FORMULAS FOR CALCULATING THE SPEED OF SOUND
Revision G

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Introduction

A sound wave is a longitudinal wave, which alternately pushes and pulls the material through which it propagates. The amplitude disturbance is thus parallel to the direction of propagation.

Sound waves can propagate through the air, water, Earth, wood, metal rods, stretched strings, and any other physical substance.

The purpose of this tutorial is to give formulas for calculating the speed of sound. Separate formulas are derived for a gas, liquid, and solid.

General Formula for Fluids and Gases

The speed of sound c is given by

$$c = \sqrt{\frac{B}{\rho_o}} \tag{1}$$

where

B is the adiabatic bulk modulus,
ρ_o is the equilibrium mass density.

Equation (1) is taken from equation (5.13) in Reference 1. The characteristics of the substance determine the appropriate formula for the bulk modulus.

Gas or Fluid

The bulk modulus is essentially a measure of stress divided by strain. The adiabatic bulk modulus B is defined in terms of hydrostatic pressure P and volume V as

$$B = \frac{\Delta P}{-\Delta V / V} \tag{2}$$

Equation (2) is taken from Table 2.1 in Reference 2.

An adiabatic process is one in which no energy transfer as heat occurs across the boundaries of the system.

An alternate adiabatic bulk modulus equation is given in equation (5.5) in Reference 1.

$$B = \rho_o \left(\frac{\partial P}{\partial \rho} \right)_{\rho_o} \tag{3}$$

Note that

$$\left(\frac{\partial P}{\partial \rho} \right) = \gamma \frac{P}{\rho} \tag{4}$$

where

γ is the ratio of specific heats.

The ratio of specific heats is explained in Appendix A.

The speed of sound can thus be represented as

$$c = \sqrt{\gamma \frac{P_o}{\rho_o}} \tag{5}$$

Equation (5) is the same as equation (5.18) in Reference 1.

Perfect Gas

An alternate formula for the speed of sound in a perfect gas is

$$c = \sqrt{\gamma \left(\frac{R}{M} \right) T_k} \tag{6a}$$

where

γ is the ratio of specific heats,
M is the molecular mass,
R is the universal gas constant,
T_k is the absolute temperature in Kelvin.

Molecular mass is explained in Appendix A. The speed of sound in the atmosphere is given in Appendix B.

Equation (6a) is taken from equations (5.19) and (A9.10) in Reference 1.

The speed of sound in a gas is directly proportional to absolute temperature.

$$\frac{c_1}{c_2} = \sqrt{\frac{T_{k,1}}{T_{k,2}}} \tag{6b}$$

Liquid

A special formula for the speed of sound in a liquid is

$$c = \sqrt{\frac{\gamma B_T}{\rho_o}} \tag{7}$$

where

- γ is the ratio of specific heats,
- B_T is the isothermal bulk modulus,
- ρ_o is the equilibrium mass density.

Equation (7) is taken from equation (5.21) in Reference 1.

The isothermal bulk modulus is related to the adiabatic bulk modulus.

$$B = \gamma B_T \tag{8}$$

Solid

The speed of sound in a solid material with a large cross-section is given by

$$c = \sqrt{\frac{B + \left(\frac{4}{3}\right)G}{\rho}} \tag{9}$$

where

G is the shear modulus,
ρ is the mass per unit volume.

Equation (9) is taken from equation (6.41) in Reference 1. The c term is referred to as the bulk or plate speed of longitudinal waves.

The shear modulus can be expressed as

$$G = \frac{E}{2(1+\nu)} \tag{10}$$

where

E is the modulus of elasticity,
ν is Poisson's ratio.

Equation (10) is taken from Table 2.2 in Reference 2.

Substitute equation (10) into (9).

$$c = \sqrt{\frac{B + \left(\frac{4}{3}\right)\left(\frac{E}{2(1+\nu)}\right)}{\rho}} \tag{11a}$$

$$c = \sqrt{\frac{B + \left(\frac{2}{3}\right)\left(\frac{E}{1+\nu}\right)}{\rho}} \tag{11b}$$

The bulk modulus for an isotropic solid is

$$B = \frac{E}{3(1-2\nu)} \tag{12}$$

where

E is the modulus of elasticity,
ν is Poisson's ratio.

The modulus of elasticity is also called Young's modulus.

Equation (12) is taken from the Definition Chapter in Reference 3. It is also given in Table 2.2 of Reference 2.

Substitute equation (12) into equation (11b).

$$c = \sqrt{\frac{\left(\frac{E}{3(1-2\nu)}\right) + \left(\frac{2}{3}\right)\left(\frac{E}{1+\nu}\right)}{\rho}} \tag{13}$$

The next steps simplify the algebra.

$$c = \sqrt{\left(\frac{E}{\rho}\right)\left(\left(\frac{1}{3(1-2\nu)}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{1+\nu}\right)\right)} \tag{14}$$

$$c = \sqrt{\left(\frac{E}{\rho}\right)\left(\frac{(1+\nu)+2(1-2\nu)}{3(1-2\nu)(1+\nu)}\right)} \tag{15}$$

$$c = \sqrt{\left(\frac{E}{\rho}\right)\left(\frac{3-3\nu}{3(1-2\nu)(1+\nu)}\right)} \tag{16}$$

$$c = \sqrt{\left(\frac{E}{\rho}\right)\left(\frac{1-\nu}{(1-2\nu)(1+\nu)}\right)} \tag{17}$$

Equation (17) is also given in Chapter 2 of Reference 4.

The Poisson terms in equation (17) account for a lateral effect, which can be neglected if the cross-section dimension is small, compared to the wavelength. In this case, equation (17) simplifies to

$$c = \sqrt{\frac{E}{\rho}} \tag{18}$$

String

Consider a string with uniform mass per length ρ_L. The string is stretched with a tension force T. The phase speed c is given by

$$c = \sqrt{\frac{T}{\rho_L}} \tag{19}$$

This speed is the phase speed of transverse traveling waves.

Equation (19) is taken from equation (2.6) in Reference 1.

Membrane

Consider a membrane with uniform mass per area ρ_a. The membrane is assumed to be thin, with negligible stiffness.

The membrane is stretched with a tension force per length T_L. The tension is assumed to be uniform throughout the membrane.

The transverse phase speed c is given by

$$c = \sqrt{\frac{T_L}{\rho_a}} \tag{20}$$

This speed is the phase speed of transverse traveling waves.

Equation (20) is the same as equation (4.3) in Reference 1.

Special Topics

Appendix B gives the variation of the speed of sound in the atmosphere with altitude.

Appendix C gives the speed of sound in seawater.

Properties

Pertinent properties of solids, liquids, and gases are given in Tables 1a, 1b, 2, and 3, respectively.

Table 1a. Solids

Solid	Density (kg/m^3)	Elastic Modulus (Pa)	Shear Modulus (Pa)	Poisson's Ratio	Speed of Sound (m/sec)	
					Bar	Bulk
Aluminum	2700	$7.0\ (10^{10})$	$2.4\ (10^{10})$	0.33	5100	6300
Brass	8500	$10.4\ (10^{10})$	$3.8\ (10^{10})$	0.37	3500	4700
Copper	8900	$12.2\ (10^{10})$	$4.4\ (10^{10})$	0.35	3700	5000
Steel	7700	$19.5\ (10^{10})$	$8.3\ (10^{10})$	0.28	5050	6100
Ice	920	-	-	-	-	3200
Glass (Pyrex)	2300	$6.2\ (10^{10})$	$2.5\ (10^{10})$	0.24	5200	5600

Table 1b. Solids (Extreme Values from Reference 7)	
Solid	Speed of Sound (m/sec)
Granite	6000
Vulcanized Rubber at 0 °C	54

Table 2. Liquids					
Liquids	Temperature (°C)	Density (kg/m^3)	Adiabatic Bulk Modulus (Pa)	Ratio of Specific Heats	Speed of Sound (m/sec)
Water (fresh)	20	998	$2.18(10^9)$	1.004	1481
Water (sea)	13	1026	$2.28(10^9)$	1.01	1500
Mercury	20	13,600	$25.3(10^9)$	1.13	1450

Table 3. Gases at a pressure of 1 atmosphere					
Gases	Molecular Mass (kg/kgmole)	Temperature (°C)	Density (kg/m^3)	Ratio of Specific Heats	Speed of Sound (m/sec)
Air	28.97	0	1.293	1.402	332
Air	28.97	20	1.21	1.402	343
Oxygen (O_2)	32.00	0	1.43	1.40	317
Hydrogen (H_2)	2.016	0	0.09	1.41	1270
Steam	-	100	0.60	-	404.8

Note: 1 (kg/kgmole) = 1 (lbm/lbmole)

Examples

Air

Calculate the speed of sound in air for a temperature of 70 degrees F (294.26 K).

The properties for air are

$\gamma = 1.402$
M = 28.97 kg/kgmole

The universal gas constant is

$R = 8314.3 \text{ J/(kgmole·K)}$

The specific heat ratio is taken from Appendix 10 of Reference 1. The molecular mass and gas constant values are taken from Reference 5.

The formula for the speed of sound is

$$c = \sqrt{\gamma\left(\frac{R}{M}\right)T_k}$$

$$c = \sqrt{1.402\left(\frac{\text{kgmole}}{28.97\text{ kg}}\right)\left(8314.3\frac{\text{J}}{\text{kgmole}\cdot\text{K}}\right)(294.26\text{ K})}$$

$c = 344$ m/sec

$c = 1130$ ft/sec

Solid, Aluminum Rod

Calculate the speed of sound in an aluminum rod. Assume that the diameter is much smaller than the wavelength.

The material properties for aluminum are:

$E = 70(10^9)$ Pa
$\rho = 2700$ kg/m^3

These properties are taken from Reference 6. The speed of sound is

$$c = \sqrt{\frac{E}{\rho}}$$

$$c = \sqrt{\frac{70(10^9)\text{ Pa}}{2700\text{ kg}/\text{m}^3}}$$

$c = 5100$ m/sec

$c = 16{,}700$ ft/sec

References

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APPENDIX A

Ratio of Specific Heats

The ratio of specific heats is defined as

$$\gamma = \frac{Cp}{Cv} \tag{A-1}$$

where

Cp is the heat capacity at constant pressure,
Cv is the heat capacity at constant volume.

Molecular Mass

Molecular mass is also called be the following names:

1. Molecular weight
2. Molal mass

Molecular mass is the mass per mole of a material.

One mole is defined as $6.023(10^{23})$ particles. This is called Avogadro's number. It is also the number of atoms in a "gram atom."

For carbon 12, the molecular mass is 12 kg/kgmole = 12 g/gmole = 12 lbm/lbmole. The mole is defined in such a way that one kgmole of a substance contains the same number of molecules as 12 kg of carbon 12. Likewise, one lbmole contains the same number of molecules as 12 lbm of carbon 12.

APPENDIX B

Variation of the Speed of Sound in the Atmosphere with Altitude

The pressure, temperature, density and speed of sound for the international standard atmosphere (ISA) can be calculated for a range of altitudes from sea level upward. These parameters are obtained from the hydrostatic equation for a column of air. The air is assumed to be a perfect gas.

The atmosphere consists of two regions.

The troposphere is the region between sea level and an altitude of approximately 11 km (36,089 feet). In reality, the boundary may be at 10 to 15 km depending on latitude and time of year. The temperature lapse rate in the troposphere is taken as L= 6.5 Kelvin/km. The actual value depends on the season, weather conditions, and other variables.

The stratosphere is the region above 11 km and below 50 km. The stratosphere is divided into two parts for the purpose of this tutorial.

The lower stratosphere extends from 11 km to 20 km. The temperature remains constant at 217 Kelvin (-69.1 F) in the lower stratosphere.

The upper stratosphere extends from 20 km to 50 km. The temperature rises in the upper stratosphere.

Basic Equations

The hydrostatic equation for pressure P and altitude h is

$$\frac{dP}{dh} = -\rho g \tag{B-1}$$

where

ρ = mass density,
g = gravitational acceleration.

The perfect gas equation is

$$P = \rho \frac{R}{M} T_k \tag{B-2}$$

where

R is the universal gas constant,
M is the molecular weight,
T_k is the absolute temperature in Kelvin.

Note that for air,

$$\frac{R}{M} = \frac{8314.3 \text{ J/(kgmole} \cdot \text{K)}}{28.97 \text{ kg / kgmole}} \tag{B-3}$$

$$\frac{R}{M} = 287 \ \frac{\text{J}}{\text{kg} \cdot \text{K}} \tag{B-4}$$

$$\frac{R}{M} = 287 \ \frac{\text{m}^2 / \text{sec}^2}{\text{K}} \tag{B-5}$$

<u>Troposphere</u>

The temperature lapse equation for the troposphere is

$$T = T_o - Lh \tag{B-6}$$

Recall the formula for the speed of sound in a perfect gas.

$$c = \sqrt{\gamma \left(\frac{R}{M} \right) T_k} \tag{B-7}$$

The speed of sound in the troposphere is thus

$$c = \sqrt{\gamma \left(\frac{R}{M} \right) \left(T_o - Lh \right)} \tag{B-8}$$

The standard sea level temperature is $T_o = 288$ Kelvin. Again, L= 6.5 Kelvin / km for the troposphere.

Substitute equation (B-6) into (B-2) to obtain the perfect gas law for the troposphere.

$$P = \rho \ \frac{R}{M} \left[T_o - Lh \right] \tag{B-9}$$

The density in the troposphere can thus be expressed as

$$\rho = \frac{P}{\frac{R}{M}\left[T_o - Lh\right]} \tag{B-10}$$

Solve the hydrostatic equation for a constant lapse rate. The resulting equation gives the pressure variation with altitude. Neglect the variation of gravity with altitude. Rewrite equation (B-1).

$$dP = -\rho g \, dh \tag{B-11}$$

Substitute equation (B-10) into (B-11).

$$dP = -\frac{P}{\frac{R}{M}\left[T_o - Lh\right]} g \, dh \tag{B-12}$$

$$\frac{dP}{P} = -\frac{g}{\frac{R}{M}\left[T_o - Lh\right]} dh \tag{B-13}$$

The hat sign is added in order to prevent confusion between the integration variables and the limits.

$$\int_{P_o}^{P} \frac{d\hat{P}}{\hat{P}} = -\int_0^h \frac{g}{\frac{R}{M}\left[T_o - L\hat{h}\right]} d\hat{h} \tag{B-14}$$

$$\ln\left[\hat{P}\right]\Big|_{P_o}^{P} = \frac{Mg}{LR} \ln\left[T_o - L\hat{h}\right]\Big|_0^h \tag{B-15}$$

$$\ln\left[\frac{P}{P_o}\right] = \frac{Mg}{LR}\left\{\ln\left[T_o - Lh\right] - \ln\left[T_o\right]\right\} \tag{B-16}$$

$$\ln\left[\frac{P}{P_o}\right] = \frac{Mg}{LR}\left\{\ln\left[\frac{T_o - Lh}{T_o}\right]\right\} \tag{B-17}$$

$$\left[\frac{P}{P_o}\right] = \left[\frac{T_o - Lh}{T_o}\right]^{\left[\frac{Mg}{LR}\right]} \tag{B-18}$$

The pressure in the troposphere is thus

$$P = P_o\left[\frac{T_o - Lh}{T_o}\right]^{\left[\frac{Mg}{LR}\right]} \tag{B-19}$$

Note that the sea level pressure is $P_o = 101.3\text{kPa}$.

The density in the troposphere is obtained from equations (B-10) and (B-19).

$$\rho = \frac{P_o\left[\frac{T_o - Lh}{T_o}\right]^{\left[\frac{Mg}{LR}\right]}}{\frac{R}{M}\left[T_o - Lh\right]} \tag{B-20}$$

<u>Lower Stratosphere</u>

Again, the temperature is constant in the lower stratosphere. The speed of sound is thus constant in the lower stratosphere.

$$dP = -\rho g\, dh \tag{B-21}$$

Let T_c be the constant temperature in the lower stratosphere.

$$dP = -\frac{P}{\frac{RT_c}{M}} g\, dh \tag{B-22}$$

$$\frac{dP}{P} = -\frac{Mg}{RT_c}\, dh \tag{B-23}$$

The hat sign is added in order to prevent confusion between the integration variables and the limits.

$$\int_{P_1}^{P} \frac{d\hat{P}}{\hat{P}} = -\int_{h_1}^{h} \frac{M g}{R T_c} d\hat{h} \tag{B-24}$$

$$\ln\left[\hat{P}\right]\Big|_{P_1}^{P} = \frac{-M g}{R T_c} \hat{h}\Big|_{h_1}^{h} \tag{B-25}$$

$$\ln\left[\frac{P}{P_1}\right] = \frac{-M g}{R T_c}\left[h - h_1\right] \tag{B-26}$$

$$\frac{P}{P_1} = \exp\left\{\frac{-M g}{R T_c}\left[h - h_1\right]\right\} \tag{B-27}$$

The pressure in the lower stratosphere is thus

$$P = P_1 \exp\left\{-\frac{M g}{R T_c}\left[h - h_1\right]\right\} \tag{B-27}$$

Note that P_1 is the pressure at the lower altitude limit of the stratosphere.

The density in the lower stratosphere is thus

$$\rho = \frac{M}{R T_k} P_1 \exp\left\{-\frac{M g}{R T_c}\left[h - h_1\right]\right\} \tag{B-28}$$

<u>Summary</u>

The pressure, density, and speed of sound are given in Table B-1 for an altitude up to 20 km.

Table B-1. Atmospheric Properties					
Altitude (km)	Pressure (kPa)	Mass Density (kg/m^3)	Temp. (Kelvin)	Temp. (°C)	Speed of Sound (m/sec)
0	101.3	1.226	288	14.9	340.2
1	89.85	1.112	282	8.4	336.3
2	79.47	1.007	275	1.9	332.4
3	70.09	0.9096	269	-4.7	328.5
4	61.62	0.8195	262	-11.2	324.5
5	54.00	0.7365	256	-17.7	320.4
6	47.17	0.6600	249	-24.2	316.3
7	41.05	0.5898	243	-30.7	312.1
8	35.59	0.5254	236	-37.2	307.9
9	30.73	0.4666	230	-43.7	303.7
10	26.43	0.4129	223	-50.2	299.3
11	22.62	0.3641	217	-56.2	295
12	19.33	0.3104	217	-56.2	295
13	16.51	0.2652	217	-56.2	295
14	14.11	0.2266	217	-56.2	295
15	12.06	0.1936	217	-56.2	295
16	10.30	0.1654	217	-56.2	295
17	8.801	0.1413	217	-56.2	295
18	7.519	0.1207	217	-56.2	295
19	6.424	0.1032	217	-56.2	295
20	5.489	0.0881	217	-56.2	295

Again, the values in Table B-1 are approximate. The actual values depend on the time of day, season, weather conditions, etc.

APPENDIX C

<u>The Speed of Sound in Seawater</u>

The speed of sound in seawater at 13 °C is 1500 m/sec, per Table 2. This is a nominal value. The actual value depends on the depth, salinity, and temperature.

A number of empirical equations exist for determining the speed of sound in seawater. Reference 8 gives the following "Leroy" equation.

$$\begin{aligned} c = \; & 1492.9 + 3(T-10) - 6\,(10^{-3})(T-10)^2 \\ & - 4\,(10^{-2})(T-18)^2 + 1.2\,(S-35) \\ & - (10^{-2})(T-18)(S-35) + Z/61 \end{aligned} \tag{C-1}$$

where

- c is the speed of sound in m/sec
- T is the temperature in °C
- S is the salinity in parts per thousand
- Z is the depth in meters.

Equation (1) is accurate to 0.1 m/sec for T less than 20 °C and Z less than 8000 m.

An alternate equation is the Lovett equation given in Chapter 15 of Reference 6.