THE STEADY-STATE RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A HARMONIC FORCE Revision A

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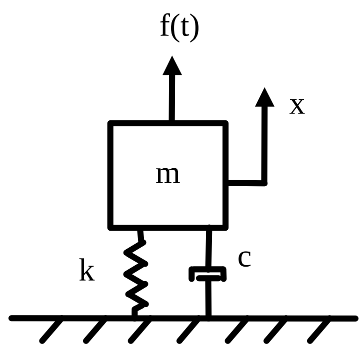
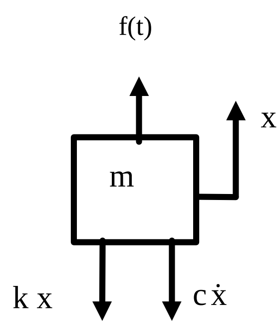
Consider a single-degree-of-freedom system.



where

m is the mass,
c is the viscous damping coefficient,
k is the stiffness,
x is the absolute displacement of the mass,
f(t) is the applied force.

The free-body diagram is



Sum the forces in the vertical direction.

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = -c\dot{x} - kx + f(t) \tag{2}$$

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad (3)$$

Divide through by m,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)f(t) \quad (4)$$

By convention,

$$(c/m) = 2\xi\omega_n \quad (5)$$

$$(k/m) = \omega_n{}^2 \quad (6)$$

where

ω_n is the natural frequency in (radians/sec),
ξ is the damping ratio.

By substitution,

$$\ddot{x} + 2\,\xi\,\omega_n\,\dot{x} + \omega_n{}^2\,x = \frac{1}{m}f(t) \quad (7)$$

Now assume a sinusoidal forcing function, which is representative of harmonic motion.

$$f(t) = f_o \sin(\omega t) \quad (8)$$

The governing equation becomes.

$$\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n^2\,x = \frac{1}{m}f_o \sin(\omega t) \quad (9)$$

The right-hand-side can be rewritten as

$$\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n^2\,x = \frac{\omega_n{}^2}{k}f_o \sin(\omega t) \quad (10)$$

Now take the Fourier transform of each side

$$\int_{-\infty}^{\infty}\left\{\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2 x\right\}e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}\left\{\frac{\omega_n^2}{k}f_o\sin(\omega t)\right\}e^{-j\omega t}\,dt \tag{11}$$

Note that the approach used here is rigorous. Simpler approaches are often used in other references.

Let

$$X(\omega)=\int_{-\infty}^{\infty}\{x(t)\}e^{-j\omega t}\,dt \tag{12}$$

$$F(\omega)=\int_{-\infty}^{\infty}\{f_o\sin(\omega t)\}e^{-j\omega t}\,dt \tag{13}$$

Now take the Fourier transform of the velocity term

$$\int_{-\infty}^{\infty}\{\dot{x}(t)\}e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}\left\{\frac{dx(t)}{dt}\right\}e^{-j\omega t}\,dt \tag{14}$$

Integrate by parts

$$\int_{-\infty}^{\infty}\{\dot{x}(t)\}e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}d\left\{x(t)e^{-j\omega t}\right\}-\int_{-\infty}^{\infty}[x(t)](-j\omega)e^{-j\omega t}\,dt \tag{15}$$

$$\int_{-\infty}^{\infty}\{\dot{x}(t)\}e^{-j\omega t}\,dt=x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty}+(j\omega)\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt \tag{16}$$

$$x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty}=0 \quad \text{as } t \text{ approaches the} \pm\infty \text{ limits per Reference 1.} \tag{17}$$

$$\int_{-\infty}^{\infty}\{\dot{x}(t)\}e^{-j\omega t}\,dt=(j\omega)\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt \tag{18}$$

$$\int_{-\infty}^{\infty} \{\dot{x}(t)\} e^{-j\omega t}\, dt = (j\omega) X(\omega) \tag{19}$$

Furthermore

$$\int_{-\infty}^{\infty} \{\ddot{x}(t)\} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \left\{ \frac{d^2 x(t)}{dt^2} \right\} e^{-j\omega t}\, dt \tag{20}$$

$$\int_{-\infty}^{\infty} \{\ddot{x}(t)\} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} d\left\{ \frac{dx(t)}{dt} e^{-j\omega t} \right\} - \int_{-\infty}^{\infty} \frac{dx(t)}{dt} (-j\omega) e^{-j\omega t}\, dt \tag{21}$$

$$\int_{-\infty}^{\infty} \{\ddot{x}(t)\} e^{-j\omega t}\, dt = \left. \frac{dx(t)}{dt} e^{-j\omega t} \right|_{-\infty}^{\infty} + (j\omega) \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t}\, dt \tag{22}$$

$$\left. \frac{dx(t)}{dt} e^{-j\omega t} \right|_{-\infty}^{\infty} = 0 \qquad \text{per Reference 1.} \tag{23}$$

$$\int_{-\infty}^{\infty} \{\ddot{x}(t)\} e^{-j\omega t}\, dt = (j\omega) \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t}\, dt \tag{24}$$

$$\int_{-\infty}^{\infty} \{\ddot{x}(t)\} e^{-j\omega_n t}\, dt = (j\omega)(j\omega) X(\omega) \tag{25}$$

$$\int_{-\infty}^{\infty} \{\ddot{x}(t)\} e^{-j\omega t}\, dt = -\omega^2 X(\omega) \tag{26}$$

Recall

$$\int_{-\infty}^{\infty} \left\{ \ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x \right\} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \left\{ \frac{\omega_n^2}{k} f_o \sin(\omega t) \right\} e^{-j\omega t}\, dt \tag{27}$$

By substitution,

$$-\omega^2 X(\omega) + j\omega(2\xi\omega_n)X(\omega) + \omega_n{}^2\ X(\omega) = \frac{\omega_n{}^2}{k}\ F(\omega) \tag{28}$$

$$\left\{-\omega^2 + j\omega(2\xi\omega_n) + \omega_n{}^2\right\}X(\omega) = \frac{\omega_n{}^2}{k}\ F(\omega) \tag{29}$$

$$X(\omega) = \frac{\omega_n{}^2}{k}\left[\frac{1}{-\omega^2 + j\omega(2\xi\omega_n) + \omega_n{}^2}\right]F(\omega) \tag{30}$$

$$\frac{k\,X(\omega)}{F(\omega)} = \left[\frac{\omega_n{}^2}{-\omega^2 + j\omega(2\xi\omega_n) + \omega_n{}^2}\right] \tag{31}$$

$$\frac{k\,X(\omega)}{F(\omega)} = \left[\frac{\omega_n{}^2}{\omega_n{}^2 - \omega^2 + j\omega(2\xi\omega_n)}\right] \tag{32}$$

$$\frac{k\,X(\omega)}{F(\omega)} = \left[\frac{1}{\left[\frac{\omega_n{}^2 - \omega^2}{\omega_n{}^2}\right] + j2\xi\frac{\omega}{\omega_n}}\right] \tag{33}$$

Let

$$\rho = \frac{\omega}{\omega_n} \tag{34}$$

$$\frac{k\,X(\rho)}{F(\rho)} = \left[\frac{1}{\left[1-\rho^2\right] + j\,2\xi\rho} \right] \tag{35}$$

The transfer function is often represented by $H(\rho)$.

$$H(\rho) = \frac{k\,X(\rho)}{F(\rho)} \tag{36}$$

$$H(\rho) = \left[\frac{1}{\left[1-\rho^2\right] + j\,2\xi\rho} \right] \tag{37}$$

This transfer function, which represents displacement over force, is sometimes called the receptance function.

The transfer function can be represented in terms of magnitude $\left| H(\rho) \right|$ and phase angle ϕ as

$$\left| H(\rho) \right| = \left[\frac{1}{\sqrt{\left[1-\rho^2\right]^2 + \left[2\xi\rho\right]^2}} \right] \tag{38}$$

$$\phi = \arctan\left[\frac{2\xi\rho}{1-\rho^2} \right] \tag{39}$$

The transfer function magnitude is plotted for three damping cases in Figure 1. The corresponding phase angles are shown in Figure 2.

The transfer function can also be represented in terms of real and imaginary components.

$$H(\rho) = \left[\frac{1}{\left[1-\rho^2\right] + j2\xi\rho}\right]\left[\frac{\left[1-\rho^2\right] - j2\xi\rho}{\left[1-\rho^2\right] - j2\xi\rho}\right] \tag{40}$$

$$H(\rho) = \left[\frac{\left[1-\rho^2\right] - j2\xi\rho}{\left[1-\rho^2\right]^2 + [2\xi\rho]^2}\right] \tag{41}$$

The transfer function can thus be represented in terms of real and imaginary components as

$$H(\rho) = \left[\frac{\left[1-\rho^2\right]}{\left[1-\rho^2\right]^2 + [2\xi\rho]^2}\right] - j\left[\frac{2\xi\rho}{\left[1-\rho^2\right]^2 + [2\xi\rho]^2}\right] \tag{42}$$

The force transmitted to the floor, or mounting surface, is given in Appendix A.

<u>Reference</u>

1. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.

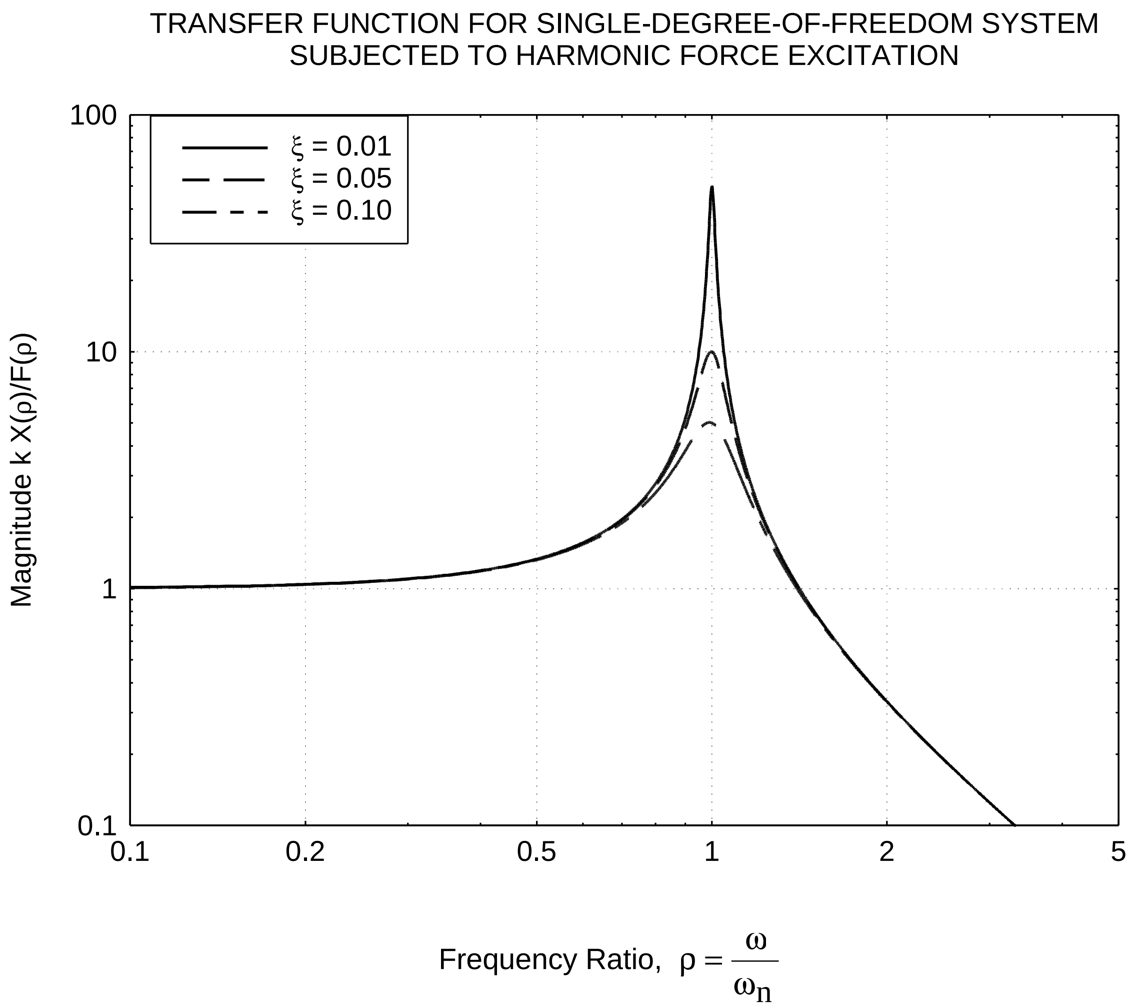
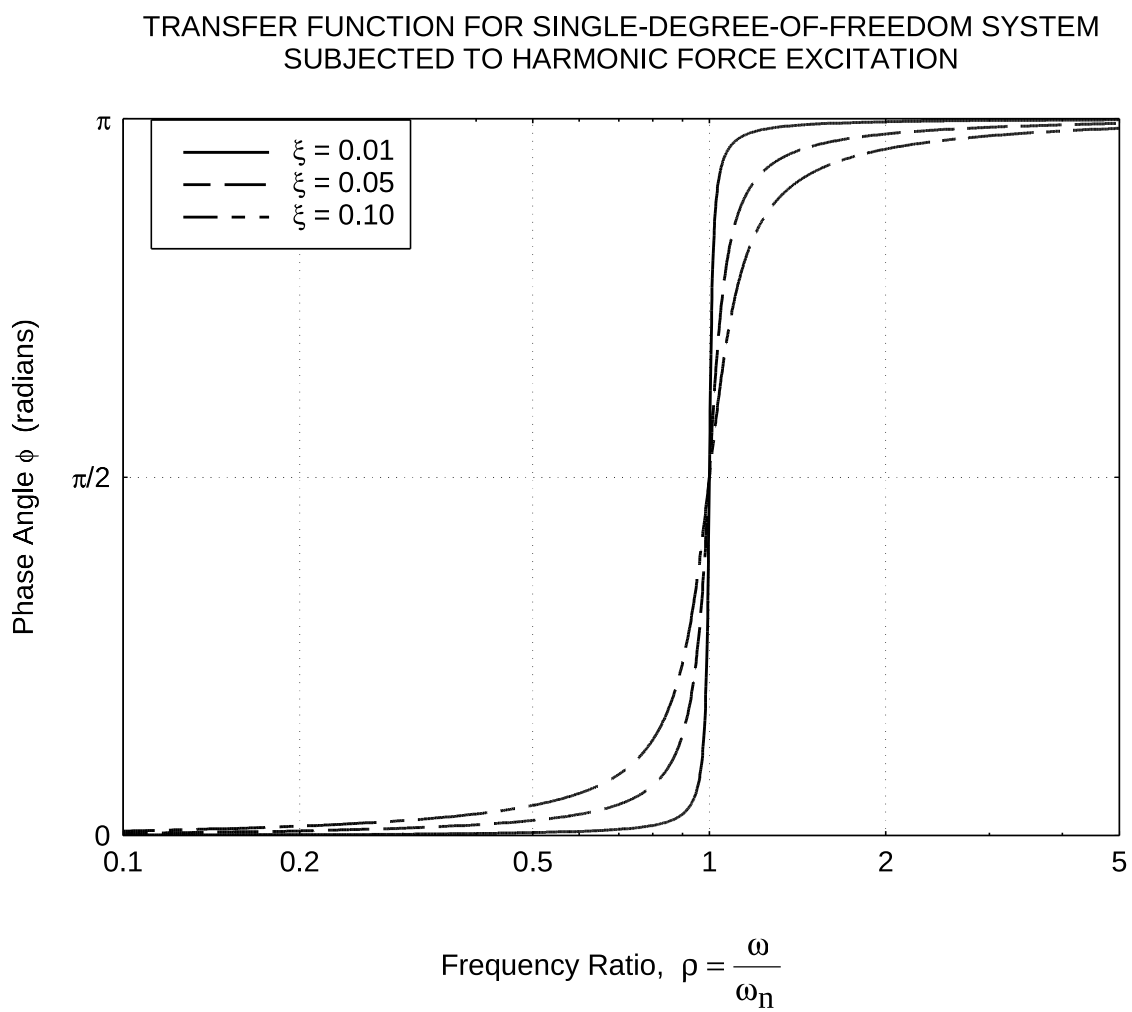

TRANSFER FUNCTION FOR SINGLE-DEGREE-OF-FREEDOM SYSTEM
SUBJECTED TO HARMONIC FORCE EXCITATION
ξ = 0.01
ξ = 0.05
ξ = 0.10
Magnitude k X(ρ)/F(ρ)
100
10
1
0.1
0.1
0.2
0.5
1
2
5
Frequency Ratio, $\rho = \frac{\omega}{\omega_n}$

Figure 1.



TRANSFER FUNCTION FOR SINGLE-DEGREE-OF-FREEDOM SYSTEM
SUBJECTED TO HARMONIC FORCE EXCITATION
π
π/2
0
0.1
0.2
0.5
1
2
5
ξ = 0.01
ξ = 0.05
ξ = 0.10
Phase Angle φ (radians)
Frequency Ratio, ρ = ω/ω_n

Figure 2.

APPENDIX A

The transmitted force F_t through the spring and damper is

$$|F_t| = \sqrt{(kX)^2 + (c\omega X)^2} \tag{A-1}$$

$$|F_t| = X\sqrt{(k)^2 + (c\omega)^2} \tag{A-2}$$

Recall

$$(c/m) = 2\xi\omega_n \tag{A-3}$$

$$(k/m) = \omega_n{}^2 \tag{A-4}$$

$$|F_t| = X\sqrt{\left(\omega_n{}^2 m\right)^2 + (2\xi\omega_n\omega m)^2} \tag{A-5}$$

$$|F_t| = X\omega_n m\sqrt{\omega_n{}^2 + (2\xi\omega)^2} \tag{A-6}$$

Recall

$$\frac{kX(\omega)}{F(\omega)} = \left[\frac{\omega_n{}^2}{\omega_n{}^2 - \omega^2 + j\omega(2\xi\omega_n)}\right] \tag{A-7}$$

$$X(\omega) = \left[\frac{F}{k}\right]\left[\frac{\omega_n{}^2}{\omega_n{}^2 - \omega^2 + j\omega(2\xi\omega_n)}\right] \tag{A-8}$$

$$X(\omega) = \left[\frac{F}{k}\right]\left[\frac{\omega_n{}^2}{\omega_n{}^2 - \omega^2 + j\omega(2\xi\omega_n)}\right] \tag{A-9}$$

$$X(\omega) = \left[\frac{F}{k}\right]\left[\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\left(\frac{\omega}{\omega_n}\right)}\right] \qquad \text{(A-10)}$$

The displacement magnitude is

$$\left|X(\omega)\right| = \left[\frac{F}{k}\right]\left[\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}\right] \qquad \text{(A-11)}$$

By substitution,

$$\left|F_t\right| = \omega_n{}^2\, m\left[\frac{F}{k}\right]\left[\frac{\sqrt{1+\left(2\xi\left(\frac{\omega}{\omega_n}\right)\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}\right] \qquad \text{(A-12)}$$

$$\left|\frac{F_t}{F}\right| = \frac{\sqrt{1+\left(2\xi\left(\frac{\omega}{\omega_n}\right)\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \qquad \text{(A-13)}$$

The magnitude is thus

$$\left|\frac{F_t}{F}\right| = \frac{\sqrt{1+(2\xi\rho)^2}}{\sqrt{\left[1-\rho^2\right]^2+[2\xi\rho]^2}} \quad , \qquad \rho = \frac{\omega}{\omega_n} \tag{A-14}$$