SINE AND RANDOM VIBRATION EQUIVALENCE
Revision B

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April 5, 2012

Introduction

Avionics components are subjected to vibration qualification testing prior to flight. The purpose of this testing is to verify the integrity of the component design. A qualification test level must meet two criteria:

1. It must envelop the flight vibration environment plus some safety factor.
2. It must envelop the minimum requirement of the appropriate military standard.

The primary vibration environment of concern is *random vibration.* A second is *sine vibration*. These two forms of vibration are inherently different, as evidenced by a comparison of their respective histograms, or probability density functions.

Nevertheless, engineers must occasionally make a comparison between the two forms of vibration. There are a number of scenarios in which this need arises. Here are two examples:

1. A component has been qualified to a sine vibration specification. It must withstand random vibration in flight, however. Does the component need re-qualification?
2. A component has been qualified to a random vibration specification for flight on a certain vehicle. Plans are made to fly the component on a new vehicle which has a sinusoidal motor resonance. Does the component need re-qualification?

A variety of Vibration Equivalence techniques for comparing vibration environments are given in Reference 1.

The purpose of this report is to outline a comparison method, which is similar to the Magnitude Equivalence methods given in Reference 1.

The method in this report is based on histograms of the vibration response spectra. An example from an actual case history is considered.

RESPONSE SPECTRA

Model

The response spectra method models the avionics component as a single-degree-of-freedom (SDOF) system subjected to base excitation. The natural frequency is allowed to vary as an independent variable. The damping is fixed at a constant value, typically at 5% which is equivalent to an amplification factor of Q=10.

A thorough explanation of vibration response spectra is given in Reference 2. A diagram of the response spectrum model is given in Figure 1.

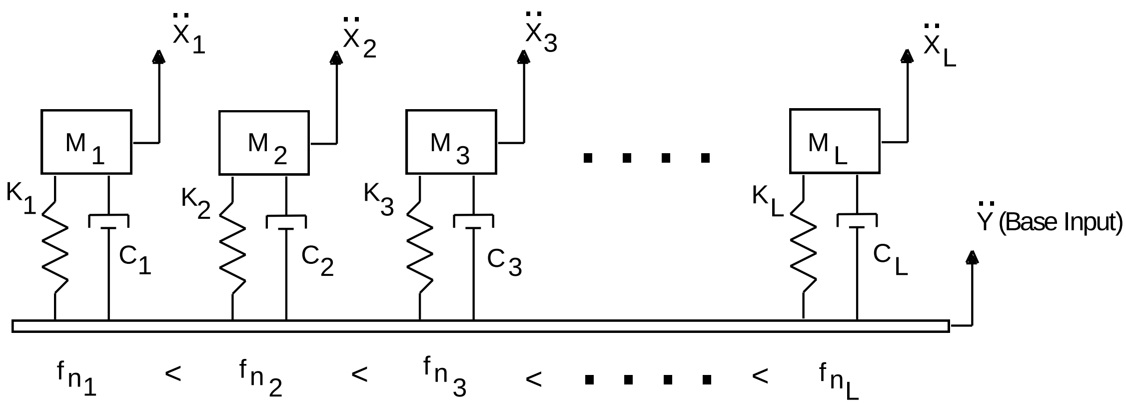

Figure 1. Response Spectrum Model

$\ddot{Y}$ is the common base input for each system, and $\ddot{X}_i$ is the absolue response of each system to the input. The double-dot denotes acceleration. M_i is the mass, C_i is the damping coefficient, and K_i is the stiffness for each system. f_{n_i} is the natural frequency for each system.

Steady-State Random Vibration

From Reference 2, the overall response $\ddot{x}_{GRMS}$ to base input random vibration is given by

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\sum_{i=1}^{N} \frac{\left\{1+(2\xi\rho_i)^2\right\}}{\left\{\left[1-\rho_i^{\,2}\right]^2+\left[2\xi\rho_i\right]^2\right\}} \hat{Y}_{APSD}(f_i)\Delta f_i} \quad , \rho_i = f_i / f_n \tag{1}$$

where

ξ is the damping ratio

f_i is the driving frequency

f_n is the natural frequency

$\hat{Y}_{APSD}(f_i)$ is the base input power spectral density as a function of frequency.

The GRMS in $\ddot{x}_{GRMS}$ stands for G root-mean-square.

Also note that Q = 1/(2ξ).

Steady-State Sine Vibration

From Reference 3, the steady-state response to base input sine vibration is

$$\left|\ddot{x}(f_n,\xi)\right| = \left|\ddot{Y}\right| \sqrt{\frac{1+(2\xi\rho_i)^2}{(1-\rho_i^{\,2})^2+(2\xi\rho_i)^2}} \quad , \quad \rho_i = f_i / f_n \tag{2}$$

where

$\left|\ddot{Y}\right|$ is the peak magnitude of the sine input

and

$\left|\ddot{x}(f_n,\xi)\right|$ is the peak response magnitude

Transient Response to an Abitrary Base Input

An equation for the response of a system to an arbitrary base input is given in Appendix A.

EXAMPLE

Specifications

An avionics component has been qualified for a certain vehicle to the level shown in Table 1.

Table 1. Random Vibration PSD, 15.4 GRMS overall, 225 second duration	
Frequency (Hz)	Accel (G^2/Hz)
20	0.021
72	0.218
660	0.218
2000	0.0279

The component is being used on a new vehicle which has a motor resonance. It must thus be qualified to the level shown in Table 2.

Table 2. Sine Vibration
1.8 Gpeak from 50 to 80 Hz, swept at 6 Hz/min over 5 minute duration

Was the random vibration qualification test sufficient to cover the sine vibration specification?

Vibration Response Spectra

The first step is to calculate response spectra of the random vibration input, as shown in Figure 2. This calculation is performed via Equation (1).

Two curves for the random input are shown. Note that s represents standard deviation.

The GRMS value is equal to 1σ assuming zero mean. A value of 3σ is often taken as the peak value for a variable with a normal distribution. Again, refer to References 1 and 3 for further statistical theory.

The second step is to calculate the response spectra of the sine input. The sine is swept, so the response at each of the boundary frequencies can be calculated as a first approach.

The sine response spectra are shown in Figure 2. This calculation is performed via equation (2). The calculation assumes that the sweep rate is slow enough that a steady-state formula may be used.

The comparison of response spectra in Figure 2 shows that the random GRMS response is approximately equal to the sine peak response at a natural frequency of 80 Hz. Again, a pure comparison of random and sine vibration can never be made.[1] Nevertheless, the random qualification level can be considered to nearly cover the sine requirement at 80 Hz, but note that duration must still be accounted for.

At a natural frequency of 50 Hz, however, the sine peak response is nearly at the midpoint of the random 3σ response and the random GRMS response. The situation at 50 Hz is thus ambiguous. Further analysis is required. A histogram is an appropriate tool.

<u>Response Histograms</u>

Random Vibration Histogram

The random vibration response is 9.23 GRMS at 50 Hz, as shown in Figure 2. Thus, the standard deviation σ is 9.23 G, since the mean is zero. Again, this result was obtained by equation (1).

As an exercise, a sample base input time history is synthesized to meet the specification in Table 1. The time history is not unique since the power spectral density specification discards phase angle.

The power spectral density function of the synthesized time history is shown in Figure 3.

A segment of the sample time history is shown in Figure 4. It is sampled at 20,000 samples per second. The response has a narrowband random character.

The sample time history was applied as a base input to a single-degree-freedom system with a natural frequency of 50 Hz and an amplification factor Q=10. The response calculation was made in the time domain using the digital recursive filtering relationship in Appendix A.

A segment of the time history response is shown in Figure 5. The response data is decimated to a sample rate of 4000 samples per second.

The response histogram is plotted in Figure 6, for a sampling rate of 4000 samples per second and with Δx equal to 1 G for each band. Again, the response histogram is calculated from the complete response time history represented by the segment in Figure 5.

The response time history histogram could have also been calculated directly from the theoretical equations in Appendix B, although further justification would be desired.

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[1] Again, the respective probability density functions differ. A sine wave reaches its positive and negative extremes once per cycle. A random signal exceeds its $\pm 3\sigma$ limits 0.3% of the time.

Sine Sweep Histogram

Next, a base input sine sweep time history was generated to meet the specification in Table 2. The sine sweep was calculated using the algorithm in Reference 4. It is essentially unique.

A segment of the sine sweep time history is shown in Figure 7.

This sweep was applied as a base input to a single-degree-freedom system with a natural frequency of 50 Hz and an amplification factor Q=10. The response calculation was made in the time domain using the digital recursive filtering relationship in Appendix A.

A segment of the response is also shown in Figure 7.

A GRMS time history of the response is given in Figure 8.

The histogram of the response to the sine sweep input is shown in Figure 6, along with the corresponding histogram for the random input.

Response Histogram Comparison

Note that the histograms in Figure 5 fully account for duration.

The comparison in Figure 4 shows that the random vibration yields a greater number of response occurrences outside of the ±4 G interval than the sine vibration yields. Recall that this is for a natural frequency of 50 Hz and a damping value equivalent to Q=10. The random vibration is thus more severe than the sine vibration assuming that the ±4 G limits can be treated as endurance limits, within which no fatigue occurs.

On this basis, the random vibration qualification level envelops the sine vibration level.[2] The avionics component can thus be flown on the new vehicle without further qualification.

Conclusion

The response spectrum method is useful for comparing sine and random vibration. Histograms of the respective response functions are taken to complete the comparison.

As a postscript, consider the probability density functions of the random base input and the sine sweep base input, as shown in Figures 9 and 10, respectively. The random base input function has a bell shape. The sine sweep base input function has a bathtub shape. This qualitative difference hinders the comparison sine and random levels in their base input form. Hence, the need for the vibration response spectrum method.

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[2] The histogram analysis should be repeated at 80 Hz and perhaps at an intermediate frequency for rigor.

References

1. W. Fackler, Equivalence Techniques for Vibration Testing, SVM-9, The Shock and Vibration Information Center, Naval Research Laboratory, United States Department of Defense, Washington D.C., 1972.
2. T. Irvine, An Introduction to Vibration Response Spectrum, Vibrationdata, 2000.
3. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.
4. T. Irvine, Sine Sweep Frequency Parameters, Vibrationdata, 2000.
5. T. Irvine, An Introduction to the Shock Response Spectrum, Vibrationdata, 2000.

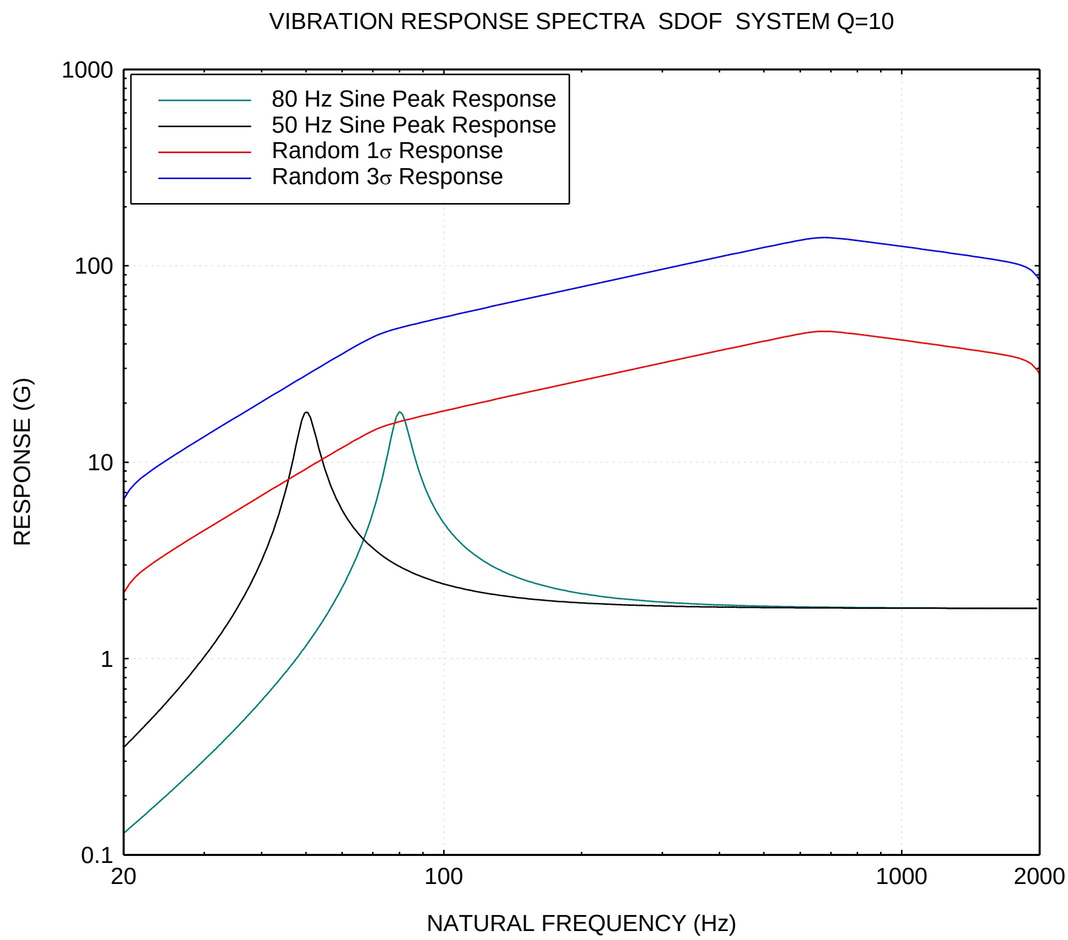

Figure 2.

The 50 Hz sine curve represents a base excitation frequency at 50 Hz.

Likewise, the 80 Hz sine curve represents a base excitation frequency at 80 Hz.

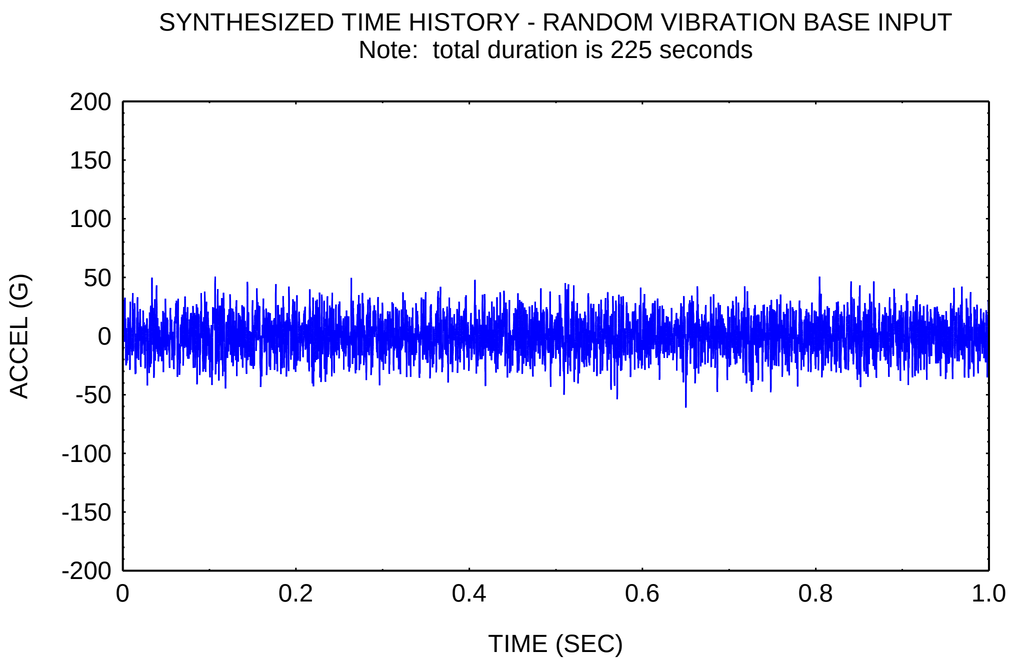

Figure 3.

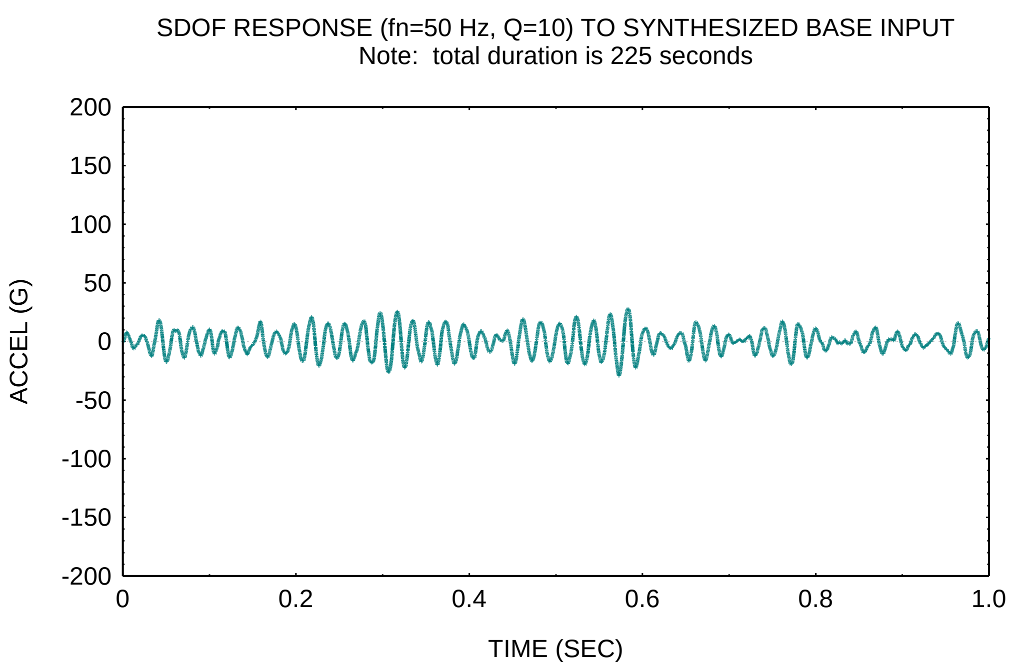

Figure 4.

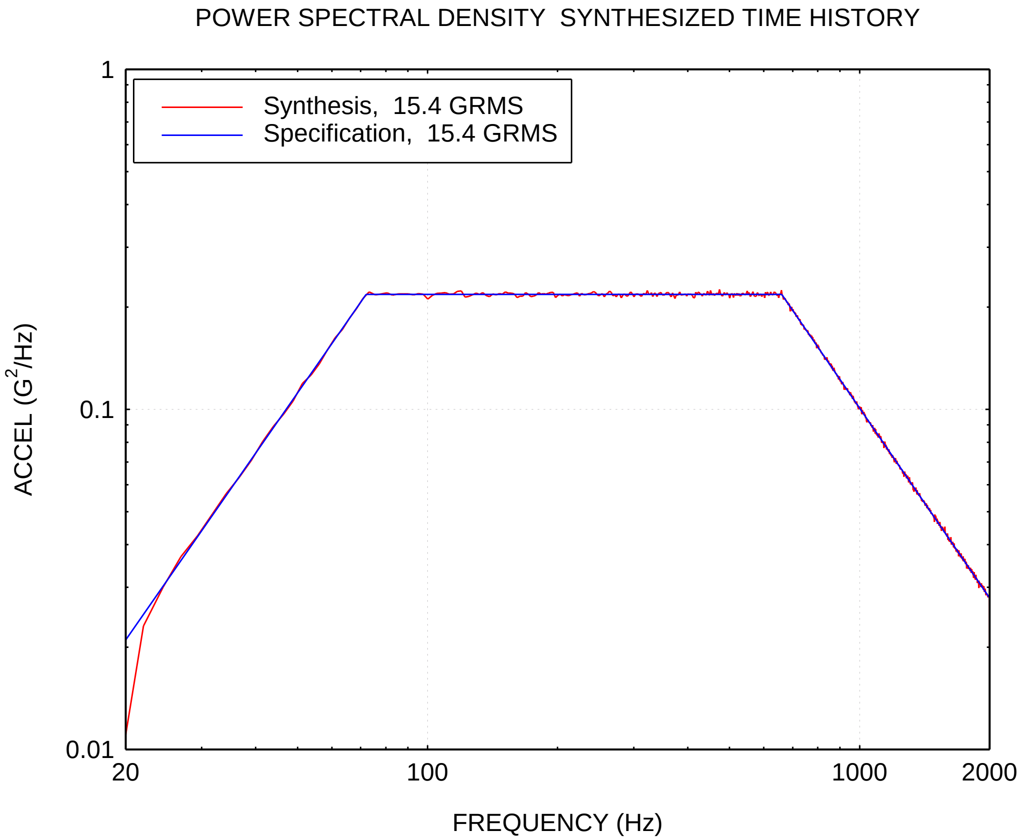
POWER SPECTRAL DENSITY SYNTHESIZED TIME HISTORY
Synthesis, 15.4 GRMS
Specification, 15.4 GRMS
ACCEL (G²/Hz)
1
0.1
0.01
20
100
1000
2000
FREQUENCY (Hz)

Figure 5.

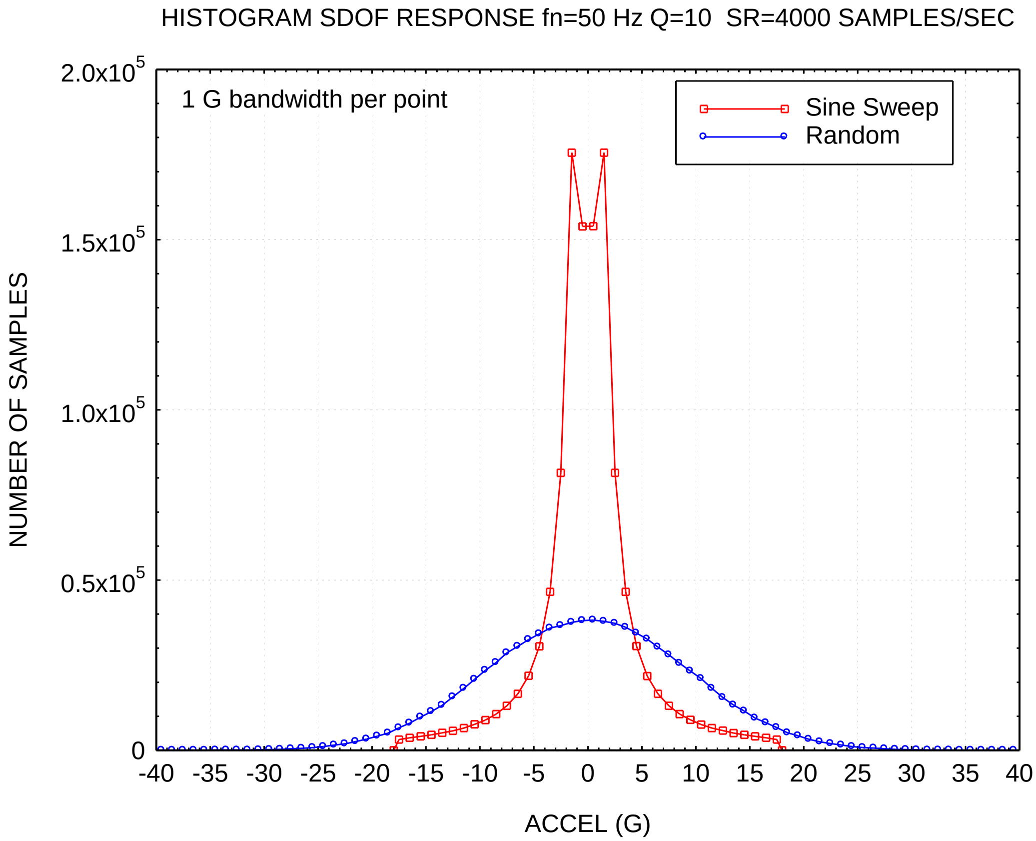
HISTOGRAM SDOF RESPONSE fn=50 Hz Q=10 SR=4000 SAMPLES/SEC
1 G bandwidth per point
Sine Sweep
Random
NUMBER OF SAMPLES
2.0x10^5
1.5x10^5
1.0x10^5
0.5x10^5
0
-40
-35
-30
-25
-20
-15
-10
-5
0
5
10
15
20
25
30
35
40
ACCEL (G)

Figure 6.

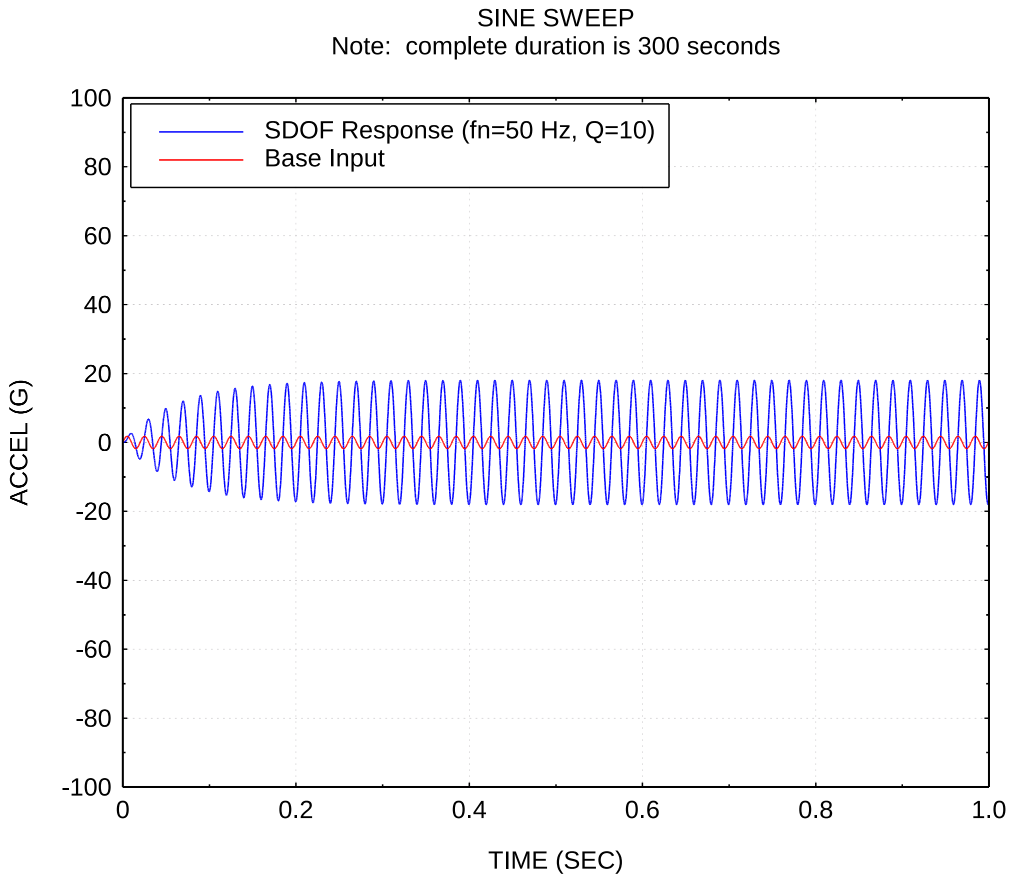
SINE SWEEP
Note: complete duration is 300 seconds
SDOF Response (fn=50 Hz, Q=10)
Base Input
ACCEL (G)
TIME (SEC)
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
0.2
0.4
0.6
0.8
1.0

Figure 7.

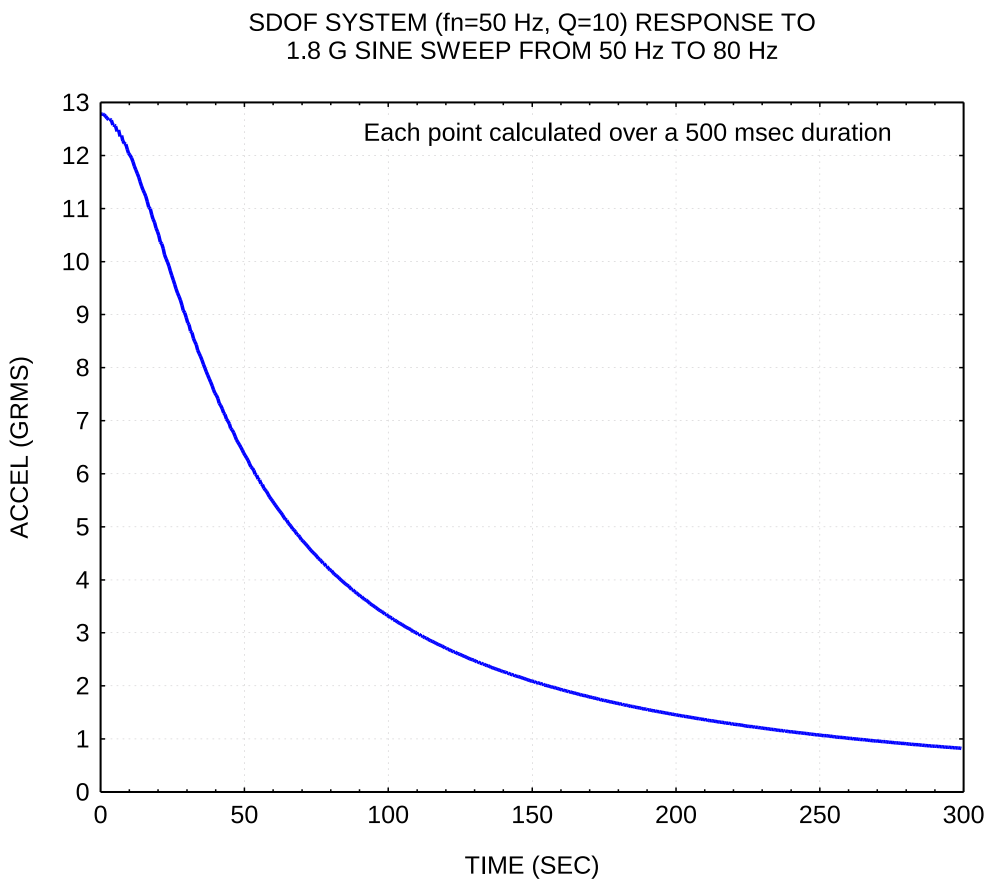
SDOF SYSTEM (fn=50 Hz, Q=10) RESPONSE TO
1.8 G SINE SWEEP FROM 50 Hz TO 80 Hz
Each point calculated over a 500 msec duration
ACCEL (GRMS)
TIME (SEC)
0
1
2
3
4
5
6
7
8
9
10
11
12
13
0
50
100
150
200
250
300

Figure 8.

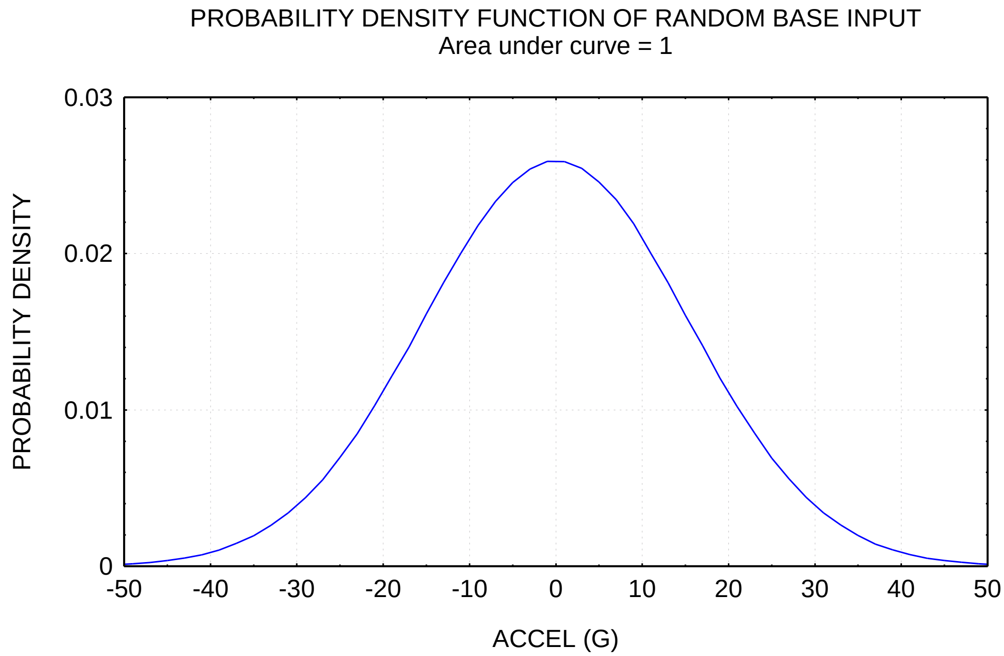

Figure 9.

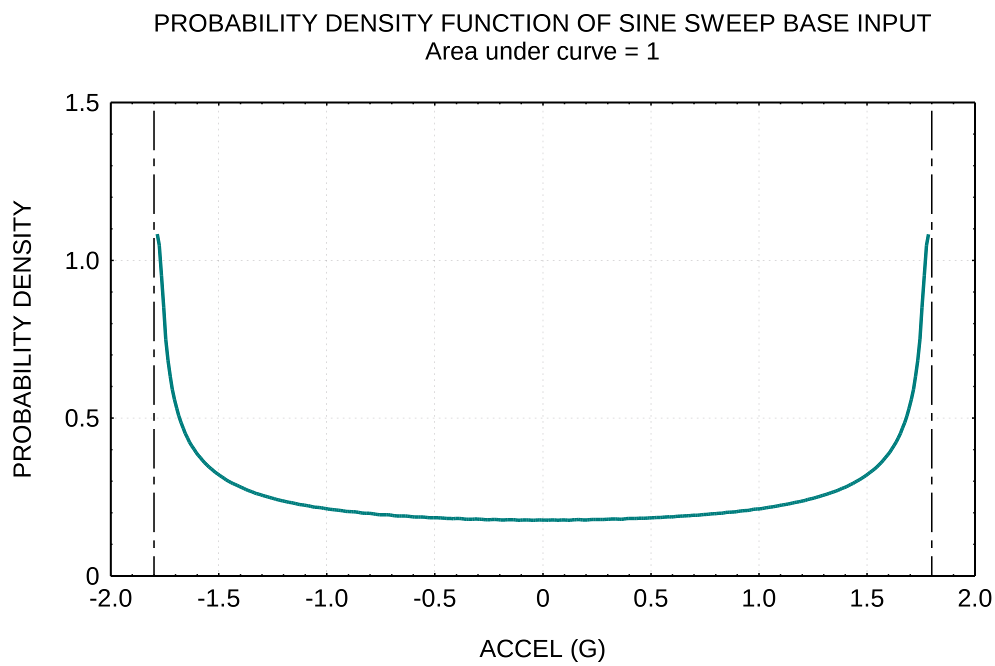

Figure 10.

APPENDIX A

<u>Response to an Arbitrary Base Input</u>

As a review, consider the single-degree-of-freedom system subjected to base excitation shown in Figure A-1. The free-body diagram is shown in Figure A-2.

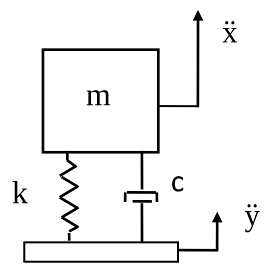

Figure A-1. Single-degree-of-freedom System

The variables are

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
x	=	absolute displacement of the mass
y	=	base input displacement

The double-dot notation indicates acceleration

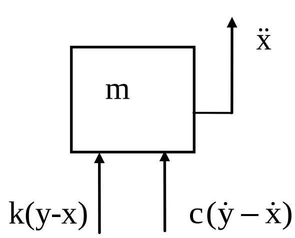

Figure A-2. Free-body Diagram

The equation of motion is

$$m\ddot{x}=c(\dot{y}-\dot{x})+k(y-x) \tag{A-1}$$

Define a relative displacement z. Let z = x -y.

The following equation of motion for the relative displacement z is derived in References 2 and 5.

$$\ddot{z}+2\xi\omega_n\dot{z}+\omega_n{}^2 z=-\ddot{y} \tag{A-2}$$

Equation (A-2) cannot be solved exactly for the case where the base input is an arbitrary pulse. A convolution integral approach is needed as described in Reference 5.

After many steps, the response can be represented in terms of a digital recursive filtering relationship, as derived in Reference 5.

$$\begin{aligned}
\ddot{x}_i &= \\
&+2\exp[-\xi\omega_n\Delta t]\ddot{x}_{i-1} \\
&-\exp[-2\xi\omega_n\Delta t]\ddot{x}_{i-2} \\
&+2\xi\omega_n\Delta t\,\ddot{y}_i \\
&+\omega_n\Delta t\exp[-\xi\omega_n\Delta t]\left\{\left[\frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\right]\sin[\omega_d\Delta t]-2\xi\cos[\omega_d\Delta t]\right\}\ddot{y}_{i-1}
\end{aligned}$$

(A-3)

where

ξ	damping ratio
ω_d =	$\omega_n\sqrt{1-\xi^2}$
Δt =	time step
$\ddot{x}_i$	is the response at time t
$\ddot{x}_{i-1}$	is the response at time t-Δt
$\ddot{x}_{i-2}$	is the response at time t-2Δt
$\ddot{y}_i$	is the base input at time t
$\ddot{y}_{i-1}$	is the base input at time t-Δt

APPENDIX B

Normal Distribution Theory

The equation for the probability density p(x) for a normal distribution with zero mean is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-x^2}{2\sigma^2}\right] \tag{B-1}$$

where x is the amplitude.

This corresponding histogram equation for the number of occurrences $N(x_i)$ in each band is

$$N(x_i) = \left[\frac{T}{\Delta t}\right]\left[\frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-x^2}{2\sigma^2}\right]\right] \Delta x_i \tag{B-2}$$

where

- T is the period
- Δt is the sample duration which is the inverse of the sample rate
- x_i is a discrete amplitude value serves as the center amplitude for each Δx_i bandwidth

Also note that a probability density function can be represented either as a discrete or as a continuous function but that a histogram is always discrete.