THE VIBRATION RESPONSE OF SOME SPRING-DAMPER SYSTEMS WITH AND WITHOUT MASSES

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Introduction

Consider the single-degree-of-freedom system subjected to an applied force in Figure 1.

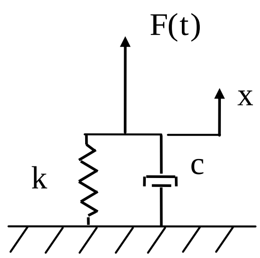

Figure 1.

where

F(t)	=	Applied force
c	=	viscous damping coefficient
k	=	stiffness
x	=	displacement

Summation of forces in the vertical direction yields the equation of motion.

$$c\dot{x} + kx = F(t) \tag{1}$$

Derive the transfer function by taking the Laplace transform.

$$L\{c\dot{x} + kx\} = L\{F(t)\} \tag{2}$$

$$c\,s\,X(s) - c\,x(0) + kX(s) \;= F(s) \tag{3}$$

$$[cs + k]X(s) \;= F(s) + c\,x(0) \tag{4}$$

$$X(s) \;= \frac{F(s) + c\,x(0)}{cs + k} \tag{5}$$

Now assume that the initial displacement is zero.

$$X(s) \;= \frac{F(s)}{cs + k} \tag{6}$$

$$\frac{X(s)}{F(s)} = \frac{1}{cs + k} \tag{7}$$

The dynamic stiffness in the Laplace domain is

$$\frac{F(s)}{X(s)} = cs + k \tag{7}$$

The dynamic stiffness in the frequency domain is

$$\frac{F(\omega)}{X(\omega)} = k + jc\omega \tag{8}$$

The initial value problem for the free response is

$$X(s) \;= \frac{c\,x(0)}{cs + k} \tag{9}$$

$$X(s) = \frac{x(0)}{s+(k/c)} \tag{10}$$

The inverse Laplace transform yields the time domain response.

$$x(t) = x(0)\ \exp[(k/c)t] \tag{11}$$

APPENDIX A

Consider the single-degree-of-freedom system subjected to an applied force.

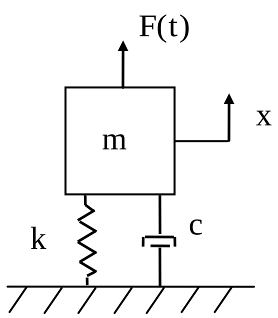

Figure A-1.

The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{A-1}$$

Derive the transfer function by taking the Laplace transform.

$$L\{m\ddot{x} + c\dot{x} + kx\} = L\{F(t)\} \tag{A-2}$$

$$ms^2 X(s) - m\dot{x}(0) - msx(0) + csX(s) - cx(0) + kX(s) = F(s) \tag{A-3}$$

$$\left[ms^2 + cs + k\right]X(s) = F(s) + c\,x(0) + m\dot{x}(0) + msx(0) \tag{A-4}$$

$$X(s) = \frac{F(s) + c\,x(0) + m\dot{x}(0) + msx(0)}{ms^2 + cs + k} \tag{A-5}$$

Now assume that the initial displacement is zero.

$$X(s) = \frac{F(s)}{ms^2 + cs + k} \tag{A-6}$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{A-7}$$

The dynamic stiffness in the Laplace domain is

$$\frac{F(s)}{X(s)} = ms^2 + cs + k \tag{A-8}$$

The dynamic stiffness in the frequency domain is

$$\frac{F(\omega)}{X(\omega)} = k - m\omega^2 + jc\omega \tag{A-9}$$

The mechanical impedance in the frequency domain is

$$\frac{F(\omega)}{V(\omega)} = \frac{k - m\omega^2 + jc\omega}{j\omega} \tag{A-10}$$

$$\frac{F(\omega)}{V(\omega)} = c + j\left[-\frac{k}{\omega} + m\omega\right] \tag{A-11}$$

The apparent mass in the frequency domain is

$$\frac{F(\omega)}{A(\omega)} = \frac{c + j\left[-\frac{k}{\omega} + m\omega\right]}{j\omega} \tag{A-12}$$

$$\frac{F(\omega)}{A(\omega)} = \left[m - \frac{k}{\omega^2}\right] - j\frac{c}{\omega} \tag{A-13}$$

$$\frac{F(\omega)}{A(\omega)} = m\left\{\left[1 - \frac{k}{m\omega^2}\right] - j\frac{c}{m\omega}\right\} \tag{A-14}$$

Note that

$$\omega_n = \sqrt{\frac{k}{m}} \tag{A-15}$$

$$c = 2\xi\omega_n m \tag{A-16}$$

$$\eta = 2\xi \tag{A-17}$$

By substitution,

$$\frac{F(\omega)}{A(\omega)} = m\left\{\left[1 - \frac{{\omega_n}^2}{\omega^2}\right] - j\frac{2\xi\omega_n}{\omega}\right\} \tag{A-18}$$

$$\frac{F(\omega)}{A(\omega)} = m\left\{\left[1 - \frac{{\omega_n}^2}{\omega^2}\right] - j\frac{\eta\omega_n}{\omega}\right\} \tag{A-19}$$

APPENDIX B

Consider the following two-degree-of-freedom system subjected to an applied force.

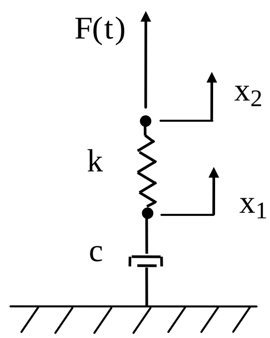

Figure B-1.

The free-body diagrams are

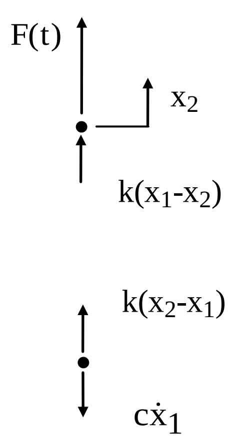

Figure B-2.

The equations of motion are

$$k(x_1 - x_2) + F(t) = 0 \tag{B-1}$$

$$k(x_2 - x_1) = F(t) \tag{B-2}$$

$$c\dot{x}_1 + k(x_1 - x_2) = 0 \tag{B-3}$$

Solve for x_2 using Equation (B-2).

$$x_2 = \frac{F(t)}{k} + x_1 \tag{B-4}$$

By substitution,

$$c\dot{x}_1 + k(x_1 - x_2) = 0 \tag{B-5}$$

$$c\dot{x}_1 + k\left(x_1 - \frac{F(t)}{k} - x_1\right) = 0 \tag{B-6}$$

$$c\dot{x}_1 + k\left(-\frac{F(t)}{k}\right) = 0 \tag{B-7}$$

$$c\dot{x}_1 = F(t) \tag{B-8}$$

Take the Laplace transform.

$$L\{c\dot{x}_1\} = L\{F(t)\} \tag{B-9}$$

$$c s X_1(s) - c x_1(0) = F(s) \tag{B-10}$$

$$c s X_1(s) = F(s) + c x_1(0) \tag{B-11}$$

$$X_1(s) = \frac{F(s) + c x_1(0)}{c s} \tag{B-12}$$

Recall

$$k(x_2 - x_1) = F(t) \tag{B-13}$$

Take the Laplace transform.

$$L\{k(x_2 - x_1)\} = L\{F(t)\} \tag{B-14}$$

$$k X_2(s) - k X_1(s) = F(s) \tag{B-15}$$

$$k X_2(s) = F(s) + k X_1(s) \tag{B-16}$$

$$k X_2(s) = F(s) + k\left\{\frac{F(s) + c x_1(0)}{c s}\right\} \tag{B-17}$$

$$k X_2(s) = F(s)\left[1 + \frac{k}{c s}\right] + \frac{k}{s} x_1(0) \tag{B-18}$$

$$X_2(s) = F(s)\left[\frac{1}{k} + \frac{1}{c s}\right] + \frac{1}{s} x_1(0) \tag{B-19}$$

Now assume the initial displacement is zero.

$$X_2(s) = F(s)\left[\frac{1}{k} + \frac{1}{cs}\right] \tag{B-20}$$

$$X_2(s) = F(s)\left[\frac{cs + k}{cks}\right] \tag{B-21}$$

The receptance in the Laplace domain is

$$\frac{X_2(s)}{F(s)} = \frac{cs + k}{cks} \tag{B-22}$$

The dynamic stiffness in the Laplace domain is

$$\frac{F(s)}{X_2(s)} = \frac{cks}{cs + k} \tag{B-22}$$

The dynamic stiffness in the frequency domain is

$$\frac{F(\omega)}{X_2(\omega)} = \frac{jck\omega}{k + jc\omega} \tag{B-22}$$

The initial value problem for the free response is

$$X_2(s) = \frac{1}{s} x_1(0) \tag{B-23}$$

The inverse Laplace transform yields the time domain response in terms of the unit step function u(t).

$$x_2(t) = x_1(0)\ u(t) \tag{B-24}$$

This can be simplified as

$$x_2(t) = x_1(0) \tag{B-25}$$

Likewise

$$x_1(t) = x_1(0) \tag{B-26}$$

APPENDIX C

Consider the following two-degree-of-freedom system subjected to an applied force.

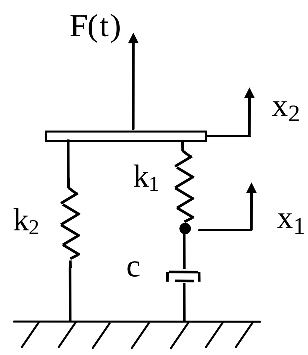

Figure C-1.

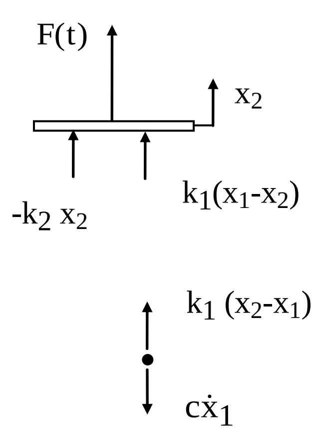

Figure C-2.

The equations of motion are

$$k_1(x_1 - x_2) - k_2 x_2 + F(t) = 0 \tag{C-1}$$

$$(k_1 + k_2)x_2 - k_1 x_1 = F(t) \tag{C-2}$$

$$c\dot{x}_1 + k_1(x_1 - x_2) = 0 \tag{C-3}$$

Solve for x_2 using Equation (C-2).

$$x_2 = \frac{F(t)}{k_1 + k_2} + \left[\frac{k_1}{k_1 + k_2}\right] x_1 \tag{C-4}$$

By substitution,

$$c\dot{x}_1 + k_1(x_1 - x_2) = 0 \tag{C-5}$$

$$c\dot{x}_1 + k_1\left(x_1 - \frac{F(t)}{k_1 + k_2} - \left[\frac{k_1}{k_1 + k_2}\right] x_1\right) = 0 \tag{C-6}$$

$$c\dot{x}_1 + k_1\left(1 - \left[\frac{k_1}{k_1 + k_2}\right]\right) x_1 - \left(\frac{k_1}{k_1 + k_2}\right) F(t) = 0 \tag{C-7}$$

$$c\dot{x}_1 + k_1\left(\frac{k_1 + k_2 - k_1}{k_1 + k_2}\right) x_1 - \left(\frac{k_1}{k_1 + k_2}\right) F(t) = 0 \tag{C-8}$$

$$c\dot{x}_1 + \left(\frac{k_1 k_2}{k_1 + k_2}\right) x_1 = \left(\frac{k_1}{k_1 + k_2}\right) F(t) \tag{C-9}$$

Take the Laplace transform.

$$L\left\{ c\dot{x}_1 + \left(\frac{k_1 k_2}{k_1 + k_2}\right) x_1 \right\} = L\left\{ \left(\frac{k_1}{k_1 + k_2}\right) F(t) \right\} \tag{C-10}$$

$$c\, s\, X_1(s) - c\, x_1(0) + \left(\frac{k_1 k_2}{k_1 + k_2}\right) X_1(s) = \left(\frac{k_1}{k_1 + k_2}\right) F(s) \tag{C-11}$$

$$\left(c\, s + \left(\frac{k_1 k_2}{k_1 + k_2}\right)\right) X_1(s) = F(s) + c\, x_1(0) \tag{C-12}$$

$$X_1(s) = \frac{F(s) + c\, x_1(0)}{c\, s + \left(\frac{k_1 k_2}{k_1 + k_2}\right)} \tag{C-13}$$

Recall

$$(k_1 + k_2) x_2 - k_1 x_1 = F(t) \tag{C-14}$$

Take the Laplace transform.

$$L\{(k_1 + k_2)x_2 - k_1 x_1 \} = L\{F(t)\} \tag{C-15}$$

$$(k_1 + k_2)X_2(s) - k_1 X_1(s) = F(s) \tag{C-16}$$

$$(k_1 + k_2)\, X_2(s) = F(s) + k_1 X_1(s) \tag{C-17}$$

$$(k_1 + k_2)X_2(s) = F(s) + k_1 \left\{ \frac{F(s) + c\,x_1(0)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \tag{C-18}$$

$$X_2(s) = \frac{1}{(k_1 + k_2)} F(s) + \frac{k_1}{(k_1 + k_2)} \left\{ \frac{F(s) + c\,x_1(0)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \tag{C-19}$$

$$X_2(s) = \frac{1}{(k_1 + k_2)} F(s) + \frac{k_1}{(k_1 + k_2)} \left\{ \frac{F(s)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} + \frac{k_1}{(k_1 + k_2)} \left\{ \frac{c\,x_1(0)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\}$$

(C-20)

$$X_2(s) = \left\{ \frac{1}{(k_1 + k_2)} + \frac{k_1}{(k_1 + k_2)} \left\{ \frac{1}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \right\} F(s) + \frac{k_1}{(k_1 + k_2)} \left\{ \frac{c\,x_1(0)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\}$$

(C-21)

$$X_2(s) = \frac{1}{(k_1 + k_2)} \left\{ 1 + \left\{ \frac{k_1}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \right\} F(s) + \frac{k_1}{(k_1 + k_2)} \left\{ \frac{c\,x_1(0)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\}$$

(C-22)

Now assume the initial displacement is zero.

$$X_2(s) = \frac{1}{(k_1 + k_2)} \left\{ 1 + \left\{ \frac{k_1}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \right\} F(s) \qquad \text{(C-23)}$$

The receptance in the Laplace domain is

$$\frac{X_2(s)}{F(s)} = \frac{1}{(k_1 + k_2)} \left\{ 1 + \left\{ \frac{k_1}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \right\} \qquad \text{(C-24)}$$

$$\frac{X_2(s)}{F(s)} = \left\{ \frac{1}{(k_1 + k_2)} + \left\{ \frac{k_1}{c(k_1 + k_2)s + k_1 k_2} \right\} \right\} \tag{C-25}$$

$$\frac{X_2(s)}{F(s)} = \frac{c(k_1 + k_2)s + k_1 k_2 + k_1(k_1 + k_2)}{c(k_1 + k_2)^2 s + k_1 k_2 (k_1 + k_2)} \tag{C-26}$$

$$\frac{X_2(s)}{F(s)} = \frac{c(k_1 + k_2)s + k_1(k_1 + 2k_2)}{c(k_1 + k_2)^2 s + k_1 k_2 (k_1 + k_2)} \tag{C-27}$$

The dynamic stiffness in the Laplace domain is

$$\frac{F(s)}{X_2(s)} = \frac{c(k_1 + k_2)^2 s + k_1 k_2 (k_1 + k_2)}{c(k_1 + k_2)s + k_1(k_1 + 2k_2)} \tag{C-28}$$

The initial value problem for the free response is

$$X_2(s) = \frac{k_1}{(k_1 + k_2)} \left\{ \frac{c\, x_1(0)}{cs + \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \tag{C-29}$$

$$X_2(s) = \frac{k_1}{(k_1 + k_2)} \left\{ \frac{x_1(0)}{s + \frac{1}{c}\left(\frac{k_1 k_2}{k_1 + k_2} \right)} \right\} \tag{C-30}$$

The inverse Laplace transform yields the time domain response.

$$x_2(t) = x_1(0)\left[\frac{k_1}{k_1 + k_2}\right]\exp\left\{-\frac{1}{c}\left(\frac{k_1 k_2}{k_1 + k_2}\right)t\right\} \tag{C-31}$$

$$(k_1 + k_2)x_2 - k_1 x_1 = 0 \tag{C-32}$$

$$k_1 x_1 = (k_1 + k_2)x_2 \tag{C-33}$$

$$x_1 = \left[\frac{k_1 + k_2}{k_1}\right]x_2 \tag{C-34}$$

$$x_1(t) = x_1(0)\ \exp\left\{-\frac{1}{c}\left(\frac{k_1 k_2}{k_1 + k_2}\right)t\right\} \tag{C-35}$$

APPENDIX D

Consider the following two-degree-of-freedom system subjected to an applied force.

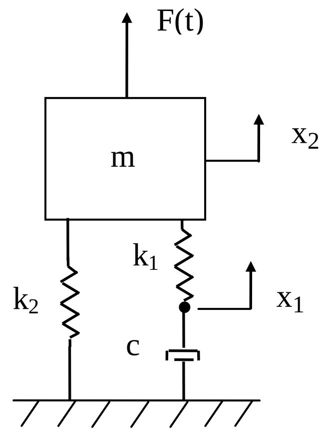

Figure D-1.

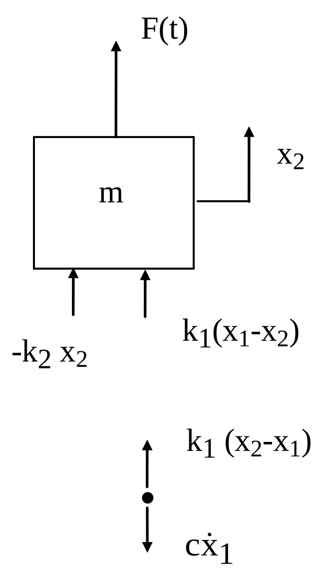

Figure D-2.

The equations of motion are

$$m\ddot{x}_2 = F(t) + k_1(x_1 - x_2) - k_2 x_2 \quad \text{(D-1)}$$

$$m\ddot{x}_2 + (k_1 + k_2)x_2 - k_1 x_1 = F(t) \quad \text{(D-2)}$$

$$c\dot{x}_1 + k_1(x_1 - x_2) = 0 \quad \text{(D-3)}$$

Take the Laplace transform of equation (D-2).

$$L\{m\ddot{x}_2 + (k_1 + k_2)x_2 - k_1 x_1\} = L\{F(t)\} \quad \text{(D-4)}$$

$$ms^2 X_2(s) - m\dot{x}_2(0) - ms\,x_2(0) + (k_1 + k_2)X_2(s) - k_1 X_1(s) = F(s) \quad \text{(D-5)}$$

$$\{ms^2 + (k_1 + k_2)\}X_2(s) - k_1 X_1(s) = F(s) + m\dot{x}_2(0) + ms\,x_2(0) \quad \text{(D-6)}$$

Take the Laplace transform of equation (D-3).

$$L\{c\dot{x}_1 + k_1(x_1 - x_2)\} = 0 \quad \text{(D-7)}$$

$$cs\,X_1(s) - c\,x_1(0) + k_1 X_1(s) - k_2 X_2(s) = 0 \quad \text{(D-8)}$$

$$\{cs + k_1\}X_1(s) - k_2 X_2(s) = c\,x_1(0) \quad \text{(D-9)}$$

Consider equations (D-6) and (D-9) for the case of zero initial conditions.

$$\{ms^2 + (k_1 + k_2)\}X_2(s) - k_1 X_1(s) = F(s) \quad \text{(D-10)}$$

$$\left\{cs+k_1\right\}X_1(s) - k_2 X_2(s) = 0 \tag{D-11}$$

$$\left\{cs+k_1\right\}X_1(s) = k_2 X_2(s) \tag{D-12}$$

$$X_1(s) = \left[\frac{k_2}{cs+k_1}\right]X_2(s) \tag{D-13}$$

$$\left[ms^2+\left(k_1+k_2\right)\right]X_2(s) - k_1\left[\frac{k_2}{cs+k_1}\right]X_2(s) = F(s) \tag{D-14}$$

$$\left\{\left[ms^2+\left(k_1+k_2\right)\right] - \left[\frac{k_1 k_2}{cs+k_1}\right]\right\}X_2(s) = F(s) \tag{D-15}$$

$$X_2(s) = \frac{F(s)}{\left[ms^2+\left(k_1+k_2\right)\right] - \left[\frac{k_1 k_2}{cs+k_1}\right]} \tag{D-16}$$

The receptance in the Laplace domain is

$$\frac{X_2(s)}{F(s)} = \frac{1}{\left[ms^2+\left(k_1+k_2\right)\right] - \left[\frac{k_1 k_2}{cs+k_1}\right]} \tag{D-17}$$

The initial value problem for the free response is

$$m\ddot{x}_2 + \left(k_1+k_2\right)x_2 - k_1 x_1 = 0 \tag{D-18}$$

$$c\dot{x}_1 + k_1(x_1 - x_2) = 0 \tag{D-19}$$

The corresponding Laplace transforms are

$$\left\{ms^2 + (k_1 + k_2)\right\}X_2(s) - k_1X_1(s) = m\dot{x}_2(0) + ms\,x_2(0) \tag{D-20}$$

$$\left\{cs + k_1\right\}X_1(s) - k_2\,X_2(s) = c\,x_1(0) \tag{D-21}$$

$$X_1(s) = \frac{k_2\,X_2(s) + c\,x_1(0)}{cs + k_1} \tag{D-22}$$

$$\left[ms^2 + (k_1 + k_2)\right]X_2(s) - k_1\left[\frac{k_2\,X_2(s) + c\,x_1(0)}{cs + k_1}\right] = m\dot{x}_2(0) + ms\,x_2(0) \tag{D-23}$$

$$\left[ms^2 + (k_1 + k_2)\right]X_2(s) - \left[\frac{k_1k_2}{cs + k_1}\right]X_2(s) - k_1\left[\frac{c\,x_1(0)}{cs + k_1}\right] = m\dot{x}_2(0) + ms\,x_2(0) \tag{D-24}$$

$$\left\{\left[ms^2 + (k_1 + k_2)\right] - \left[\frac{k_1k_2}{cs + k_1}\right]\right\}X_2(s) = m\dot{x}_2(0) + ms\,x_2(0) + \left[\frac{c\,k_1\,x_1(0)}{cs + k_1}\right] \tag{D-25}$$

$$\left\{\left[ms^2 + (k_1 + k_2)\right]\left[cs + k_1\right] - k_1k_2\right\}X_2(s) =$$

$$m(cs + k_1)\,\dot{x}_2(0) + ms(cs + k_1)x_2(0) + c\,k_1\,x_1(0) \tag{D-26}$$

$$X_2(s) = \frac{m(cs + k_1)\dot{x}_2(0) + ms(cs + k_1)x_2(0) + ck_1 x_1(0)}{\left[ms^2 + (k_1 + k_2)\right]\left[cs + k_1\right] - k_1k_2} \qquad \text{(D-27)}$$

$$X_2(s) = \frac{(mcs + mk_1)\dot{x}_2(0) + \left(mcs^2 + msk_1\right)x_2(0) + ck_1 x_1(0)}{\left[ms^2 + (k_1 + k_2)\right]cs + \left[ms^2 + (k_1 + k_2)\right]k_1 - k_1k_2} \qquad \text{(D-28)}$$

$$X_2(s) = \frac{mcs\dot{x}_2(0) + mk_1\dot{x}_2(0) + mcs^2x_2(0) + msk_1x_2(0) + ck_1 x_1(0)}{mcs^3 + c(k_1 + k_2)s + mk_1 s^2 + (k_1 + k_2)k_1 - k_1k_2} \qquad \text{(D-29)}$$

$$X_2(s) = \frac{mcs^2x_2(0) + msk_1x_2(0) + mcs\dot{x}_2(0) + mk_1\dot{x}_2(0) + ck_1 x_1(0)}{mcs^3 + mk_1 s^2 + c(k_1 + k_2)s + k_1^2} \qquad \text{(D-30)}$$

$$X_2(s) = \frac{mcs^2x_2(0) + \left[k_1x_2(0) + c\dot{x}_2(0)\right]m\,s + mk_1\dot{x}_2(0) + ck_1 x_1(0)}{mcs^3 + mk_1 s^2 + c(k_1 + k_2)s + k_1^2} \qquad \text{(D-31)}$$

$$X_2(s) = \frac{s^2x_2(0) + \left[\frac{k_1}{c}x_2(0) + \dot{x}_2(0)\right]s + k_1\left[\frac{1}{c}\dot{x}_2(0) + \frac{1}{m}x_1(0)\right]}{s^3 + \left[\frac{k_1}{c}\right]s^2 + \left[\frac{k_1 + k_2}{m}\right]s + \frac{k_1^2}{mc}} \qquad \text{(D-32)}$$

APPENDIX E

Consider the following two-degree-of-freedom system subjected to base excitation.

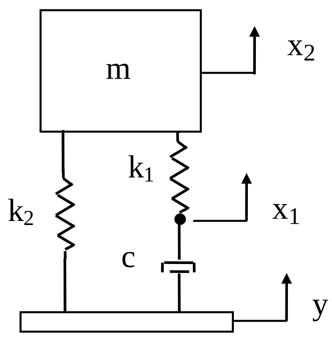

Figure E-1.

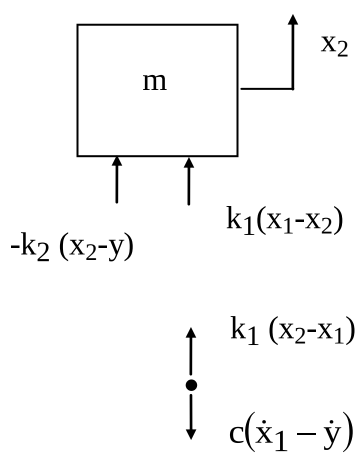

Figure E-2.

$$m\ddot{x}_2 = k_1(x_1 - x_2) - k_2(x_2 - y) \quad \text{(E-1)}$$

$$m\ddot{x}_2 + k_1(x_2 - x_1) + k_2(x_2 - y) = 0 \quad \text{(E-2)}$$

$$c(\dot{x}_1 - \dot{y}) + k_1(x_1 - x_2) = 0 \quad \text{(E-3)}$$

Let

$$z_1 = x_1 - y \quad \text{(E-4)}$$

$$z_2 = x_2 - y \quad \text{(E-5)}$$

$$z_1 - z_2 = x_1 - x_2 \quad \text{(E-6)}$$

By substitution,

$$m\ddot{z}_2 + k_1(z_2 - z_1) + k_2 z_2 = -m\ddot{y} \quad \text{(E-7)}$$

$$m\ddot{z}_2 + (k_1 + k_2)z_2 - k_1 z_1 = -m\ddot{y} \quad \text{(E-8)}$$

$$c\dot{z}_1 + k_1(z_1 - z_2) = 0 \quad \text{(E-9)}$$

Take the Laplace transform of equation (E-2).

$$L\{m\ddot{z}_2 + (k_1 + k_2)z_2 - k_1 z_1\} = -L\{m\ddot{y}\} \quad \text{(E-10)}$$

$$ms^2 Z_2(s) - m\dot{z}_2(0) - ms\, z_2(0) + (k_1 + k_2)Z_2(s) - Z_1 X_1(s) = -m\hat{Y}(s) \quad \text{(E-11)}$$

$$\{ms^2 + (k_1 + k_2)\}Z_2(s) - k_1 Z_1(s) = m\dot{z}_2(0) + ms\, z_2(0) \quad \text{(E-12)}$$

Take the Laplace transform of equation (E-3).

$$L\left\{c\dot{z}_1 + k_1(z_1 - z_2)\right\} = 0 \tag{E-13}$$

$$cs\,Z_1(s) - c\,z_1(0) + k_1\,Z_1(s) - k_2\,Z_2(s) = 0 \tag{E-14}$$

$$\left\{cs + k_1\right\}Z_1(s) - k_2\,Z_2(s) = c\,z_1(0) \tag{E-15}$$

Consider equations (E-6) and (E-9) for the case of zero initial conditions.

$$\left\{ms^2 + \left(k_1 + k_2\right)\right\}Z_2(s) - k_1 Z_1(s) = -m\hat{Y}(s) \tag{E-16}$$

$$\left\{cs + k_1\right\}Z_1(s) - k_2\,Z_2(s) = 0 \tag{E-17}$$

$$\left\{cs + k_1\right\}Z_1(s) = k_2\,Z_2(s) \tag{E-18}$$

$$Z_1(s) = \left[\frac{k_2}{cs + k_1}\right]Z_2(s) \tag{E-19}$$

$$\left[ms^2 + \left(k_1 + k_2\right)\right]Z_2(s) - k_1\left[\frac{k_2}{cs + k_1}\right]Z_2(s) = -m\hat{Y}(s) \tag{E-20}$$

$$\left\{\left[ms^2 + \left(k_1 + k_2\right)\right] - \left[\frac{k_1 k_2}{cs + k_1}\right]\right\}Z_2(s) = -m\hat{Y}(s) \tag{E-21}$$

$$Z_2(s) = \frac{-m\hat{Y}(s)}{\left[ms^2 + (k_1 + k_2)\right] - \left[\frac{k_1 k_2}{cs + k_1}\right]} \tag{E-22}$$

The relative acceleration $\hat{Z}_2(s)$ is

$$\hat{Z}_2(s) = s^2 Z_2 \tag{E-23}$$

$$\hat{Z}_2(s) = \frac{-ms^2\,\hat{Y}(s)}{\left[ms^2 + (k_1 + k_2)\right] - \left[\frac{k_1 k_2}{cs + k_1}\right]} \tag{E-24}$$

$$\hat{X}_2(s) = \frac{-ms^2\,\hat{Y}(s)}{\left[ms^2 + (k_1 + k_2)\right] - \left[\frac{k_1 k_2}{cs + k_1}\right]} + \hat{Y}(s) \tag{E-25}$$

$$\hat{X}_2(s) = \left\{ \frac{-ms^2}{\left[ms^2 + (k_1 + k_2)\right] - \left[\frac{k_1 k_2}{cs + k_1}\right]} + 1 \right\} \hat{Y}(s) \tag{E-26}$$

$$\frac{\hat{X}_2(s)}{\hat{Y}(s)} = \left\{ \frac{-ms^2}{\left[ms^2 + (k_1 + k_2)\right] - \left[\frac{k_1 k_2}{cs + k_1}\right]} + 1 \right\} \tag{E-27}$$