FREE VIBRATION OF A SEMI-DEFINITE SYSTEM SUBJECTED TO INITIAL CONDITIONS

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Two-degree-of-freedom System

Consider a two-degree-of-freedom system, as shown in Figure 1.

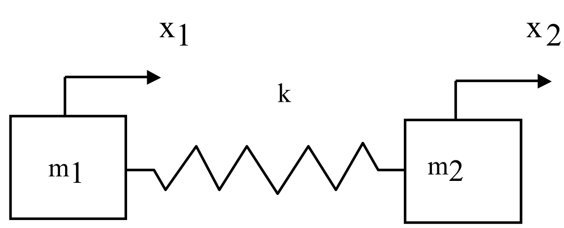

Figure 1.

The dashpot element between the masses is omitted for brevity.

The equation of motion from Reference 1 is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

Represent as

$$M\,\ddot{\bar{x}} + C\,\dot{\bar{x}} + K\,\bar{x} = F \quad (2)$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad (3)$$

$$C = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \quad (4)$$

$$K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad (5)$$

$$F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

Consider the undamped, homogeneous form of equation (2).

$$M\ \bar{\ddot{x}} + K\ \bar{x} = \bar{0} \quad (7)$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(j\omega t) \quad (8)$$

The q vector is the generalized coordinate vector.

Note that

$$\bar{\dot{x}} = j\omega\,\bar{q}\exp(j\omega t) \quad (9)$$

$$\bar{\ddot{x}} = -\omega^2\ \bar{q}\exp(j\omega t) \quad (10)$$

Substitute these equations into equation (2).

$$-\omega^2 M\ \bar{q}\exp(j\omega t) + K\,\bar{q}\exp(j\omega t) = \bar{0} \quad (11)$$

$$\left\{-\omega^2 M + K\right\}\bar{q}\exp(j\omega t) = \bar{0} \quad (12)$$

$$\left\{-\omega^2 M + K\right\} \bar{q} = \bar{0} \tag{13}$$

$$\left\{K - \omega^2 M\right\} \bar{q} = \bar{0} \tag{14}$$

Equation (14) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{15}$$

$$\det\left\{\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{16}$$

$$\det\left\{\begin{matrix} k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{matrix}\right\} = 0 \tag{17}$$

$$\left[k_2 - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2{}^2 = 0 \tag{18}$$

$$\omega^4 m_1 m_2 - \omega^2 [m_2 k_2 + m_1 k_2] = 0 \tag{19}$$

$$\omega_1 = 0 \tag{20}$$

$$\omega^2 m_1 m_2 - [m_2 k_2 + m_1 k_2] = 0 \tag{21}$$

$$\omega^2 = \frac{m_2 k_2 + m_1 k_2}{m_1 m_2} \tag{22}$$

$$\omega_2 = \sqrt{\frac{m_2 k_2 + m_1 k_2}{m_1 m_2}} \tag{23}$$

The eigenvectors are found via the following equations.

$$K \ \bar{q}_1 = \bar{0} \tag{24}$$

$$\left\{ K - \omega_2{}^2 \ M \right\} \bar{q}_2 = \bar{0} \tag{25}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{26}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{27}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\ \bar{q}_1 \mid \bar{q}_2 \ ] \tag{28}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{29}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T \ M \ \hat{Q} = I \tag{30}$$

$$\hat{Q}^T \ K \ \hat{Q} = \Omega \tag{31}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$Q = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{32}$$

$$Q^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{33}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in the references.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{34}$$

Substitute equation (34) into equation (2).

$$M\hat{Q}\,\bar{\ddot{\eta}} + C\hat{Q}\,\bar{\dot{\eta}} + K\hat{Q}\,\bar{\eta} = F \tag{35}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\bar{\ddot{\eta}} + \hat{Q}^T C\hat{Q}\,\bar{\dot{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T F \tag{36}$$

The orthogonality relationships yield

$$I\ \bar{\ddot{\eta}} + \hat{Q}^T C\hat{Q}\,\bar{\dot{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T F \tag{37}$$

Furthermore, the following assumption is made.

$$\hat{Q}^T C \hat{Q} \bar{\eta} = \begin{bmatrix} 0 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \tag{38}$$

where ξ_i is the modal damping ratio for mode i.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega_2{}^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega_2{}^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{40}$$

The two equations are now decoupled in terms of the modal coordinate.

$$\ddot{\eta}_1 = 0 \tag{41}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 \dot{\eta}_2 + \omega_2{}^2 \eta_2 = 0 \tag{42}$$

The solution to the first is

$$\eta_1(t) = \dot{\eta}_1(0)\, t + \eta_1(0) \tag{43a}$$

$$\eta_1(t) = \dot{\eta}_1(0) \tag{43b}$$

The following equation is obtained via Reference 5.

$$\eta_2(t) = \exp(-\xi_2 \omega_2 t) \left\{ \eta_2(0) \cos(\omega_{d2} t) + \frac{1}{\omega_{d2}} [\xi_2 \omega_2 \eta_2(0) + \dot{\eta}_2(0)] \sin(\omega_{d2} t) \right\} \tag{44}$$

$$\dot{\eta}_2(t) =$$

$$-\xi_2\,\omega_2 \exp(-\xi_2\,\omega_2\,t)\left\{\eta_2(0)\cos(\omega_{d2}\,t) + \frac{1}{\omega_{d2}}[\xi_2\omega_2\eta_2(0) + \dot{\eta}_2(0)]\sin(\omega_{d2}\,t)\right\}$$

$$+\omega_{d2} \exp(-\xi_2\,\omega_2\,t)\left\{-\eta_2(0)\sin(\omega_{d2}\,t) + \frac{1}{\omega_{d2}}[\xi_2\omega_2\eta_2(0) + \dot{\eta}_2(0)]\cos(\omega_{d2}\,t)\right\}$$

(45)

$$\dot{\eta}_2(t) = -\xi_2\,\omega_2\,\eta_2(t)$$

$$+\omega_{d2} \exp(-\xi_2\,\omega_2\,t)\left\{-\eta_2(0)\sin(\omega_{d2}\,t) + \frac{1}{\omega_{d2}}[\xi_2\omega_2\eta_2(0) + \dot{\eta}_2(0)]\cos(\omega_{d2}\,t)\right\}$$

(46)

$$\ddot{\eta}_2(t) = -2\xi_2\,\omega_2\,\dot{\eta}_2(t) - \omega_2{}^2\,\eta_2(t) \tag{47}$$

Recall

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{48}$$

The displacements are

$$x_1(t) = v_1\,\eta_1(t) + w_1\,\eta_{2(t)} \tag{49}$$

$$x_2(t) = v_2\,\eta_1(t) + w_2\,\eta_2(t) \tag{50}$$

The velocities are

$$\dot{x}_1(t) = v_1\,\dot{\eta}_1(t) + w_1\,\dot{\eta}_{2(t)} \tag{51}$$

$$\dot{x}_2(t) = v_2\,\dot{\eta}_1(t) + w_2\,\dot{\eta}_2(t) \tag{52}$$

$$\dot{x}_1(t) = v_1\,\dot{\eta}_1(0)\,t + w_1\,\dot{\eta}_{2(t)} \tag{53}$$

$$\dot{x}_2(t) = v_2\, \dot{\eta}_1(0)\, t + w_2\, \dot{\eta}_2(t) \tag{54}$$

The accelerations are

$$\ddot{x}_1(t) = v_1\, \ddot{\eta}_1(t) + w_1\, \ddot{\eta}_{2(t)} \tag{55}$$

$$\ddot{x}_2(t) = v_2\, \ddot{\eta}_1(t) + w_2\, \ddot{\eta}_2(t) \tag{56}$$

Recall equation (41). The accelerations simplify to

$$\ddot{x}_1(t) = w_1\, \ddot{\eta}_{2(t)} \tag{57}$$

$$\ddot{x}_2(t) = w_2\, \ddot{\eta}_2(t) \tag{58}$$

Now consider the initial conditions. Recall

$$\overline{x} = \hat{Q}\, \overline{\eta} \tag{59}$$

Thus

$$\overline{x}(0) = \hat{Q}\, \overline{\eta}(0) \tag{60}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\, \overline{x}(0) = \hat{Q}^T M \hat{Q}\;\; \overline{\eta}(0) \tag{61}$$

Recall

$$\hat{Q}^T M \hat{Q} = I \tag{62}$$

$$\hat{Q}^T M\, \overline{x}(0) = I\;\; \overline{\eta}(0) \tag{63}$$

$$\hat{Q}^T M\, \overline{x}(0) = \overline{\eta}(0) \tag{64}$$

Finally, the transformed initial displacement matrix is

$$\bar{\eta}(0) = \hat{Q}^{T} M \, \bar{x}(0) \tag{65}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^{T} M \, \bar{\dot{x}}(0) \tag{66}$$

A basis for a solution is thus derived.

<u>References</u>

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