VIBRATION ANALYSIS OF A MASS WITH SIX ISOLATORS
Revision B

By Tom Irvine
Email: tomirvine@aol.com

October 24, 2011

Introduction

An avionics component may be mounted with isolator grommets, which act as soft springs. The goal of the isolator design is to provide attenuation of shock and vibration energy. This is achieved by lowering the natural frequency of the component system.

Consider a component with a complex geometry that is to be mounted via six isolators, as shown in Figures 1 and 2. Assume that the component's hardmounted natural frequency is at least one octave greater than any of its isolation frequencies.

The objective is to derive the equations of motion for this system, accounting for six degrees-of-freedom.

Derivation

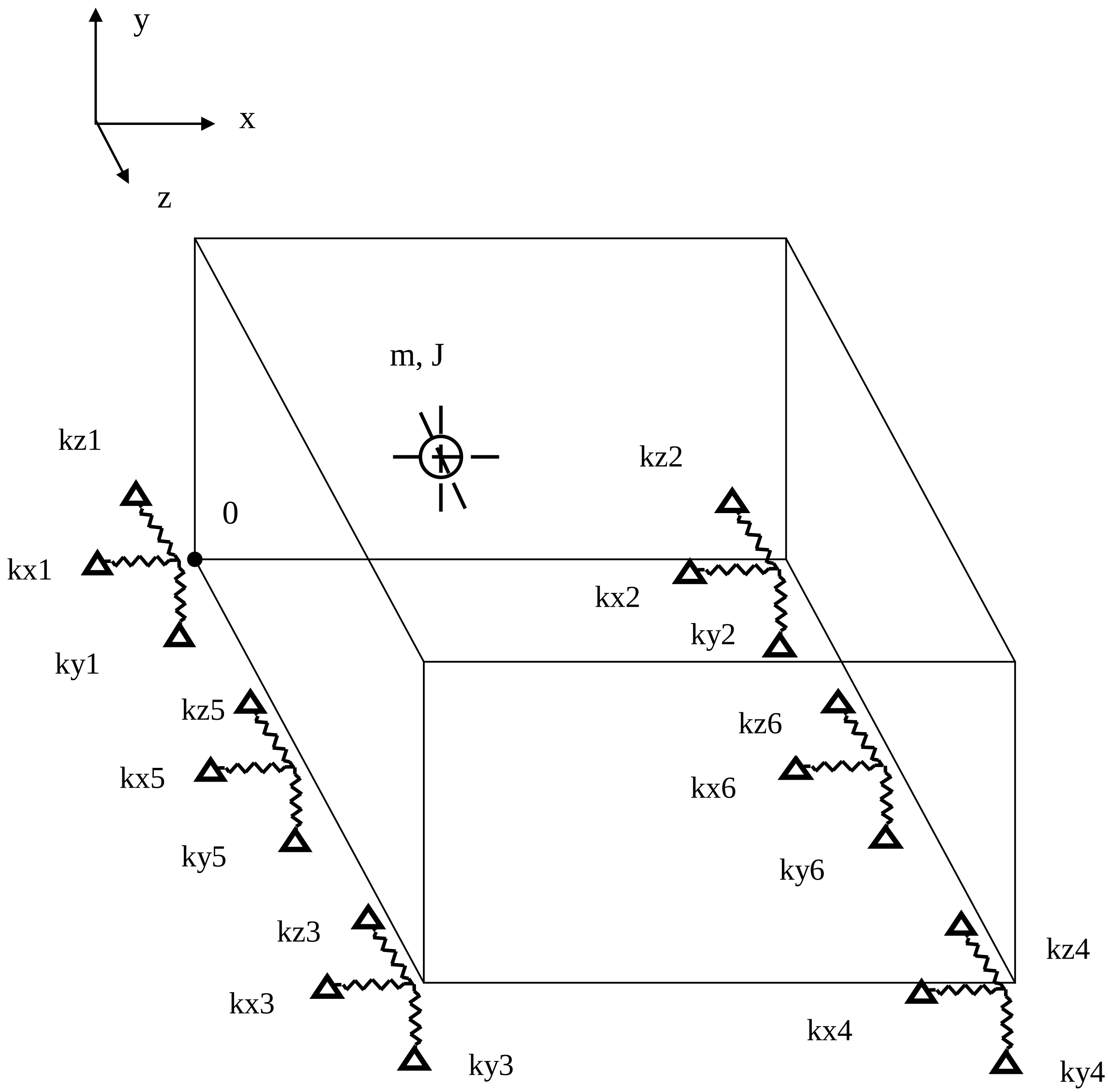

Figure 1. Isolated Avionics Component Model

The mass and inertia are represented at a point with the circle symbol. Each isolator is modeled by three orthogonal DOF springs. The springs are mounted at each corner. The springs are shown with an offset from the corners for clarity. The triangles indicate fixed constraints. “0” indicates the origin.

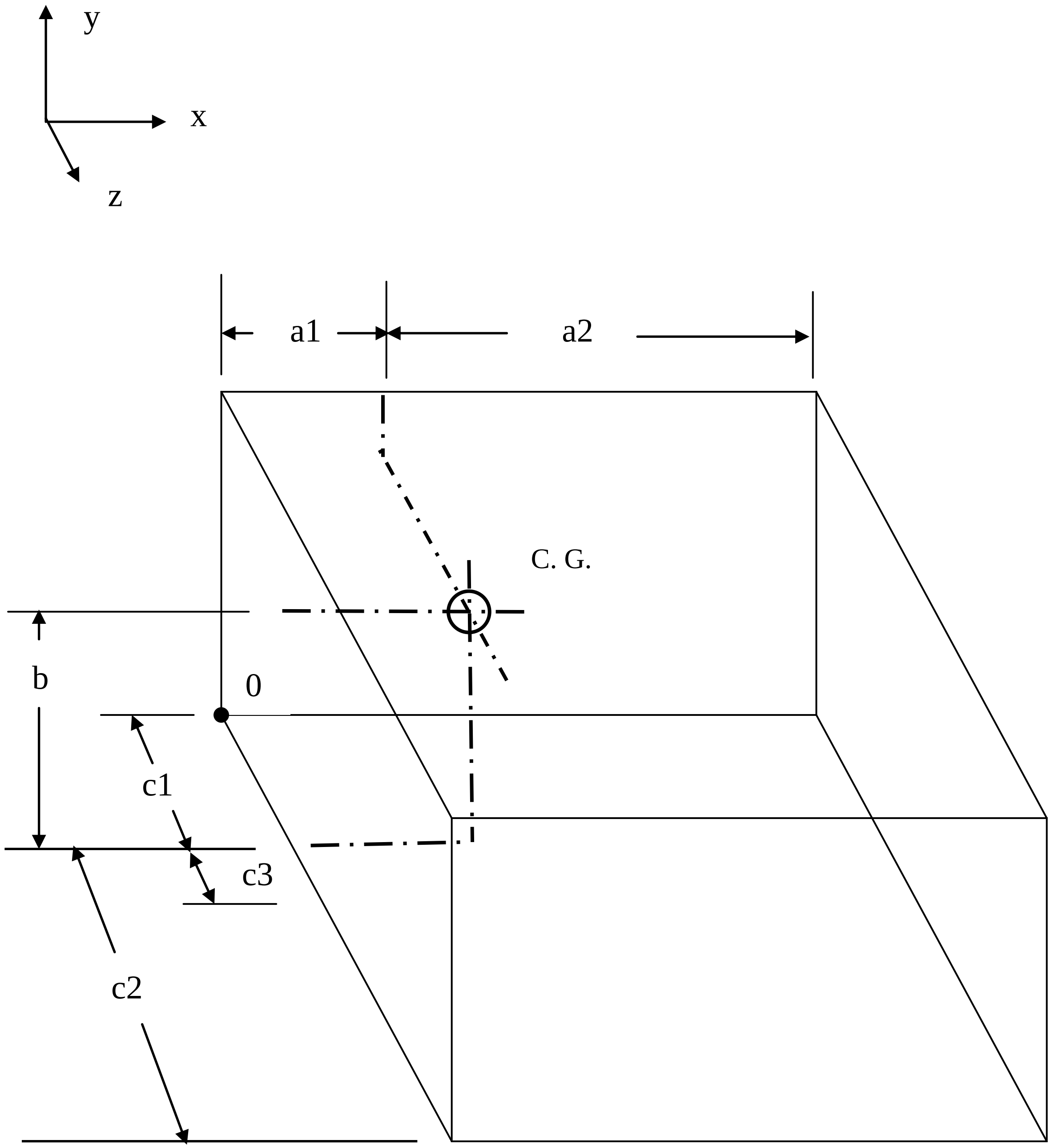

Figure 2. Isolated Avionics Component Model with Dimensions

All dimensions are positive as long as the C.G. is “inside the box.” At least one dimension will be negative otherwise.

The variables $\alpha, \beta, \text{and } \theta$ represent rotations about the X, Y, and Z axes, respectively, using the right-hand rule convention.

c3 is the distance from the C.G. to the intermediate springs.

The total kinetic energy is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}J_x\,\dot{\alpha}^2 + \frac{1}{2}J_y\,\dot{\beta}^2 + \frac{1}{2}J_z\,\dot{\theta}^2 \tag{1}$$

The total potential energy is

$$
\begin{aligned}
V = \\
&+\frac{1}{2}k_{x1}(x - c_1\beta + b\theta)^2 + \frac{1}{2}k_{x2}(x - c_1\beta + b\theta)^2 \\
&+\frac{1}{2}k_{x3}(x + c_2\beta + b\theta)^2 + \frac{1}{2}k_{x4}(x + c_2\beta + b\theta)^2 \\
&+\frac{1}{2}k_{x5}(x + c_3\beta + b\theta)^2 + \frac{1}{2}k_{x6}(x + c_3\beta + b\theta)^2 \\
&+\frac{1}{2}k_{y1}(y + c_1\alpha - a_1\theta)^2 + \frac{1}{2}k_{y2}(y + c_1\alpha + a_2\theta)^2 \\
&+\frac{1}{2}k_{y3}(y - c_2\alpha - a_1\theta)^2 + \frac{1}{2}k_{y4}(y - c_2\alpha + a_2\theta)^2 \\
&+\frac{1}{2}k_{y5}(y - c_3\alpha - a_1\theta)^2 + \frac{1}{2}k_{y6}(y - c_3\alpha + a_2\theta)^2 \\
&+\frac{1}{2}k_{z1}(z + a_1\beta - b\alpha)^2 + \frac{1}{2}k_{z2}(z - a_2\beta - b\alpha)^2 \\
&+\frac{1}{2}k_{z3}(z + a_1\beta - b\alpha)^2 + \frac{1}{2}k_{z4}(z - a_2\beta - b\alpha)^2 \\
&+\frac{1}{2}k_{z5}(z + a_1\beta - b\alpha)^2 + \frac{1}{2}k_{z6}(z - a_2\beta - b\alpha)^2
\end{aligned}
\tag{2}
$$

The energy is

$$
\begin{aligned}
E = &\\
&+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\dot{\alpha}^2+\frac{1}{2}J_y\dot{\beta}^2+\frac{1}{2}J_z\dot{\theta}^2\\
&+\frac{1}{2}k_{x1}(x-c_1\beta+b\theta)^2+\frac{1}{2}k_{x2}(x-c_1\beta+b\theta)^2\\
&+\frac{1}{2}k_{x3}(x+c_2\beta+b\theta)^2+\frac{1}{2}k_{x4}(x+c_2\beta+b\theta)^2\\
&+\frac{1}{2}k_{x5}(x+c_3\beta+b\theta)^2+\frac{1}{2}k_{x6}(x+c_3\beta+b\theta)^2\\
&+\frac{1}{2}k_{y1}(y+c_1\alpha-a_1\theta)^2+\frac{1}{2}k_{y2}(y+c_1\alpha+a_2\theta)^2\\
&+\frac{1}{2}k_{y3}(y-c_2\alpha-a_1\theta)^2+\frac{1}{2}k_{y4}(y-c_2\alpha+a_2\theta)^2\\
&+\frac{1}{2}k_{y5}(y-c_3\alpha-a_1\theta)^2+\frac{1}{2}k_{y6}(y-c_3\alpha+a_2\theta)^2\\
&+\frac{1}{2}k_{z1}(z+a_1\beta-b\alpha)^2+\frac{1}{2}k_{z2}(z-a_2\beta-b\alpha)^2\\
&+\frac{1}{2}k_{z3}(z+a_1\beta-b\alpha)^2+\frac{1}{2}k_{z4}(z-a_2\beta-b\alpha)^2\\
&+\frac{1}{2}k_{z5}(z+a_1\beta-b\alpha)^2+\frac{1}{2}k_{z6}(z-a_2\beta-b\alpha)^2
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
E = & \\
& + \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2} J_x \dot{\alpha}^2 + \frac{1}{2} J_y \dot{\beta}^2 + \frac{1}{2} J_z \dot{\theta}^2 \\
& + \frac{1}{2} k_{x1}(x - c_1\beta + b\theta)(x - c_1\beta + b\theta) + \frac{1}{2} k_{x2}(x - c_1\beta + b\theta)(x - c_1\beta + b\theta) \\
& + \frac{1}{2} k_{x3}(x + c_2\beta + b\theta)(x + c_2\beta + b\theta) + \frac{1}{2} k_{x4}(x + c_2\beta + b\theta)(x + c_2\beta + b\theta) \\
& + \frac{1}{2} k_{x5}(x + c_3\beta + b\theta)(x + c_3\beta + b\theta) + \frac{1}{2} k_{x6}(x + c_3\beta + b\theta)(x + c_3\beta + b\theta) \\
& + \frac{1}{2} k_{y1}(y + c_1\alpha - a_1\theta)(y + c_1\alpha - a_1\theta) + \frac{1}{2} k_{y2}(y + c_1\alpha + a_2\theta)(y + c_1\alpha + a_2\theta) \\
& + \frac{1}{2} k_{y3}(y - c_2\alpha - a_1\theta)(y - c_2\alpha - a_1\theta) + \frac{1}{2} k_{y4}(y - c_2\alpha + a_2\theta)(y - c_2\alpha + a_2\theta) \\
& + \frac{1}{2} k_{y5}(y - c_3\alpha - a_1\theta)(y - c_3\alpha - a_1\theta) + \frac{1}{2} k_{y6}(y - c_3\alpha + a_2\theta)(y - c_3\alpha + a_2\theta) \\
& + \frac{1}{2} k_{z1}(z + a_1\beta - b\alpha)(z + a_1\beta - b\alpha) + \frac{1}{2} k_{z2}(z - a_2\beta - b\alpha)(z - a_2\beta - b\alpha) \\
& + \frac{1}{2} k_{z3}(z + a_1\beta - b\alpha)(z + a_1\beta - b\alpha) + \frac{1}{2} k_{z4}(z - a_2\beta - b\alpha)(z - a_2\beta - b\alpha) \\
& + \frac{1}{2} k_{z5}(z + a_1\beta - b\alpha)(z + a_1\beta - b\alpha) + \frac{1}{2} k_{z6}(z - a_2\beta - b\alpha)(z - a_2\beta - b\alpha)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
E = &+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2\\
&+\frac{1}{2}k_{x1}\left(x\left(x-c_1\beta+b\theta\right)-c_1\beta\left(x-c_1\beta+b\theta\right)+b\theta\left(x-c_1\beta+b\theta\right)\right)\\
&+\frac{1}{2}k_{x2}\left(x\left(x-c_1\beta+b\theta\right)-c_1\beta\left(x-c_1\beta+b\theta\right)+b\theta\left(x-c_1\beta+b\theta\right)\right)\\
&+\frac{1}{2}k_{x3}\left(x\left(x+c_2\beta+b\theta\right)+c_2\beta\left(x+c_2\beta+b\theta\right)+b\theta\left(x+c_2\beta+b\theta\right)\right)\\
&+\frac{1}{2}k_{x4}\left(x\left(x+c_2\beta+b\theta\right)+c_2\beta\left(x+c_2\beta+b\theta\right)+b\theta\left(x+c_2\beta+b\theta\right)\right)\\
&+\frac{1}{2}k_{x5}\left(x\left(x+c_3\beta+b\theta\right)+c_3\beta\left(x+c_3\beta+b\theta\right)+b\theta\left(x+c_3\beta+b\theta\right)\right)\\
&+\frac{1}{2}k_{x6}\left(x\left(x+c_3\beta+b\theta\right)+c_2\beta\left(x+c_3\beta+b\theta\right)+b\theta\left(x+c3\beta+b\theta\right)\right)\\
&+\frac{1}{2}k_{y1}\left(y\left(y+c_1\alpha-a_1\theta\right)+c_1\alpha\left(y+c_1\alpha-a_1\theta\right)-a_1\theta\left(y+c_1\alpha-a_1\theta\right)\right)\\
&+\frac{1}{2}k_{y2}\left(y\left(y+c_1\alpha+a_2\theta\right)+c_1\alpha\left(y+c_1\alpha+a_2\theta\right)+a_2\theta\left(y+c_1\alpha+a_2\theta\right)\right)\\
&+\frac{1}{2}k_{y3}\left(y\left(y-c_2\alpha-a_1\theta\right)-c_2\alpha\left(y-c_2\alpha-a_1\theta\right)-a_1\theta\left(y-c_2\alpha-a_1\theta\right)\right)\\
&+\frac{1}{2}k_{y4}\left(y\left(y-c_2\alpha+a_2\theta\right)-c_2\alpha\left(y-c_2\alpha+a_2\theta\right)+a_2\theta\left(y-c_2\alpha+a_2\theta\right)\right)\\
&+\frac{1}{2}k_{y5}\left(y\left(y-c_3\alpha-a_1\theta\right)-c_3\alpha\left(y-c_3\alpha-a_1\theta\right)-a_1\theta\left(y-c_3\alpha-a_1\theta\right)\right)\\
&+\frac{1}{2}k_{y6}\left(y\left(y-c_3\alpha+a_2\theta\right)-c_3\alpha\left(y-c_3\alpha+a_2\theta\right)+a_2\theta\left(y-c_3\alpha+a_2\theta\right)\right)\\
&+\frac{1}{2}k_{z1}\left(z\left(z+a_1\beta-b\alpha\right)+a_1\beta\left(z+a_1\beta-b\alpha\right)-b\alpha\left(z+a_1\beta-b\alpha\right)\right)\\
&+\frac{1}{2}k_{z2}\left(z\left(z-a_2\beta-b\alpha\right)-a_2\beta\left(z-a_2\beta-b\alpha\right)-b\alpha\left(z-a_2\beta-b\alpha\right)\right)\\
&+\frac{1}{2}k_{z3}\left(z\left(z+a_1\beta-b\alpha\right)+a_1\beta\left(z+a_1\beta-b\alpha\right)-b\alpha\left(z+a_1\beta-b\alpha\right)\right)\\
&+\frac{1}{2}k_{z4}\left(z\left(z-a_2\beta-b\alpha\right)-a_2\beta\left(z-a_2\beta-b\alpha\right)-b\alpha\left(z-a_2\beta-b\alpha\right)\right)\\
&+\frac{1}{2}k_{z5}\left(z\left(z+a_1\beta-b\alpha\right)+a_2\beta\left(z+a_1\beta-b\alpha\right)-b\alpha\left(z+a_1\beta-b\alpha\right)\right)\\
&+\frac{1}{2}k_{z6}\left(z\left(z-a_2\beta-b\alpha\right)-a_2\beta\left(z-a_2\beta-b\alpha\right)-b\alpha\left(z-a_2\beta-b\alpha\right)\right)
\end{aligned}
\qquad (5)
$$

$$
\begin{aligned}
E = &+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2\\
&+\frac{1}{2}k_{x1}\left(\left(x^2-c_1\beta x+b\theta x\right)+\left(-c_1x\beta+c_1{}^2\beta^2-c_1b\theta\beta\right)+\left(bx\theta-bc_1\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x2}\left(\left(x^2-c_1\beta x+b\theta x\right)+\left(-c_1x\beta+c_1{}^2\beta^2-c_1b\theta\beta\right)+\left(bx\theta-bc_1\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x3}\left(\left(x^2+c_2\beta x+b\theta x\right)+\left(c_2x\beta+c_2{}^2\beta^2+c_2b\theta\beta\right)+\left(bx\theta+bc_2\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x4}\left(\left(x^2+c_2\beta x+b\theta x\right)+\left(c_2x\beta+c_2{}^2\beta^2+c_2b\theta\beta\right)+\left(bx\theta+bc_2\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x5}\left(\left(x^2+c_3\beta x+b\theta x\right)+\left(c_3\beta x+c_3{}^2\beta^2+c_3b\theta\beta\right)+\left(b\theta x+bc_3\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x6}\left(\left(x^2+c_3\beta x+b\theta x\right)+\left(c_3\beta x+c_3{}^2\beta^2+c_3b\theta\beta\right)+\left(b\theta x+bc_3\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y1}\left(\left(y^2+c_1\alpha y-a_1\theta y\right)+\left(c_1y\alpha+c_1{}^2\alpha^2-c_1a_1\theta\alpha\right)+\left(-a_1y\theta-a_1c_1\alpha\theta+a_1{}^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y2}\left(\left(y^2+c_1\alpha y+a_2\theta y\right)+\left(c_1y\alpha+c_1{}^2\alpha^2+a_2c_1\theta\alpha\right)+\left(a_2y\theta+a_2c_1\alpha\theta+a_2{}^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y3}\left(\left(y^2-c_2\alpha y-a_1\theta y\right)+\left(-c_2y\alpha+c_2{}^2\alpha^2+a_1c_2\theta\alpha\right)+\left(-a_1y\theta+a_1c_2\alpha\theta+a_1{}^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y4}\left(\left(y^2-c_2\alpha y+a_2\theta y\right)+\left(-c_2y\alpha+c_2{}^2\alpha^2-a_2c_2\theta\alpha\right)+\left(a_2y\theta-a_2c_2\alpha\theta+a_2{}^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y5}\left(\left(y^2-c_3\alpha y-a_1\theta y\right)+\left(-c_3\alpha y+c_3{}^2\alpha^2+c_3\alpha a_1\theta\right)+\left(-a_1\theta y+a_1c_3\theta\alpha+a_1{}^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y6}\left(\left(y^2-c_3\alpha y+a_2\theta y\right)+\left(-c_3\alpha y+c_3{}^2\alpha^2-c_3\alpha a_2\theta\right)+\left(-a_2\theta y-a_2\theta c_3\alpha+a_2{}^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{z1}\left(\left(z^2+a_1\beta z-b\alpha z\right)+\left(a_1z\beta+a_1{}^2\beta^2-a_1b\alpha\beta\right)+\left(-bz\alpha-ba_1\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z2}\left(\left(z^2-a_2\beta z-b\alpha z\right)+\left(-a_2z\beta+a_2{}^2\beta^2+a_2b\alpha\beta\right)+\left(-bz\alpha+ba_2\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z3}\left(\left(z^2+a_1\beta z-b\alpha z\right)+\left(a_1z\beta+a_1{}^2\beta^2-a_1b\alpha\beta\right)+\left(-bz\alpha-ba_1\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z4}\left(\left(z^2-a_2\beta z-b\alpha z\right)+\left(-a_2z\beta+a_2{}^2\beta^2+a_2b\alpha\beta\right)+\left(-bz\alpha+ba_2\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z5}\left(\left(z^2+a_1\beta z-b\alpha z\right)+\left(a_1z\beta+a_1{}^2\beta^2-a_1b\alpha\beta\right)+\left(-bz\alpha-ba_1\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z6}\left(\left(z^2-a_2\beta z-b\alpha z\right)+\left(-a_2z\beta+a_2{}^2\beta^2+a_2b\alpha\beta\right)+\left(-bz\alpha+ba_2\beta\alpha+b^2\alpha^2\right)\right)
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
E = &+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2 \\
&+\frac{1}{2}\left(k_{x1}+k_{x2}\right)\left(x^2-2c_1\beta x+2b\theta x+c_1^2\beta^2-2c_1b\theta\beta+b^2\theta^2\right) \\
&+\frac{1}{2}\left(k_{x3}+k_{x4}\right)\left(x^2+2c_2\beta x+2b\theta x+c_2{}^2\beta^2+2c_2b\theta\beta+b^2\theta^2\right) \\
&+\frac{1}{2}\left(k_{x5}+k_{x6}\right)\left(x^2+2c_3\beta x+2b\theta x+c_3{}^2\beta^2+2c_3b\beta\theta+b^2\theta^2\right) \\
&+\frac{1}{2}k_{y1}\left(y^2-2c_1\alpha y-2a_1\theta y+c_1^2\alpha^2+2a_1c_1\theta\alpha+a_1{}^2\theta^2\right) \\
&+\frac{1}{2}k_{y2}\left(y^2-2c_1\alpha y+2a_2\theta y+c_1^2\alpha^2-2a_2c_1\theta\alpha+a_2{}^2\theta^2\right) \\
&+\frac{1}{2}k_{y3}\left(y^2+2c_2y\alpha-2a_1\theta y+c_2{}^2\alpha^2-2a_1c_2\theta\alpha+a_1{}^2\theta^2\right) \\
&+\frac{1}{2}k_{y4}\left(y^2+2c_2\alpha y+2a_2\theta y+c_2{}^2\alpha^2+2a_2c_2\theta\alpha+a_2{}^2\theta^2\right) \\
&+\frac{1}{2}k_{y5}\left(y^2-2c_3\alpha y-2a_1\theta y+c_3^2\alpha^2+2a_1c_3\theta\alpha+a_1{}^2\theta^2\right) \\
&+\frac{1}{2}k_{y6}\left(y^2-2c_3\alpha y+2a_2\theta y+c_3{}^2\alpha^2-2a_2\theta c_3\alpha+a_2{}^2\theta^2\right) \\
&+\frac{1}{2}\left(k_{z1}+k_{z3}+k_{z5}\right)\left(z^2+2a_1\beta z-2b\alpha z+a_1^2\beta^2-2a_1b\alpha\beta+b^2\alpha^2\right) \\
&+\frac{1}{2}\left(k_{z2}+k_{z4}+k_{z6}\right)\left(z^2-2a_2\beta z-2b\alpha z+a_2^2\beta^2+2a_2b\alpha\beta+b^2\alpha^2\right)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
E \;=\; & \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2 \\[1em]
& +\frac{1}{2}(k_{x1}+k_{x2}+k_{x3}+k_{x4}+k_{x5}+k_{x6})x^2 \\
& +(-(k_{x1}+k_{x2})c_1+(k_{x3}+k_{x4})c_2+(k_{x5}+k_{x6})c_3)\beta x \\
& +(k_{x1}+k_{x2}+k_{x3}+k_{x4}+k_{x5}+k_{x6})b\theta x \\
& +\frac{1}{2}\left((k_{x1}+k_{x2})c_1{}^2+(k_{x3}+k_{x4})c_2{}^2+(k_{x5}+k_{x6})c_3{}^2\right)\beta^2 \\
& +(-(k_{x1}+k_{x2})c_1+(k_{x3}+k_{x4})c_2+(k_{x5}+k_{x6})c_3)b\theta\beta \\
& +\frac{1}{2}(k_{x1}+k_{x2}+k_{x3}+k_{x4}+k_{x5}+k_{x6})b^2\theta^2 \\[1em]
& +\frac{1}{2}(k_{y1}+k_{y2}+k_{y3}+k_{y4}+k_{y5}+k_{y6})y^2 \\
& +\frac{1}{2}\left((k_{y1}+k_{y2})c_1{}^2+(k_{y3}+k_{y4})c_2{}^2+(k_{y5}+k_{y6})c_3{}^2\right)\alpha^2 \\
& +\frac{1}{2}\left((k_{y1}+k_{y3}+k_{y5})a_1{}^2+(k_{y2}+k_{y4}+k_{y6})a_2{}^2\right)\theta^2 \\
& +((k_{y1}+k_{y2})c_1-(k_{y3}+k_{y4})c_2-(k_{y5}+k_{y6})c_3)\alpha y \\
& +((-k_{y1}-k_{y3}-k_{y5})a_1+(k_{y2}+k_{y4}+k_{y6})a_2)\theta y \\
& +(-k_{y1}a_1c_1+k_{y2}a_2c_1+k_{y3}a_1c_2-k_{y4}a_2c_2+k_{y5}a_1c_3-k_{y6}a_2\theta c_3\,\alpha)\theta\alpha \\[1em]
& +\frac{1}{2}(k_{z1}+k_{z2}+k_{z3}+k_{z4}+k_{z5}+k_{z6})z^2 \\
& +\frac{1}{2}\left((k_{z1}+k_{z3}+k_{z5})a_1{}^2+(k_{z2}+k_{z4}+k_{z6})a_2{}^2\right)\beta^2 \\
& +\frac{1}{2}(k_{z1}+k_{z2}+k_{z3}+k_{z4}+k_{z5}+k_{z6})b^2\alpha^2 \\
& +((k_{z1}+k_{z3}+k_{z5})\,a_1-(k_{z2}+k_{z4}+k_{z6})\,a_2)\beta z \\
& +(-k_{z1}-k_{z2}-k_{z3}-k_{z4}-k_{z5}-k_{z6})b\alpha z \\
& +(-(k_{z1}+k_{z3}+k_{z5})a_1+(k_{z2}+k_{z4}+k_{z6})a_2)b\alpha\beta
\end{aligned}
\tag{8}
$$

The energy method is based on conservation of energy.

$$\frac{d}{dt} E = 0 \tag{9}$$

Apply the method.

$$+ m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) + J_x \dot{\alpha}\ddot{\alpha} + J_y \dot{\beta}\ddot{\beta} + J_z \dot{\theta}\ddot{\theta}$$

$$
\begin{aligned}
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4} + k_{x5} + k_{x6}) x\dot{x} \\
&+ \left((k_{x1} + k_{x2}) c_1^2 + (k_{x3} + k_{x4}) c_2^{\ 2} + (k_{x5} + k_{x6}) c_3^{\ 2}\right) \beta\dot{\beta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4} + k_{x5} + k_{x6}) b^2 \theta\dot{\theta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4} + k_{x5} + k_{x6})(b)(\theta\dot{x} + \dot{\theta}x) \\
&+ \left(-(k_{x1} + k_{x2}) c_1 + (k_{x3} + k_{x4}) c_2 + (k_{x5} + k_{x6}) c_3\right)(\dot{\beta}x + \beta\dot{x}) \\
&+ \left(-(k_{x1} + k_{x2}) c_1 + (k_{x3} + k_{x4}) c_2 + (k_{x5} + k_{x6}) c_3\right)(b)(\theta\dot{\beta} + \dot{\theta}\beta)
\end{aligned}
$$

$$
\begin{aligned}
&+ (k_{y1} + k_{y2} + k_{y3} + k_{y4} + k_{y5} + k_{y6}) y\dot{y} \\
&+ \left((k_{y1} + k_{y2}) c_1^2 + (k_{y3} + k_{y4}) c_2^{\ 2} + (k_{y5} + k_{y6}) c_3^{\ 2}\right) \alpha\dot{\alpha} \\
&+ \left((k_{y1} + k_{y3} + k_{y5}) a_1^2 + (k_{y2} + k_{y4} + k_{y6}) a_2^{\ 2}\right) \theta\dot{\theta} \\
&+ \left((k_{y1} + k_{y2}) c_1 - (k_{y3} + k_{y4}) c_2 - (k_{y5} + k_{y6}) c_3\right)(\dot{\alpha}y + \alpha\dot{y}) \\
&+ \left((-k_{y1} - k_{y3} - k_{y5}) a_1 + (k_{y2} + k_{y4} + k_{y6}) a_2\right)(\dot{\theta}y + \theta\dot{y}) \\
&+ \left(-k_{y1} a_1 c_1 + k_{y2} a_2 c_1 + k_{y3} a_1 c_2 - k_{y4} a_2 c_2 + k_{y5} a_1 c_3 - k_{y6} a_2 c_3\right)(\dot{\theta}\alpha + \theta\dot{\alpha})
\end{aligned}
$$

$$
\begin{aligned}
&+ (k_{z1} + k_{z2} + k_{z3} + k_{z4} + k_{x5} + k_{x6}) z\dot{z} \\
&+ \left((k_{z1} + k_{z3} + k_{z5}) a_1^2 + (k_{z2} + k_{z4} + k_{z6}) a_2^{\ 2}\right) \beta\dot{\beta} \\
&+ (k_{z1} + k_{z2} + k_{z3} + k_{z4} + k_{x5} + k_{z6}) b^2 \alpha\dot{\alpha} \\
&+ \left((k_{z1} + k_{z3} + k_{z5}) a_1 - (k_{z2} + k_{z4} + k_{z6}) a_2\right)(\dot{\beta}z + \beta\dot{z}) \\
&+ (-k_{z1} - k_{z2} - k_{z3} - k_{z4} - k_{z5} - k_{z6})(b)(\dot{\alpha}z + \alpha\dot{z}) \\
&+ \left(-(k_{z1} + k_{z3} + k_{z5}) a_1 + (k_{z2} + k_{z4} + k_{z6}) a_2\right)(b)(\dot{\alpha}\beta + \alpha\dot{\beta}) = 0
\end{aligned}
\tag{10}
$$

Equation (10) can be separated into six individual equations.

$$
\begin{aligned}
& m\dot{x}\ddot{x} + \left(k_{x1} + k_{x2} + k_{x3} + k_{x4} + k_{x5} + k_{x6}\right)x\dot{x} \\
& \quad + \left(k_{x1} + k_{x2} + k_{x3} + k_{x4} + k_{x5} + k_{x6}\right)b\theta\dot{x} \\
& \quad + \left(-\left(k_{x1} + k_{x2}\right)c_1 + \left(k_{x3} + k_{x4}\right)c_2 + \left(k_{x5} + k_{x6}\right)c_3\right)\beta\dot{x} = 0
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
& m\dot{y}\ddot{y} + \left(k_{y1} + k_{y2} + k_{y3} + k_{y4} + k_{y5} + k_{y6}\right)y\dot{y} \\
& \quad + \left(\left(k_{y1} + k_{y2}\right)c_1 - \left(k_{y3} + k_{y4}\right)c_2 - \left(k_{y5} + k_{y6}\right)c_3\right)\alpha\dot{y} \\
& \quad + \left(\left(-k_{y1} - k_{y3} - k_{y5}\right)a_1 + \left(k_{y2} + k_{y4} + k_{y6}\right)a_2\right)\theta\dot{y} = 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
& m\dot{z}\ddot{z} + \left(k_{z1} + k_{z2} + k_{z3} + k_{z4} + k_{z5} + k_{z6}\right)z\dot{z} \\
& \quad + \left(\left(k_{z1} + k_{z3} + k_{z5}\right)a_1 - \left(k_{z2} + k_{z4} + k_{z6}\right)a_2\right)\beta\dot{z} \\
& \quad + \left(-k_{z1} - k_{z2} - k_{z3} - k_{z4} - k_{z5} - k_{z6}\right)b\alpha\dot{z} = 0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
& J_x\dot{\alpha}\ddot{\alpha} + \left(\left(k_{y1} + k_{y2}\right)c_1^2 + \left(k_{y3} + k_{y4}\right)c_2^{\,2} + \left(k_{y5} + k_{y6}\right)c_3^{\,2}\right)\alpha\dot{\alpha} \\
& + \left(\left(k_{y1} + k_{y2}\right)c_1 - \left(k_{y3} + k_{y4}\right)c_2 - \left(k_{y5} + k_{y6}\right)c_3\right)\dot{\alpha}y \\
& + \left(-k_{y1}a_1c_1 + k_{y2}a_2c_1 + k_{y3}a_1c_2 - k_{y4}a_2c_2 + k_{y5}a_1c_3 - k_{y6}a_2c_3\right)\theta\dot{\alpha} \\
& + \left(k_{z1} + k_{z2} + k_{z3} + k_{z4} + k_{z5} + k_{z6}\right)b^2\alpha\dot{\alpha} \\
& + \left(-k_{z1} - k_{z2} - k_{z3} - k_{z4} - k_{z5} - k_{z6}\right)b\dot{\alpha}z \\
& + \left(\left(k_{z1} + k_{z3} + k_{z5}\right)a_1 - \left(k_{z2} + k_{z4} + k_{z6}\right)a_2\right)b\dot{\alpha}\beta = 0
\end{aligned}
\tag{14}
$$

$$J_y\,\dot\beta\ddot\beta$$

$$+\left(\left(k_{x1}+k_{x2}\right)c_1{}^2+\left(k_{x3}+k_{x4}\right)c_2{}^2+\left(k_{x5}+k_{x6}\right)c_3{}^2\right)\beta\dot\beta$$
$$+\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2+\left(k_{x5}+k_{x6}\right)c_3\right)\dot\beta x$$
$$+\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2+\left(k_{x5}+k_{x6}\right)c_3\right)b\theta\dot\beta$$

$$+\left(\left(k_{z1}+k_{z3}+k_{z5}\right)a_1{}^2+\left(k_{z2}+k_{z4}+k_{z6}\right)a_2{}^2\right)\beta\dot\beta$$
$$+\left(\left(k_{z1}+k_{z3}+k_{z5}\right)a_1-\left(k_{z2}+k_{z4}+k_{z6}\right)a_2\right)\dot\beta z$$
$$+\left(-\left(k_{z1}+k_{z3}+k_{z5}\right)a_1+\left(k_{z2}+k_{z4}+k_{z6}\right)a_2\right)b\alpha\dot\beta=0$$

(15)

$$J_z\,\dot\theta\ddot\theta$$
$$+\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}+k_{x5}+k_{x6}\right)b^2\theta\dot\theta$$
$$+\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}+k_{x5}+k_{x6}\right)b\dot\theta x$$
$$+\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2+\left(k_{x5}+k_{x6}\right)c_2\right)b\dot\theta\beta$$

$$+\left(\left(k_{y1}+k_{y3}+k_{y5}\right)a_1{}^2+\left(k_{y2}+k_{y4}+k_{y6}\right)a_2{}^2\right)\theta\dot\theta$$
$$+\left(\left(-k_{y1}-k_{y3}-k_{y5}\right)a_1+\left(k_{y2}+k_{y4}+k_{y6}\right)a_2\right)\dot\theta y$$
$$+\left(-k_{y1}a_1c_1+k_{y2}a_2c_1+k_{y3}a_1c_2-k_{y4}a_2c_2+k_{y5}a_1c_3-k_{y6}a_2c_3\right)\dot\theta\alpha=0$$

(16)

Typically,

$$k_{x1}=k_{x2}=k_{x3}=k_{x4}=k_{x5}=k_{x6}=k_x \quad (17)$$

$$k_{y1}=k_{y2}=k_{y3}=k_{y4}=k_{y5}=k_{y6}=k_y \quad (18)$$

$$k_{z1}=k_{z2}=k_{z3}=k_{z4}=k_{z5}=k_{z6}=k_z \quad (19)$$

Thus

$$m\dot{x}\ddot{x}+6k_x x\dot{x}+6k_x b\theta\dot{x}+2k_x(-c_1+c_2+c_3)\beta\dot{x}=0 \tag{20}$$

$$m\dot{y}\ddot{y}+6k_y y\dot{y}+2k_y(c_1-c_2-c_3)\alpha\dot{y}+3k_y(-a_1+a_2)\theta\dot{y}=0 \tag{21}$$

$$m\dot{z}\ddot{z}+6k_z z\dot{z}-6k_z b\alpha\dot{z}+3k_z(a_1-a_2)\beta\dot{z}=0 \tag{22}$$

$$\begin{aligned}&J_x\dot{\alpha}\ddot{\alpha}+2k_y\left(c_1^2+c_2^{\ 2}+c_3^{\ 2}\right)\alpha\dot{\alpha}+2k_y(c_1-c_2-c_3)\dot{\alpha}y\\&+k_y(-a_1+a_2)(c_1-c_2-c_3)\theta\dot{\alpha}\\&+6k_z b^2\alpha\dot{\alpha}-6k_z b\dot{\alpha}z+3k_z(a_1-a_2)b\dot{\alpha}\beta=0\end{aligned} \tag{23}$$

$$\begin{aligned}&J_y\dot{\beta}\ddot{\beta}+2k_x\left(c_1^2+c_2^{\ 2}+c_3^{\ 2}\right)\beta\dot{\beta}+2k_x(-c_1+c_2+c_3)b\theta\dot{\beta}+2k_x(-c_1+c_2+c_3)\dot{\beta}x\\&+3k_z\left(a_1^2+a_2^2\right)\beta\dot{\beta}+3k_z(a_1-a_2)\dot{\beta}z+3k_z(-a_1+a_2)b\alpha\dot{\beta}=0\end{aligned} \tag{24}$$

$$\begin{aligned}&J_z\dot{\theta}\ddot{\theta}+6k_x b^2\theta\dot{\theta}+6k_x b\dot{\theta}x+2k_x(-c_1+c_2+c_3)b\dot{\theta}\beta\\&+3k_y\left(a_1^2+a_2^2\right)\theta\dot{\theta}+3k_y(-a_1+a_2)\dot{\theta}y+k_y(-a_1+a_2)(c_1-c_2-c_3)\dot{\theta}\alpha=0\end{aligned} \tag{25}$$

The equations can be simplified as

$$m\ddot{x} + 6k_x\ x + 6k_x\ b\theta + 2k_x(-c_1 + c_2 + c_3)\beta = 0 \tag{26}$$

$$m\ddot{y} + 6k_y y + 2k_y(c_1 - c_2 - c_3)\alpha + 3k_y(-a_1 + a_2)\theta = 0 \tag{27}$$

$$m\ddot{z} + 6k_z z - 6k_z b\alpha + 3k_z(a_1 - a_2)\beta = 0 \tag{28}$$

$$\begin{aligned} &J_x\ddot{\alpha} + 2k_y\left(c_1^2 + c_2^{\ 2} + c_3^{\ 2}\right)\alpha + 2k_y(c_1 - c_2 - c_3)y \\ &+ k_y(-a_1 + a_2)(c_1 - c_2 - c_3)\theta \\ &+ 6k_z b^2\alpha - 6k_z bz + 3k_z(a_1 - a_2)b\beta = 0 \end{aligned} \tag{29}$$

$$\begin{aligned} &J_y\ddot{\beta} + 2k_x\left(c_1^2 + c_2^{\ 2} + c_3^{\ 2}\right)\beta + 2k_x(-c_1 + c_2 + c_3)b\theta + 2k_x(-c_1 + c_2 + c_3)x \\ &+ 3k_z\left(a_1^2 + a_2^2\right)\beta + 3k_z(a_1 - a_2)z + 3k_z(-a_1 + a_2)b\alpha = 0 \end{aligned} \tag{30}$$

$$\begin{aligned} &J_z\ddot{\theta} + 6k_x b^2\theta + 6k_x bx + 2k_x(-c_1 + c_2 + c_3)b\beta \\ &+ 3k_y\left(a_1^2 + a_2^2\right)\theta + 3k_y(-a_1 + a_2)y + k_y(-a_1 + a_2)(c_1 - c_2 - c_3)\alpha = 0 \end{aligned} \tag{31}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix} + \underline{K}\begin{bmatrix}x\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{32}$$

The mass and stiffness matrices are shown in upper triangular form due to symmetry.

$$\underline{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 \\ & & m & 0 & 0 & 0 \\ & & & J_x & 0 & 0 \\ & & & & J_y & 0 \\ & & & & & J_z \end{bmatrix} \tag{33}$$

$$\underline{K} = \begin{bmatrix} 6k_x & 0 & 0 & 0 & 2k_x(-c_1 + c_2 + c_3) & 6k_x b \\ & 6k_y & 0 & 2k_y(c_1 - c_2 - c_3) & 0 & 3k_y(-a_1 + a_2) \\ & & 6k_z & -6k_z b & 3k_z(a_1 - a_2) & 0 \\ & & & 6k_z b^2 + 2k_y\left(c_1^2 + c_2^2 + c_3^2\right) & 3k_z(-a_1 + a_2)b & k_y(-a_1 + a_2)(c_1 - c_2 - c_3) \\ & & & & 2k_x\left(c_1^2 + c_2^2 + c_3^2\right) + 3k_z\left(a_1^2 + a_2^2\right) & 2k_x(-c_1 + c_2 + c_3)b \\ & & & & & 6k_x b^2 + 3k_y\left(a_1^2 + a_2^2\right) \end{bmatrix} \tag{34}$$

Base Excitation

Consider three separate base excitation cases, following the convention in Reference 1.

The mass and stiffness matrices in equations (33) and (34) apply in each of the cases.

Let u, x, and w be the base displacement in the X, Y and Z-axes, respectively.

X-axis Excitation

Let

$$r_1 = x - u \tag{35}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{r}_1\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}r_1\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}-m\ddot{u}\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{36}$$

Y-axis Excitation

Let

$$r_2 = y - v \tag{37}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{r}_2\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}x\\ r_2\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ -m\ddot{v}\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{38}$$

Z-axis Excitation

Let

$$r_3 = z - w \tag{39}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{r}_3\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix} + \underline{K}\begin{bmatrix}x\\ y\\ r_3\\ \alpha\\ \beta\\ \theta\end{bmatrix} = \begin{bmatrix}0\\ 0\\ -m\ddot{w}\\ 0\\ 0\\ 0\end{bmatrix} \tag{40}$$

The acceleration transmissibility functions of equations (36), (38) and (40) can then be determined via Reference 2.

The modal transient response to a base input time history can be calculated via References 3 and 4.

<u>References</u>

1. T. Irvine, Vibration Analysis of an Isolated Mass with Six Degrees of Freedom, Revision F, Vibrationdata, 2011.

2. T. Irvine, Frequency Response Function Analysis of a Multi-degree-of-freedom System with Enforced Motion, Vibrationdata, 2011.

3. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Subjected to Base Excitation, Revision B, Vibrationdata, 2004.

4. T. Irvine, Shock Response of Multi-degree-of-freedom Systems, Revision F, Vibrationdata, 2010.

APPENDIX A

Example

A mass is mounted to a surface with six isolators. The system has the following properties.

M	=	9.0 lbm
Jx	=	90.17 lbm in^2
Jy	=	105.2 lbm in^2
Jz	=	42.08 lbm in^2
kx	=	500 lbf/in
ky	=	500 lbf/in
kz	=	500 lbf/in
a1	=	3 in
a2	=	3 in
b	=	2 in
c1	=	3 in
c2	=	5 in
c3	=	1 in

The natural frequency results are calculated using the program: six_isolators.m.

The output is

```
>> six_isolators

 six_isolators.m   ver 1.2  August 9, 2011

 by Tom Irvine   Email: tomirvine@aol.com

 This program finds the eigenvalues and eigenvectors for a
```

```
 six-degree-of-freedom system mounted on six isolators.

 Refer to six_isolators.pdf for a diagram.

 The equation of motion is:   M (d^2x/dt^2) + K x = 0


Enter m (lbm)
 9.0
 Enter Jx (lbm in^2)
 90.17
 Enter Jy (lbm in^2)
 105.20
 Enter Jz (lbm in^2)
 42.08

 Note that the stiffness values are for individual springs

 Enter kx (lbf/in)
 500

 Enter ky (lbf/in)
 500

 Enter kz (lbf/in)
 500

 Enter a1 (in)
 3

 Enter a2 (in)
 3

 Enter b (in)
 2

 Enter c1 (in)
 3

 Enter c2 (in)
 5

 Enter c3 (in)
 1
```

```
 The mass matrix is

m =

    0.0233         0         0         0         0         0
         0    0.0233         0         0         0         0
         0         0    0.0233         0         0         0
         0         0         0    0.2336         0         0
         0         0         0         0    0.2725         0
         0         0         0         0         0    0.1090


 The stiffness matrix is

k =

  Columns 1 through 5

        3000           0           0           0        3000
           0        3000           0       -3000           0
           0           0        3000       -6000           0
           0       -3000       -6000       47000           0
        3000           0           0           0       62000
        6000           0           0           0        6000

  Column 6

        6000
           0
           0
           0
        6000
       39000


 Eigenvalues

lambda =

  1.0e+005 *

    0.6707    0.7501    1.2867    2.1907    2.6280    4.1983


  Natural Frequencies =
 1.      41.22 Hz
 2.      43.59 Hz
 3.      57.09 Hz
 4.      74.49 Hz
 5.      81.59 Hz
 6.      103.1 Hz
```

Modes Shapes (rows represent modes)

	x	y	z	alpha	beta	theta
1	0	2.42	4.85	1.16	0	0
2	6.01	0	0	0	-0.273	-1.12
3	0	-5.86	2.93	0	0	0
4	-0.254	0	0	0	-1.84	0.831
5	0	-1.64	-3.29	1.71	0	0
6	2.58	0	0	0	0.455	2.69

Effective Modal Mass (rows represent modes)

	x	y	z	alpha	beta	theta
1	0	1.23	4.93	28.4	0	0
2	7.59	0	0	0	2.13	5.73
3	0	7.2	1.8	0	0	0
4	0.0135	0	0	0	97.1	3.17
5	0	0.566	2.27	61.8	0	0
6	1.39	0	0	0	5.95	33.2

Total Modal Mass

9	9	9	90.2	105	42.1

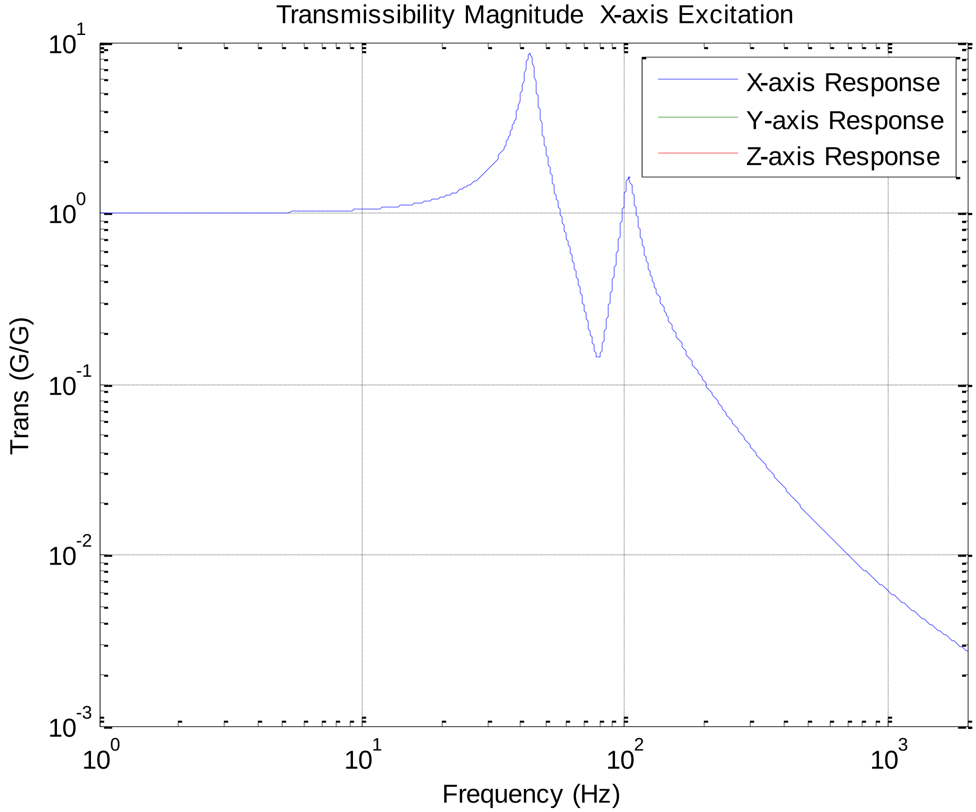

Figure A-1.

Both the Y and Z-axes responses were below the lower amplitude plot limit.

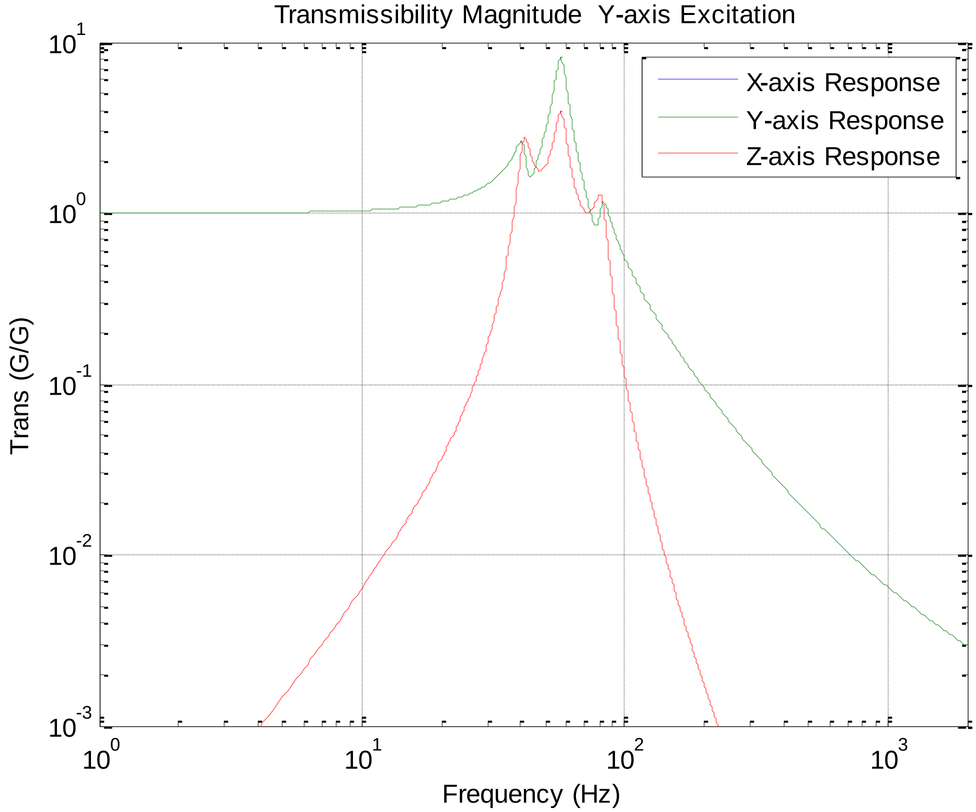

Figure A-2.

The X-axis response was below the lower amplitude plot limit.

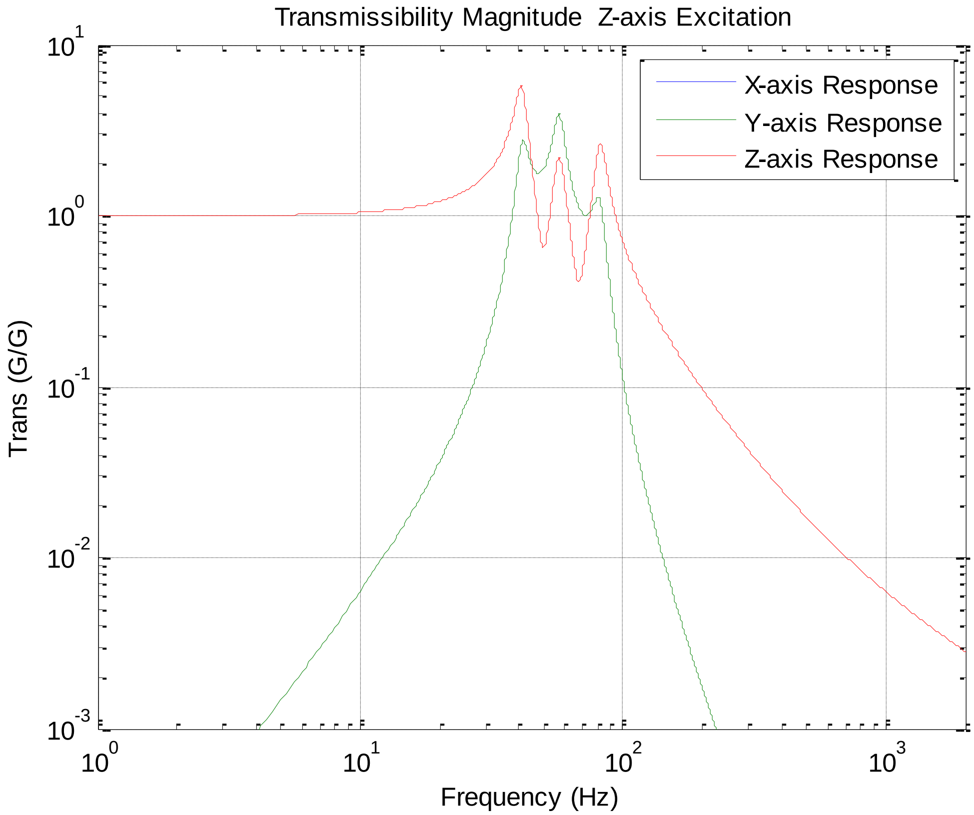

Figure A-3.

The X-axis response was below the lower amplitude plot limit.