SEMIDEFINITE TWO-DEGREE-OF-FREEDOM SYSTEM
SUBJECTED TO A SINUSOIDAL FORCE
Revision A

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Two-degree-of-freedom System

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

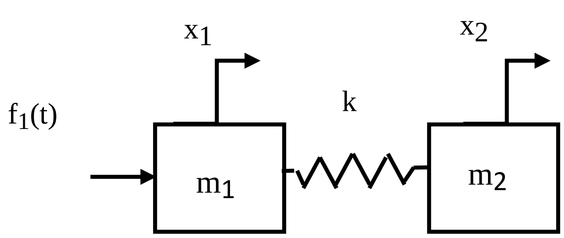

Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

Consider the case of free vibration.

The kinetic energy is

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \quad (1)$$

The potential energy is

$$U = \frac{1}{2}k(x_1 - x_2)^2 \tag{2}$$

$$\frac{d}{dt}\{T + U\} = 0 \tag{3}$$

$$\frac{d}{dt}\left\{\frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2} + \frac{1}{2}k(x_1 - x_2)^2\right\} = 0 \tag{4}$$

$$m_1\dot{x}_1\ddot{x}_1 + m_2\dot{x}_2\ddot{x}_2 + k(x_1 - x_2)\dot{x}_1 - k(x_1 - x_2)\dot{x}_2 = 0 \tag{5}$$

$$\{m_1\ddot{x}_1 + k(x_1 - x_2)\}\dot{x}_1 + \{m_2\ddot{x}_2 - k(x_1 - x_2)\}\dot{x}_2 = 0 \tag{6}$$

Equation (6) yields two equations.

$$\{m_1\ddot{x}_1 + k(x_1 - x_2)\}\dot{x}_1 = 0 \tag{7}$$

$$\{m_2\ddot{x}_2 - k(x_1 - x_2)\}\dot{x}_2 = 0 \tag{8}$$

Divide through by the respective velocity terms

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \tag{9}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) = 0 \tag{10}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

Seek a solution of the form

$$\bar{x}=\bar{q}\exp(j\omega t) \tag{12}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}}=j\omega\,\bar{q}\exp(j\omega t) \tag{13}$$

$$\ddot{\bar{x}}=-\omega^2\,\bar{q}\exp(j\omega t) \tag{14}$$

By substitution

$$-\omega^2 M\,\bar{q}\exp(j\omega t)+K\bar{q}\exp(j\omega t)=\bar{0} \tag{15}$$

$$\left\{-\omega^2 M\,\bar{q}+K\bar{q}\right\}\exp(j\omega t)=\bar{0} \tag{16}$$

$$-\omega_n{}^2 M\,\bar{q}+K\bar{q}=\bar{0} \tag{17}$$

$$\left\{-\omega^2 M+K\right\}\bar{q}=\bar{0} \tag{18}$$

$$\det\left\{K-\omega^2 M\right\}=0 \tag{19}$$

$$\det\left\{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\}=0 \tag{20}$$

$$\left(k-\omega^2 m_1\right)\left(k-\omega^2 m_2\right)-k^2=0 \tag{21}$$

$$k^2-k\omega^2(m_1+m_2)+\omega^4 m_1 m_2-k^2=0 \tag{22}$$

$$-k\omega^2(m_1+m_2)+\omega^4 m_1 m_2=0 \tag{23}$$

$$\omega^2\left[-k(m_1+m_2)+\omega^2 m_1 m_2\right]=0 \tag{24}$$

Thus the first root is

$$\omega_1=0 \tag{25}$$

$$f_1=0 \tag{26}$$

Find the second root

$$\left[-k(m_1+m_2)+\omega^2 m_1 m_2\right]=0 \tag{27}$$

$$\omega^2=\frac{k(m_1+m_2)}{m_1 m_2} \tag{28}$$

$$\omega_2=\sqrt{\frac{k(m_1+m_2)}{m_1 m_2}} \tag{29}$$

$$f_2=\frac{1}{2\pi}\sqrt{\frac{k(m_1+m_2)}{m_1 m_2}} \tag{30}$$

The eigenvectors are found via the following equations.

$$\left\{K - \omega_1^2 M\right\}\bar{q}_1 = \bar{0} \tag{31}$$

$$\left\{K - \omega_2^2 M\right\}\bar{q}_2 = \bar{0} \tag{32}$$

For the first mode,

$$\omega_1 = 0 \tag{33}$$

$$K\ \bar{q}_1 = \bar{0} \tag{34}$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\bar{q}_1 = \bar{0} \tag{35}$$

The eigenvector is

$$\bar{q}_1 = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{36}$$

Mass-normalize as follows

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = m_1 + m_2 \tag{37}$$

The mass-normalized eigenvector is

$$\bar{q}_1 = \frac{1}{\sqrt{m_1 + m_2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{38}$$

For the first mode,

$$\omega_2 = \frac{k(m_1 + m_2)}{m_1 m_2} \tag{39}$$

$$\left\{ K - \omega_2{}^2\, M \right\} \bar{q}_2 = \bar{0} \tag{40}$$

$$\left\{ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} - \frac{k(m_1 + m_2)}{m_1 m_2} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} \bar{q}_2 = \bar{0} \tag{41}$$

$$\left\{ m_1 m_2 \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} - k(m_1 + m_2) \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} \bar{q}_2 = \bar{0} \tag{42}$$

$$\left\{ k \begin{bmatrix} m_1 m_2 & -m_1 m_2 \\ -m_1 m_2 & m_1 m_2 \end{bmatrix} - k(m_1 + m_2) \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} \bar{q}_2 = \bar{0} \tag{43}$$

\

$$\left\{ \begin{bmatrix} m_1 m_2 & -m_1 m_2 \\ -m_1 m_2 & m_1 m_2 \end{bmatrix} - (m_1 + m_2) \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} \bar{q}_2 = \bar{0} \tag{44}$$

$$\left\{\begin{bmatrix} m_1 m_2 - (m_1 + m_2) m_1 & -m_1 m_2 \\ -m_1 m_2 & m_1 m_2 - (m_1 + m_2) m_2 \end{bmatrix}\right\} \bar{q}_2 = \bar{0} \tag{45}$$

$$\left\{\begin{bmatrix} -m_1^2 & -m_1 m_2 \\ -m_1 m_2 & -m_2^2 \end{bmatrix}\right\} \bar{q}_2 = \bar{0} \tag{46}$$

The unscaled mode shape is

$$\bar{q}_2 = \beta \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} \tag{47}$$

Mass-normalize as follows

$$\begin{bmatrix} m_2 & -m_1 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} = \begin{bmatrix} m_2 & -m_1 \end{bmatrix} \begin{bmatrix} m_1 m_2 \\ -m_1 m_2 \end{bmatrix} = m_1 m_2^2 + m_2 m_1^2$$

$$= m_1 m_2 (m_1 + m_2) \tag{48}$$

$$\bar{q}_2 = \frac{1}{\sqrt{m_1 m_2 (m_1 + m_2)}} \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} \tag{49}$$

$$Q = \begin{bmatrix} \bar{q}_1 & \bar{q}_2 \end{bmatrix} \tag{50}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{m_1 + m_2}} & \frac{m_2}{\sqrt{m_1 m_2 (m_1 + m_2)}} \\ \frac{1}{\sqrt{m_1 + m_2}} & \frac{-m_1}{\sqrt{m_1 m_2 (m_1 + m_2)}} \end{bmatrix} \tag{51}$$

$$Q = \frac{1}{\sqrt{m_1 + m_2}} \begin{bmatrix} 1 & \frac{m_2}{\sqrt{m_1 m_2}} \\ 1 & \frac{-m_1}{\sqrt{m_1 m_2}} \end{bmatrix} \quad (52)$$

$$Q = \frac{1}{\sqrt{m_1 + m_2}} \begin{bmatrix} 1 & \sqrt{\frac{m_2}{m_1}} \\ 1 & -\sqrt{\frac{m_1}{m_2}} \end{bmatrix} \quad (53)$$

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^T M \, \bar{r} \quad (54)$$

$$\bar{L} = \frac{1}{\sqrt{m_1 + m_2}} \begin{bmatrix} \frac{1}{\sqrt{\frac{m_2}{m_1}}} & -\frac{1}{\sqrt{\frac{m_1}{m_2}}} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (55)$$

$$\bar{L} = \frac{1}{\sqrt{m_1 + m_2}} \begin{bmatrix} \frac{1}{\sqrt{\frac{m_2}{m_1}}} & -\frac{1}{\sqrt{\frac{m_1}{m_2}}} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \quad (56)$$

$$\bar{L} = \frac{1}{\sqrt{m_1 + m_2}} \begin{bmatrix} m_1 + m_2 \\ 0 \end{bmatrix} \tag{57}$$

$$\bar{L} = \begin{bmatrix} \sqrt{m_1 + m_2} \\ 0 \end{bmatrix} \tag{58}$$

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{59}$$

Note that $\hat{m}_{ii} = 1$ for each index if the eigenvectors have been normalized with respect to the mass matrix.

$$\Gamma_1 = \sqrt{m_1 + m_2} \tag{60}$$

$$\Gamma_2 = 0 \tag{61}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\,2}}{\hat{m}_{ii}} \tag{62}$$

$$m_{eff,1} = m_1 + m_2 \tag{63}$$

$$m_{eff,2} = 0 \tag{64}$$

Assemble the equations in matrix form with the applied force.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{65}$$

<u>Decoupling</u>

Equation (65) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M \ddot{\bar{x}} + K \bar{x} = \bar{F} \tag{66}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{67}$$

$$K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{68}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{69}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{70}$$

$$Q = \begin{bmatrix} \bar{q}_1 & \bar{q}_2 \end{bmatrix} \tag{71}$$

$$\hat{Q}^T M \hat{Q} = I \tag{72}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{73}$$

where

I	is the identity matrix
Ω	is a diagonal matrix of eigenvalues

The superscript T represents transpose.

Note the mass-normalized forms

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{74}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{75}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial.

Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{76}$$

Substitute equation (76) into the equation of motion, equation (66).

$$M\hat{Q}\,\ddot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = \bar{F} \tag{77}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M \hat{Q} \ddot{\bar{\eta}} + \hat{Q}^T K \hat{Q} \bar{\eta} = \hat{Q}^T \bar{F} \tag{78}$$

The orthogonality relationships yield

$$I \ddot{\bar{\eta}} + \Omega \bar{\eta} = \hat{Q}^T \bar{F} \tag{79}$$

The equations of motion along with an added damping matrix become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{80}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{81}$$

Note that the two equations are decoupled in terms of the generalized coordinate.

Equation (81) yields two equations

$$\ddot{\eta}_1 = \hat{v}_1 f_1(t) \tag{82}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 \dot{\eta}_2 + \omega_2^2 \eta_2 = \hat{w}_1 f_1(t) \tag{83}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{x} = \hat{Q} \bar{\eta} \tag{84}$$

Thus

$$\bar{x}(0) = \hat{Q}\,\bar{\eta}(0) \tag{85}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\,\bar{x}(0) = \hat{Q}^T M \hat{Q}\,\bar{\eta}(0) \tag{86}$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \tag{87}$$

$$\hat{Q}^T M\,\bar{x}(0) = I\,\bar{\eta}(0) \tag{88}$$

$$\hat{Q}^T M\,\bar{x}(0) = \bar{\eta}(0) \tag{89}$$

Finally, the transformed initial displacement is

$$\bar{\eta}(0) = \hat{Q}^T M\,\bar{x}(0) \tag{90}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^T M\,\bar{\dot{x}}(0) \tag{91}$$

A basis for a solution is thus derived.

Sinusoidal Force

Now consider the special case of a sinusoidal force applied to mass 1 with zero initial conditions.

$$f_1(t) = A \sin(\omega t) \tag{92}$$

$$f_2(t) = 0 \tag{93}$$

Thus,

$$\ddot{\eta}_1 = \hat{v}_1 A \sin(\omega t) \tag{94}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 \dot{\eta}_2 + \omega_2^2 \eta_2 = \hat{w}_1 A \sin(\omega t) \tag{95}$$

The equations are solved using the methods in References 3 and 4.

Take the Laplace transform of equation (94).

$$\ddot{\eta}_1 = \hat{v}_1 A \sin(\omega t) \tag{96}$$

$$L\{\ddot{\eta}_1\} = L\{\hat{v}_1 A \sin(\omega t)\} \tag{97}$$

$$s^2 \hat{\eta}_1(s) - s\eta_1(0) - \dot{\eta}_1(0) = \hat{v}_1 A \left\{ \frac{\omega}{s^2 + \omega^2} \right\} \tag{98}$$

$$s^2 \hat{\eta}_1(s) - s\eta_1(0) - \dot{\eta}_1(0) \quad = \quad \hat{v}_1 A \left\{ \frac{\omega}{s^2 + \omega^2} \right\} \tag{99}$$

$$s^2 \hat{\eta}_1(s) \quad = \quad \hat{v}_1 A \left\{ \frac{\omega}{s^2 + \omega^2} \right\} + s\eta_1(0) + \dot{\eta}_1(0) \tag{100}$$

$$\hat{\eta}_1(s) \quad = \quad \hat{v}_1 A \frac{1}{s^2} \left\{ \frac{\omega}{s^2 + \omega^2} \right\} + \frac{1}{s} \eta_1(0) + \frac{1}{s^2} \dot{\eta}_1(0) \tag{101}$$

$$\hat{\eta}_1(s) \quad = \quad \hat{v}_1 A \frac{1}{\omega} \left\{ \frac{1}{s^2} + \frac{-1}{s^2 + \omega^2} \right\} + \frac{1}{s} \eta_1(0) + \frac{1}{s^2} \dot{\eta}_1(0) \tag{102}$$

The solution is found via References 3 and 4. The inverse Laplace transform for the first modal coordinate is

$$\eta_1(t) = \hat{v}_1 A \frac{1}{\omega^2}\{\omega t - \sin(\omega t)\} + \eta_1(0) + \dot{\eta}_1(0)\,t \tag{103}$$

For zero initial conditions,

$$\eta_1(t) = \hat{v}_1 A \frac{1}{\omega^2}\{\omega t - \sin(\omega t)\} \tag{104}$$

Recall the equation for the second modal coordinate.

$$\ddot{\eta}_2 + 2\xi_2\,\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = \hat{w}_1 A \sin(\omega t) \tag{105}$$

From Reference (5),

$$\begin{aligned}
&\eta_2(t) = \\
&+ \frac{\hat{w}_1 A}{\left[\left(\omega^2 - \omega_2^2\right)^2 + (2\xi_2\,\beta\,\omega_2)^2\right]}\left\{ -[2\xi_2\omega_2\,\omega]\cos(\omega t) - \frac{1}{\omega}\left[\omega^2 - \omega_2^2\right]\sin(\omega t)\right\} \\
&+ \frac{\hat{w}_1 A\left[\dfrac{\omega}{\omega_{d,2}}\right]\exp(-\xi_2\omega_2 t)}{\left[\left(\beta^2 - \omega_2^2\right)^2 + (2\xi_2\,\beta\,\omega_2)^2\right]}\left\{\left[2\xi_2\omega_2\,\omega_{d,2}\right]\cos\left(\omega_{d,2}t\right) + \left[\omega^2 - {\omega_2}^2\left(1 - 2{\xi_2}^2\right)\right]\sin\left(\omega_{d,2}t\right)\right\} \\
&+ \exp(-\xi_2\omega_2 t)\left\{\eta_2(0)\cos\left(\omega_{d,2}t\right) + \left\{\frac{\dot{\eta}_2(0) + (\xi_2\,\omega_2)\eta_2(0)}{\omega_{d,2}}\right\}\sin\left(\omega_{d,2}t\right)\right\}
\end{aligned} \tag{106}$$

For zero initial conditions,

$$\eta_2(t) =$$

$$+\frac{\hat{w}_1 A}{\left[\left(\omega^2-\omega_2^2\right)^2+\left(2\xi_2\,\beta\,\omega_2\right)^2\right]}\left\{-\left[2\xi_2\omega_2\,\omega\right]\cos(\omega t)-\frac{1}{\omega}\left[\omega^2-\omega_2^2\right]\sin(\omega t)\right\}$$

$$+\frac{\hat{w}_1 A\left[\dfrac{\omega}{\omega_{d,2}}\right]\exp\left(-\xi_2\omega_2 t\right)}{\left[\left(\beta^2-\omega_2^2\right)^2+\left(2\xi_2\,\beta\,\omega_2\right)^2\right]}\left\{\left[2\xi_2\omega_2\,\omega_{d,2}\right]\cos\left(\omega_{d,2}t\right)+\left[\omega^2-\omega_2{}^2\left(1-2\xi_2{}^2\right)\right]\sin\left(\omega_{d,2}t\right)\right\}$$

(107)

The physical displacements are found via

$$\bar{x}=\hat{Q}\,\bar{\eta} \qquad (108)$$

An example is given in Appendix A.

The transfer function can be calculated using the method in Appendix B.

<u>References</u>

1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.
2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.
3. T. Irvine, Table of Laplace Transforms, Rev J, Vibrationdata, 2001.
4. T. Irvine, Partial Fraction Expansion, Rev K, Vibrationdata, 2013.
5. T. Irvine, Two-degree-of-freedom System Subjected to a Half-sine Pulse Force, Vibrationdata, 2012.
6. R. Craig & A. Kurdila, Fundamentals of Structural Dynamics, Second Edition, Wiley, New Jersey, 2006.

APPENDIX A

Example

Consider the system in Figure 1 with the values in Table A-1.

Assume 5% damping for each mode. Assume zero initial conditions.

Table A-1. Parameters		
Variable	Value	Unit
m_1	2	lbm
m_2	1	lbm
k	2000	lbf/in
A	1	lbf
f	171.3	Hz

The analysis is performed using a Matlab script. Note that the system is driven at its second natural frequency.

```
>> semidefinite_force
 semidefinite_force.m  ver 1.4   May 2, 2014

 Response of a semi-definite two-degree-of-freedom
 system subjected to an applied sinusoidal force.

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 Enter unit:  1=English  2=metric
 1
      Mass unit: lbm
 Stiffness unit: lbf/in

 Enter mass 1
 2

 Enter mass 2
 1

 Enter stiffness for spring between masses 1 & 2
 2000
```

```
        Natural     Participation    Effective
Mode   Frequency      Factor         Modal Mass
1   5.096e-07 Hz      0.08816     0.007772
2       171.3 Hz            0            0

 modal mass sum = 0.007772


 mass matrix

m =

    0.0052         0
         0    0.0026


 stiffness matrix

k =

        2000       -2000
       -2000        2000



ModeShapes =

   11.3431   -8.0208
   11.3431   16.0416


 Enter viscous damping ratio 0.05

 Apply sinusoidal force to mass 1

 Enter force (lbf) 1

 Enter excitation frequency (Hz) 171.3

 Enter duration (sec) 0.1
```

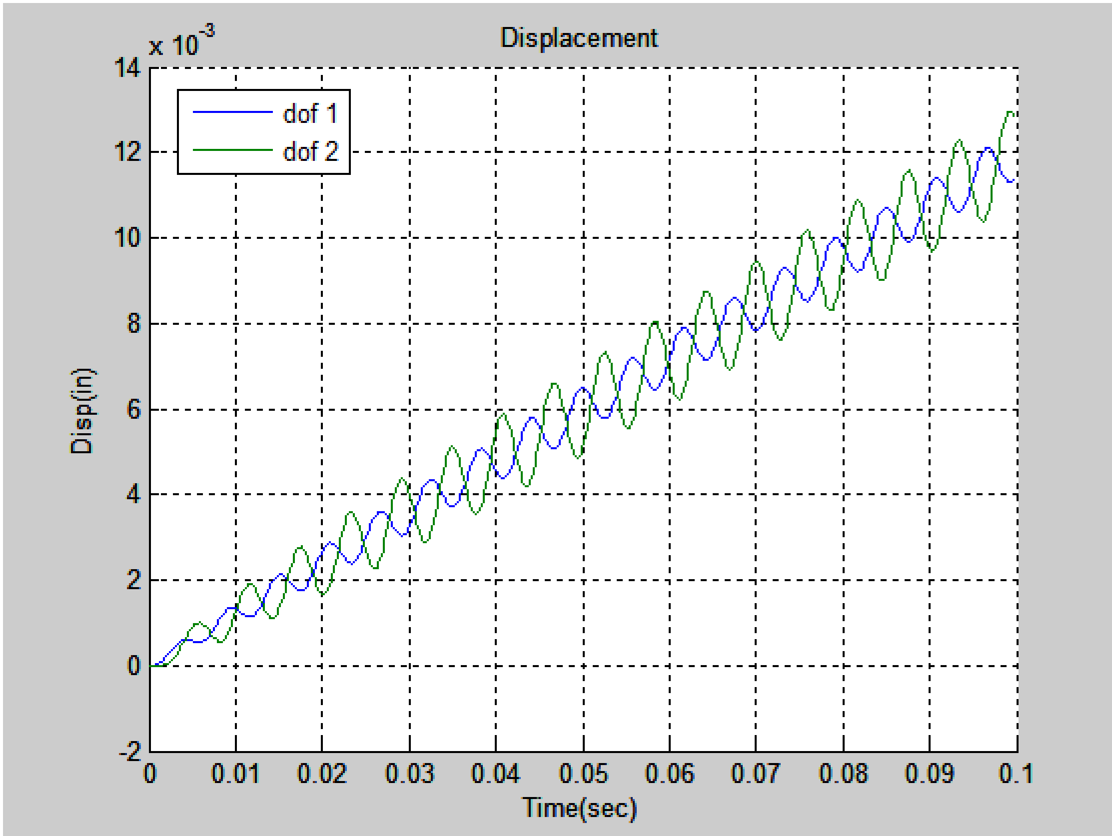
Displacement
x 10^-3
14
12
10
8
6
4
2
0
-2
dof 1
dof 2
Disp(in)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
Time(sec)

Figure A-1.

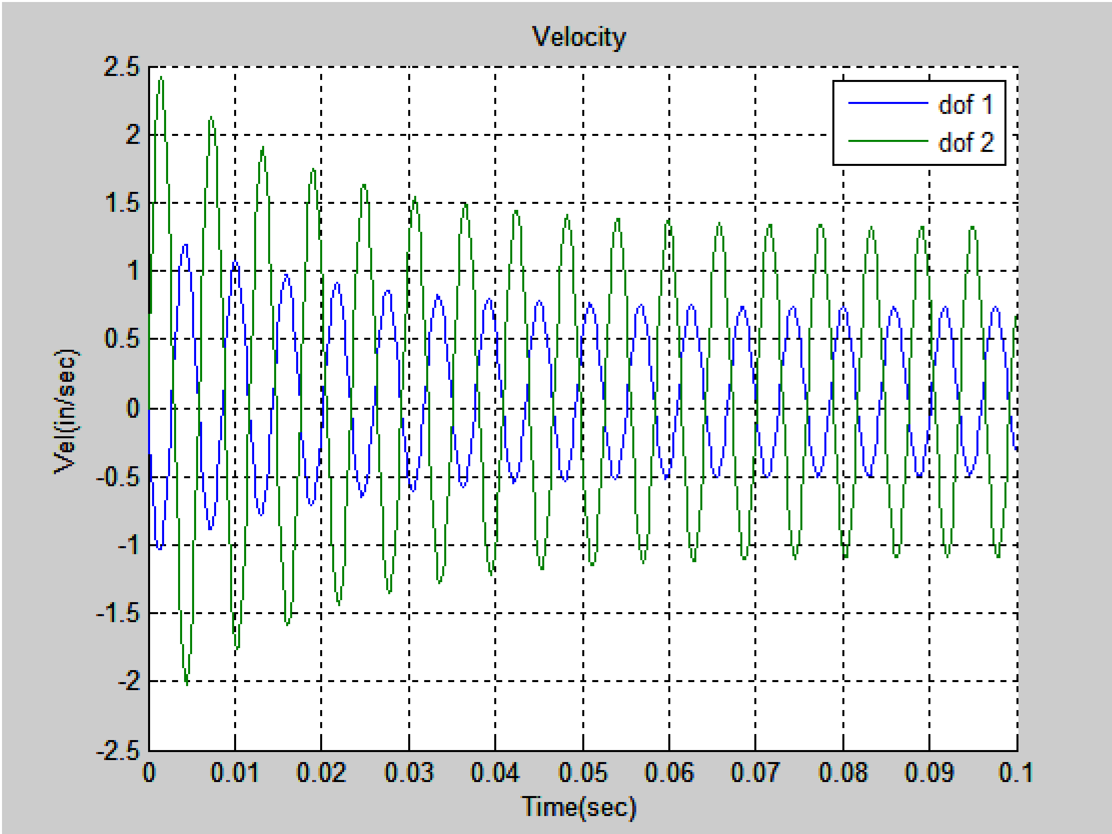
Velocity
Vel(in/sec)
Time(sec)
dof 1
dof 2
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Figure A-2.

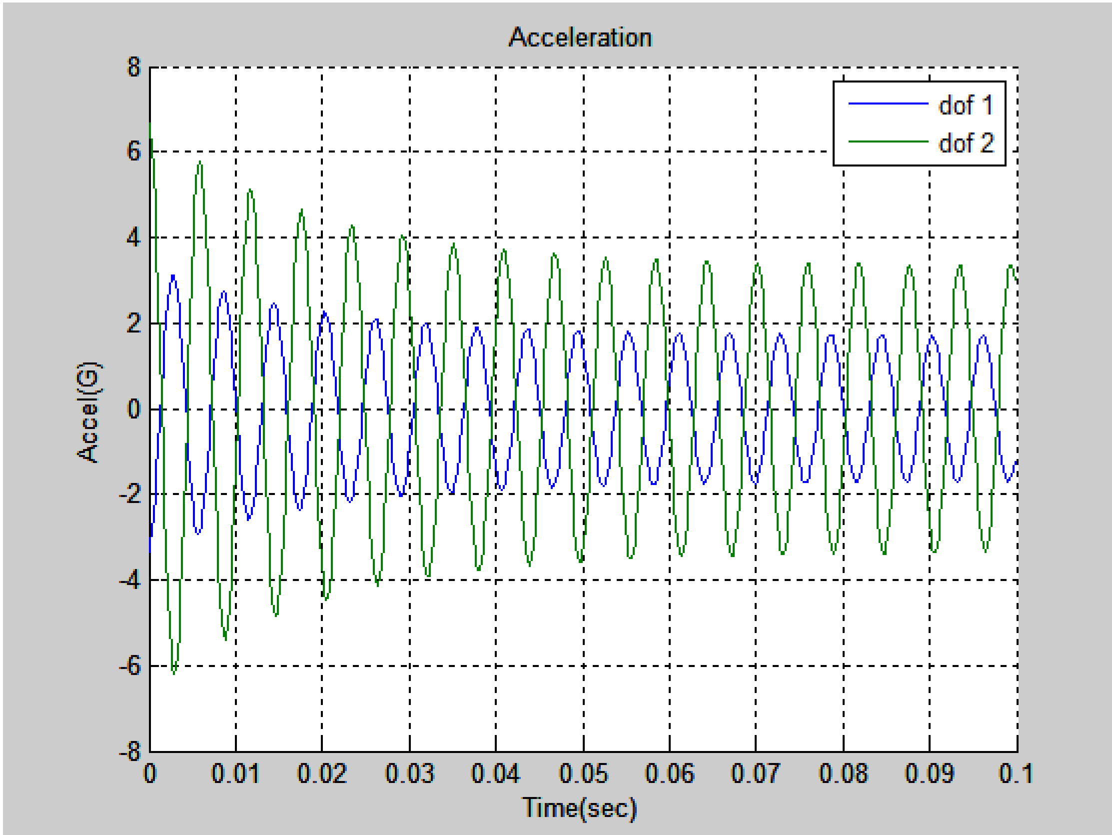
Acceleration
dof 1
dof 2
Accel(G)
8
6
4
2
0
-2
-4
-6
-8
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
Time(sec)

Figure A-3.

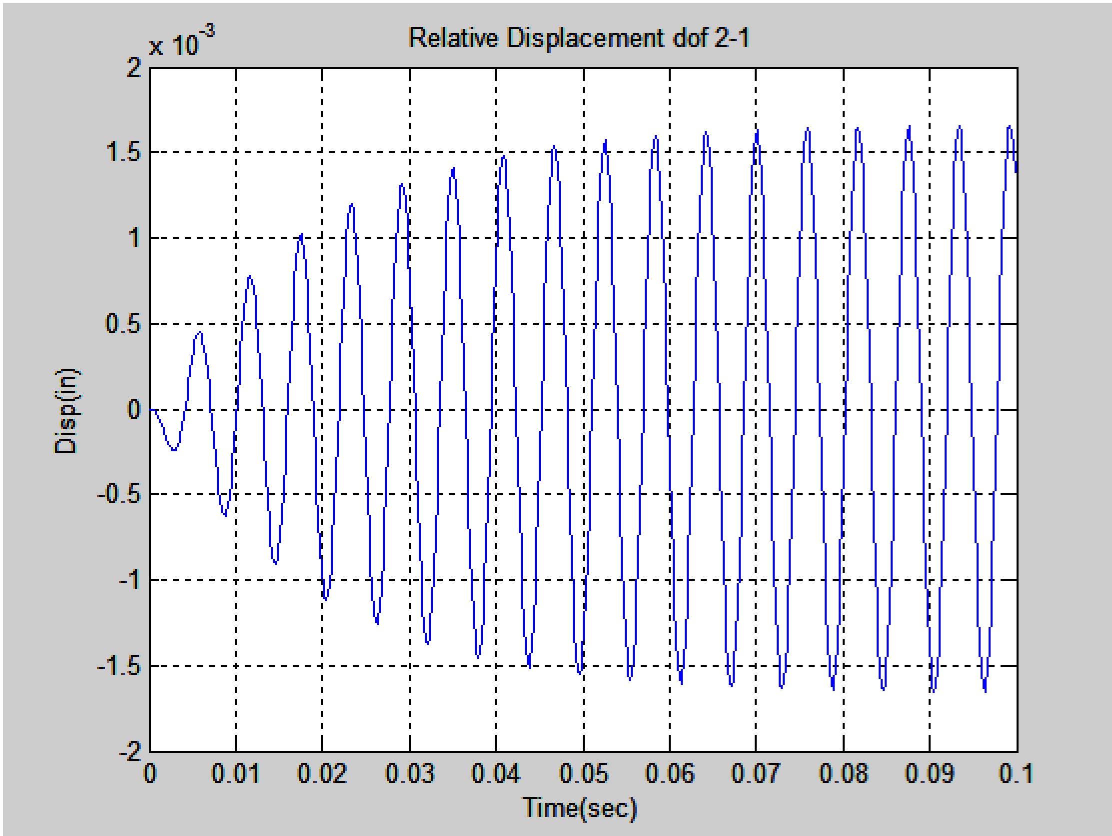
Relative Displacement dof 2-1
x 10^-3
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
Disp(in)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
Time(sec)

Figure A-4.

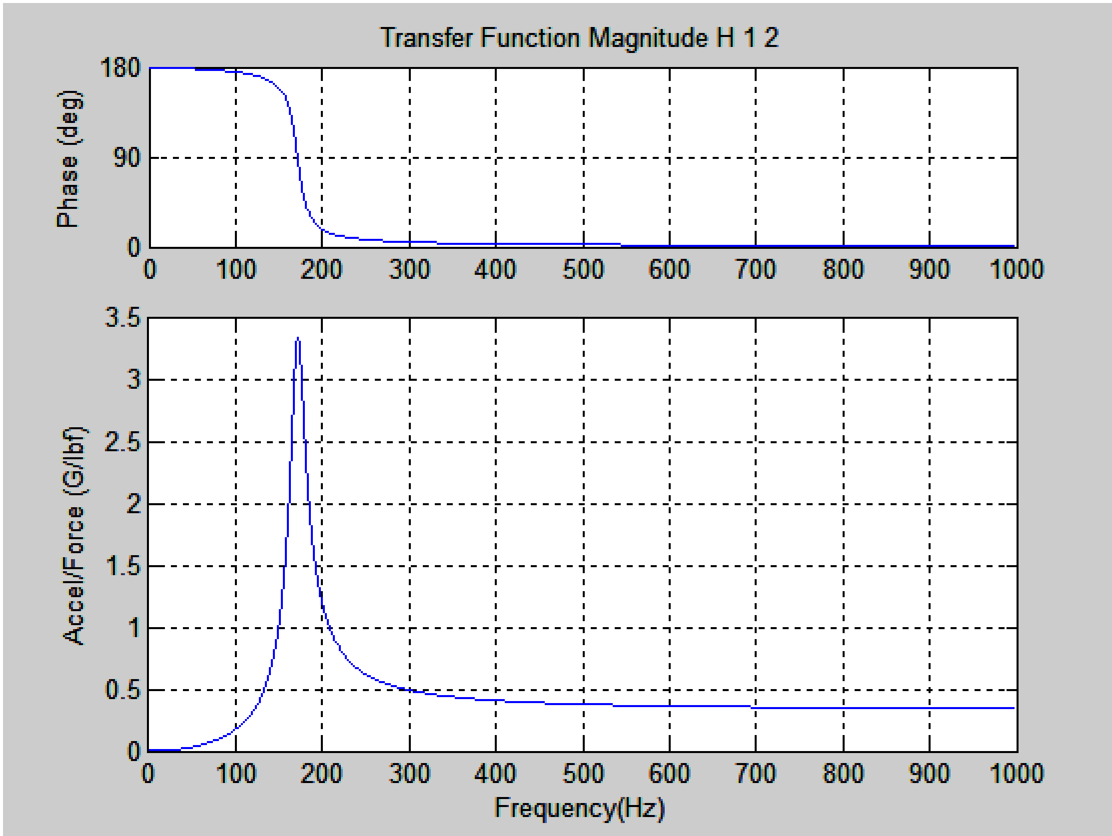

Figure A-5.

The rigid-body mode has been suppressed.

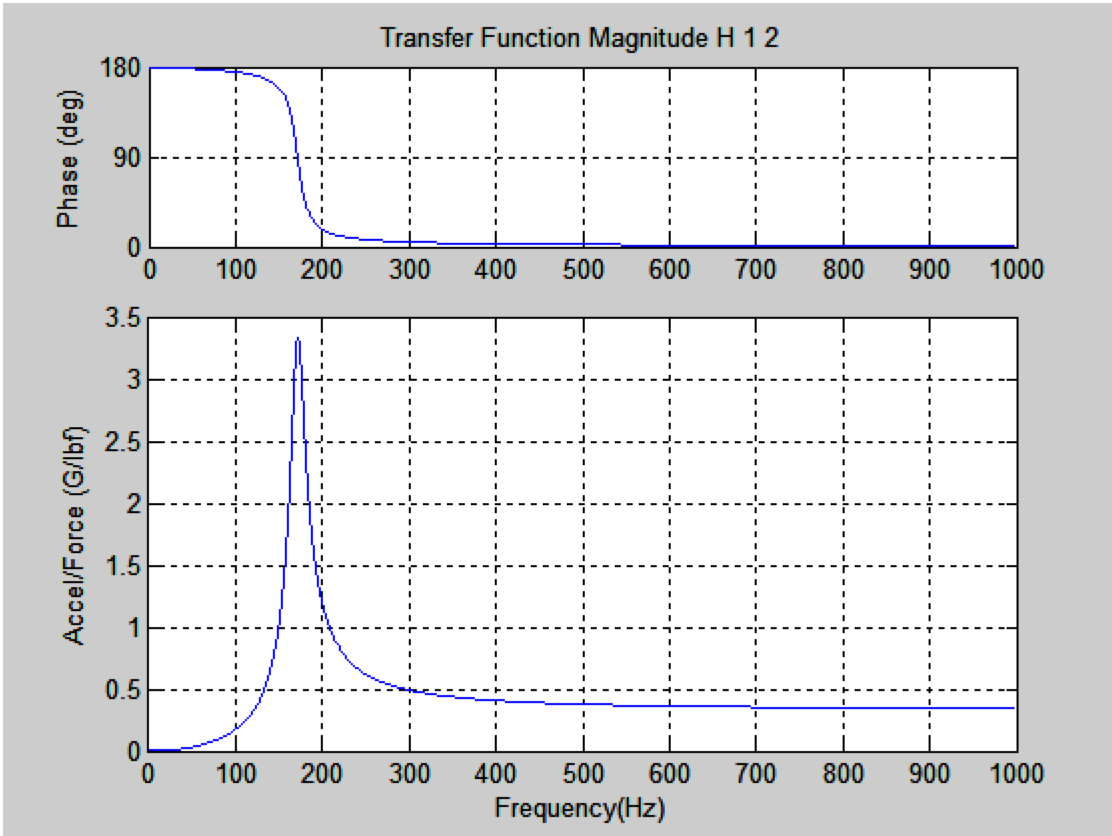

Figure A-6.

The rigid-body mode has been suppressed.

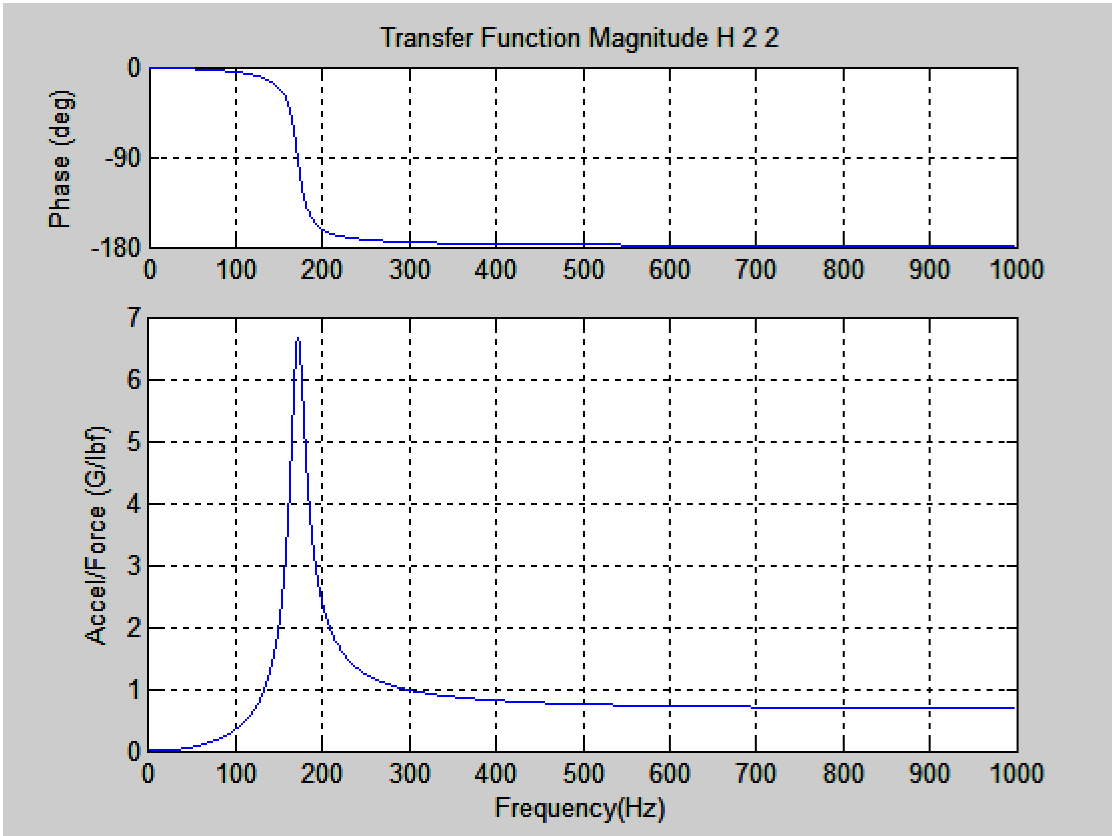

Figure A-7.

The rigid-body mode has been suppressed.

APPENDIX B

Transfer Function

The following is taken from Reference 6.

The variables are:

F	Excitation frequency
f_r	Natural frequency for mode *r*
N	Total degrees-of-freedom
$H_{ij}(f)$	The steady state displacement at coordinate *i* due to a harmonic force excitation only at coordinate *j*
ξ_r	Damping ratio for mode *r*
ϕ_{ir}	Mass-normalized eigenvector for physical coordinate *i* and mode number *r*
ω	Excitation frequency (rad/sec)
ω_r	Natural frequency (rad/sec) for mode *r*

The following equations are for a general system. Note that r should be given an initial value of 2 in order to suppress the rigid-body mode for the case of the semi-definite, two-degree-of-freedom system. This is needed since the fundamental frequency is zero, aside from numerical error.

Receptance

The steady-state displacement at coordinate *i* due to a harmonic force excitation only at coordinate *j* is:

$$H_{ij}(f) = \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{jr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right) + \hat{j}\left(2\xi_r\rho_r\right)} \right\} \tag{B-1}$$

where

$$\rho_r = f / f_r \tag{B-2}$$

$$\hat{j} = \sqrt{-1} \tag{B-3}$$

Note that the phase angle is typically represented as the angle by which force leads displacement. In terms of a C++ or Matlab type equation, the phase angle would be

$$\text{Phase} = -\,\text{atan2}(\text{imag(H)}, \text{real(H)}) \tag{B-4}$$

Note that both the phase and the transfer function vary with frequency.

A more formal equation is

$$\text{Phase}(f) = -\arctan\left\{\frac{\text{imag}\left(H_{ij}(f)\right)}{\text{real}\left(H_{ij}(f)\right)}\right\} \tag{B-5}$$

<u>Mobility</u>

The steady-state velocity at coordinate *i* due to a harmonic force excitation only at coordinate *j* is

$$\hat{H}_{ij}(f) = j\omega \sum_{r=1}^{N}\left\{\frac{\phi_{ir}\,\phi_{jr}}{\omega_r^{\,2}}\,\frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\,\left(2\xi_r\rho_r\right)}\right\} \tag{B-6}$$

<u>Accelerance</u>

The steady-state acceleration at coordinate *i* due to a harmonic force excitation only at coordinate *j* is

$$\tilde{H}_{ij}(f) = -\omega^2 \sum_{r=1}^{N}\left\{\frac{\phi_{ir}\,\phi_{jr}}{\omega_r^{\,2}}\,\frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\,\left(2\xi_r\rho_r\right)}\right\} \tag{B-7}$$

Relative Displacement

Consider two translational degrees-of-freedom *i* and *j*. A force is applied at degree-of-freedom *k*.

The steady-state relative displacement transfer function R_{ij} between *i* and *j* due to an applied force at *k* is

$$R_{ij} = H_{ik}(f) - H_{jk}(f)$$

$$= \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{kr}}{\omega_r^2} \frac{1}{\left(1-\rho_r^2\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\} - \sum_{r=1}^{N} \left\{ \frac{\phi_{jr}\ \phi_{kr}}{\omega_r^2} \frac{1}{\left(1-\rho_r^2\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\}$$

(B-8)

$$R_{ij} = \sum_{r=1}^{N} \left\{ \frac{\left(\phi_{ir} - \phi_{jr}\right)\phi_{kr}}{\omega_r^2} \frac{1}{\left(1-\rho_r^2\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\} \quad \text{(B-9)}$$

The steady-state relative displacement transfer function R_{ij} between *i* and *j* due to an applied force at *k* is

$$R_{ij} = H_{ik}(f) - H_{jk}(f)$$

$$= \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{kr}}{\omega_r^2} \frac{1}{\left(1-\rho_r^2\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\} - \sum_{r=1}^{N} \left\{ \frac{\phi_{jr}\ \phi_{kr}}{\omega_r^2} \frac{1}{\left(1-\rho_r^2\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\}$$

(B-10)

$$R_{ij} = \sum_{r=1}^{N} \left\{ \frac{\left(\phi_{ir} - \phi_{jr}\right)\phi_{kr}}{\omega_r^{\,2}} \frac{\left(1-\rho_r^{\,2}\right)}{\left(1-\rho_r^{\,2}\right)^2 + \left(2\xi_r\rho_r\right)^2} \right\}$$

$$- j \sum_{r=1}^{N} \left\{ \frac{\left(\phi_{ir} - \phi_{jr}\right)\phi_{kr}}{\omega_r^{\,2}} \frac{\left(2\xi_r\rho_r\right)}{\left(1-\rho_r^{\,2}\right)^2 + \left(2\xi_r\rho_r\right)^2} \right\}$$

(B-11)