n s + \omega_n^2\} Z(s) = \dot{z}(0) + \{s + 2\xi\omega_n\} z(0) - \frac{A \alpha}{s^2 + \alpha^2} + \frac{F \beta}{s^2 + \beta^2} \quad (15)$$

$$Z(s) = \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}}{\{s^2 + 2\xi\omega_n s + \omega_n^2\}} + \left\{ \frac{-A\alpha}{s^2 + \alpha^2} + \frac{F\beta}{s^2 + \beta^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (16)$$

Consider the denominator term.

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - (\xi\omega_n)^2 \quad (17)$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 (1 - \xi^2) \quad (18)$$

Now define the damped natural frequency,

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \quad (19)$$

By substitution,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \quad (20)$$

$$Z(s) = \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}}{(s + \xi\omega_n)^2 + \omega_d^2} + \left\{ \frac{-A\alpha}{s^2 + \alpha^2} + \frac{F\beta}{s^2 + \beta^2} \right\} \left\{ \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (21)$$

The solution for the relative displacement is found using Reference 2.

$$\begin{aligned}
z(t) = & \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\
& + \frac{A}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[(2\xi\alpha\omega_n) \cos(\alpha t) + (\alpha^2 - \omega_n^2) \sin(\alpha t) \right] \\
& - \frac{A \frac{\alpha}{\omega_d} [\exp(-\xi\omega_n t)]}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[(2\xi\omega_n \omega_d) \cos(\omega_d t) + \left(\alpha^2 - \omega_n^2 (1 - 2\xi^2) \right) \sin(\omega_d t) \right] \\
& - \frac{F}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[(2\xi\beta\omega_n) \cos(\beta t) + (\beta^2 - \omega_n^2) \sin(\beta t) \right] \\
& + \frac{F \frac{\beta}{\omega_d} [\exp(-\xi\omega_n t)]}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[(2\xi\omega_n \omega_d) \cos(\omega_d t) + \left(\beta^2 - \omega_n^2 (1 - 2\xi^2) \right) \sin(\omega_d t) \right]
\end{aligned} \tag{22}$$

The total relative velocity is

$$\begin{aligned}
\dot{z}(t) = & \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \{-\omega_n z(0) - \xi \dot{z}(0)\} \sin(\omega_d t) \right\} \\
& + \frac{A\alpha}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[(\alpha^2 - \omega_n^2) \cos(\alpha t) - (2\xi\alpha\omega_n) \sin(\alpha t) \right] \\
& + \frac{A\alpha [\exp(-\xi\omega_n t)]}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[(\omega_n^2 - \alpha^2) \cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d} (\omega_n^2 + \alpha^2) \sin(\omega_d t) \right] \\
& - \frac{F\beta}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[(\beta^2 - \omega_n^2) \cos(\beta t) - (2\xi\beta\omega_n) \sin(\beta t) \right] \\
& - \frac{F\beta [\exp(-\xi\omega_n t)]}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[(\omega_n^2 - \beta^2) \cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d} (\omega_n^2 + \beta^2) \sin(\omega_d t) \right]
\end{aligned} \tag{23}$$

The total relative acceleration is

$$\begin{aligned}
\ddot{z}(t) = & -\omega_n \exp(-\xi\omega_n t) \left\{ [\omega_n z(0) + 2\xi\dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left\{ -\xi\omega_n z(0) + (1 - 2\xi^2)\dot{z}(0) \right\} \sin(\omega_d t) \right\} \\
& + \frac{A\alpha^2}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[-(\alpha^2 - \omega_n^2) \sin(\alpha t) - (2\xi\alpha\omega_n) \cos(\alpha t) \right] \\
& + \frac{A\alpha\omega_n [\exp(-\xi\omega_n t)]}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[(2\xi\alpha^2) \cos(\omega_d t) + \left\{ \frac{\omega_n}{\omega_d} \right\} \left\{ -\omega_n^2 + \alpha^2 (1 - 2\xi^2) \right\} \sin(\omega_d t) \right] \\
& - \frac{F\beta^2}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[-(\beta^2 - \omega_n^2) \sin(\beta t) - (2\xi\beta\omega_n) \cos(\beta t) \right] \\
& - \frac{F\beta\omega_n [\exp(-\xi\omega_n t)]}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[(2\xi\beta^2) \cos(\omega_d t) + \left\{ \frac{\omega_n}{\omega_d} \right\} \left\{ -\omega_n^2 + \beta^2 (1 - 2\xi^2) \right\} \sin(\omega_d t) \right]
\end{aligned} \tag{24}$$

The total absolute acceleration is

$$\begin{aligned}
\ddot{x}(t) = & -\omega_n \exp(-\xi\omega_n t) \left\{ [\omega_n z(0) + 2\xi\dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left\{ -\xi\omega_n z(0) + (1-2\xi^2)\dot{z}(0) \right\} \sin(\omega_d t) \right\} \\
& + \frac{A\alpha^2}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[-(\alpha^2 - \omega_n^2) \sin(\alpha t) - (2\xi\alpha\omega_n) \cos(\alpha t) \right] \\
& + \frac{A\alpha\omega_n [\exp(-\xi\omega_n t)]}{\left[(\alpha^2 - \omega_n^2)^2 + (2\xi\alpha\omega_n)^2 \right]} \left[(2\xi\alpha^2) \cos(\omega_d t) + \left\{ \frac{\omega_n}{\omega_d} \right\} \left\{ -\omega_n^2 + \alpha^2 (1-2\xi^2) \right\} \sin(\omega_d t) \right] \\
& - \frac{F\beta^2}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[-(\beta^2 - \omega_n^2) \sin(\beta t) - (2\xi\beta\omega_n) \cos(\beta t) \right] \\
& - \frac{F\beta [\exp(-\xi\omega_n t)]}{\left[(\beta^2 - \omega_n^2)^2 + (2\xi\beta\omega_n)^2 \right]} \left[(2\xi\beta^2) \cos(\omega_d t) + \left\{ \frac{\omega_n}{\omega_d} \right\} \left\{ -\omega_n^2 + \beta^2 (1-2\xi^2) \right\} \sin(\omega_d t) \right] \\
& + A \sin(\alpha t)
\end{aligned} \tag{25}$$

References

1. T. Irvine, An Introduction to the Shock Response Spectrum, Rev Q, Vibrationdata, 2009.
2. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Rev A, Vibrationdata, 1999.

APPENDIX A

Example using Matlab Script

```
>> sdof_force_accel
```

```
sdof_force_accel.m version 1.1    January 13, 2011  
By Tom Irvine    Email:  tomirvine@aol.com
```

```
This program calculates the response of  
a single-degree-of-freedom system subjected  
to combined sinusoidal force and base excitation.
```

```
Assume zero initial displacement and zero initial velocity.
```

```
Enter the natural frequency (Hz) 100  
Enter amplification factor Q 10
```

```
Enter the mass(lbm) 1
```

```
Enter duration (sec) 1
```

```
Enter the sample rate (sample/sec) 4000
```

```
Enter base acceleration amplitude (G) 2  
Enter base acceleration frequency (Hz) 110
```

```
Enter force amplitude (lbf) 3  
Enter force frequency (Hz) 90
```

```
maximum acceleration =          25.2 G  
minimum acceleration =        -25.32 G
```

```
Plot the acceleration response time history ?  
1=yes  2= no  1
```

```
maximum relative disp =          0.02693 inch  
minimum relative disp =        -0.02725 inch
```

```
Plot the relative displacement time history ?  
1=yes  2= no  1
```

```
Output Files:  
acceleration  
relative_displacement
```

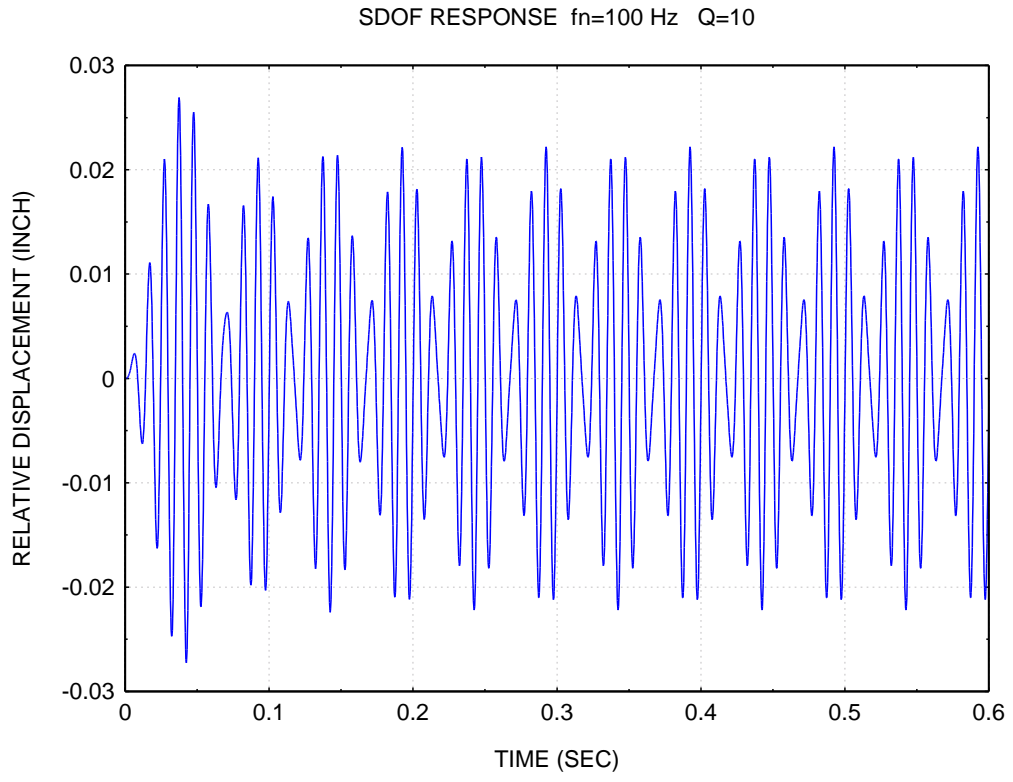


Figure A-1.

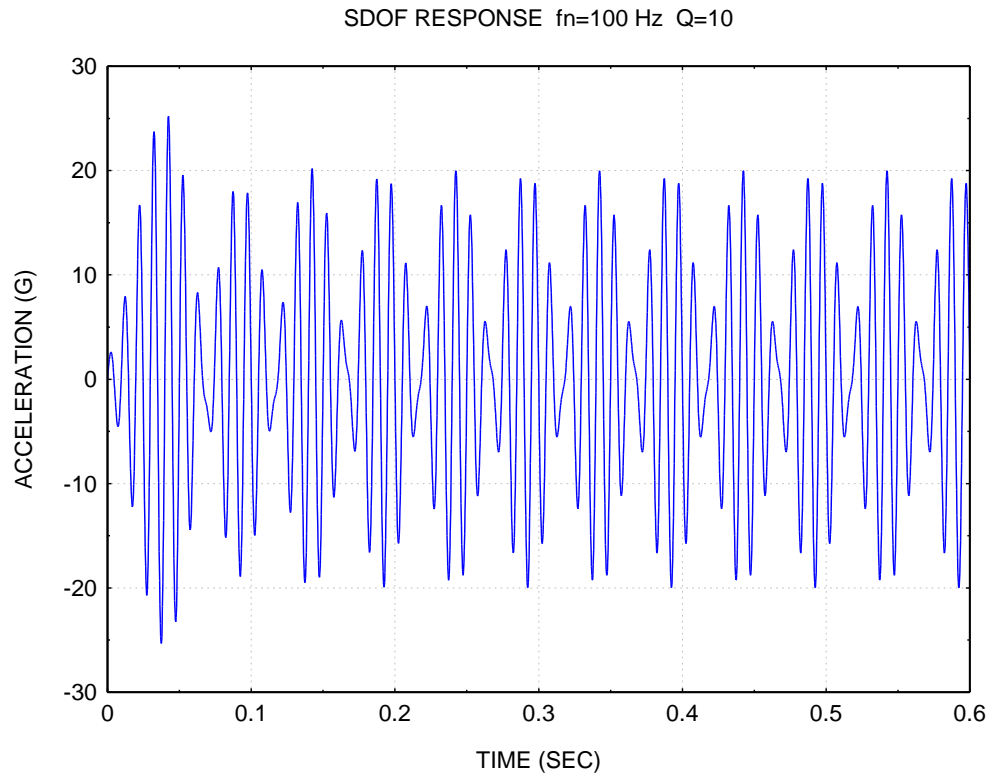


Figure A-2.