NATURAL FREQUENCIES OF SPRING-MASS SYSTEMS IN SERIES

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Two-degree-of-freedom System

Consider a two-degree-of-freedom system, as shown in Figure 1. Free-body diagrams are shown in Figure 2.

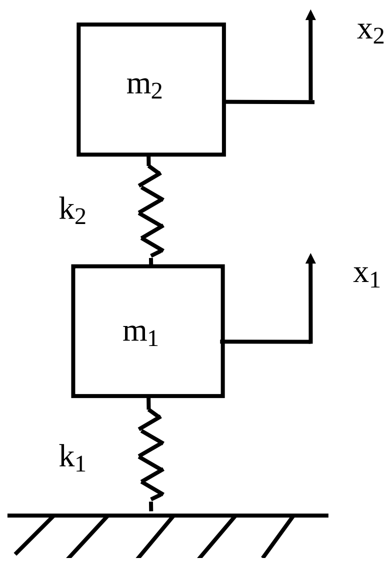


Figure 1.

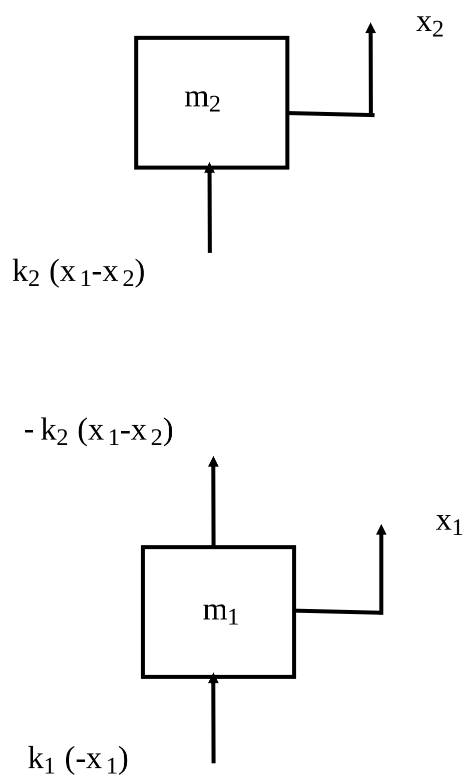


Figure 2.

Determine the equation of motion for mass 2.

$$\sum F = m_2 \ddot{x}_2 \tag{1}$$

$$m_2 \ddot{x}_2 = k_2 (x_1 - x_2) \tag{2}$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0 \tag{3}$$

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{4}$$

$$m_1 \ddot{x}_1 = - k_2 (x_1 - x_2) + k_1 (-x_1) \tag{5}$$

$$m_1 \ddot{x}_1 + k_2 (x_1 - x_2) + k_1 x_1 = 0 \tag{6}$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \tag{7}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

Represent as

$$M \ddot{\overline{x}} + K \overline{x} = F \tag{9}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{10}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{11}$$

$$F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

Consider the undamped, homogeneous form of equation (9).

$$M\ \ddot{\overline{x}} + K\ \overline{x} = \overline{0} \tag{13}$$

Seek a solution of the form

$$\overline{x} = \overline{q}\exp(j\omega t) \tag{14}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\overline{x}} = j\omega\ \overline{q}\exp(j\omega t) \tag{15}$$

$$\ddot{\overline{x}} = -\omega^2\ \overline{q}\exp(j\omega t) \tag{16}$$

Substitute these equations into equation (14).

$$-\omega^2 M\ \ \overline{q}\exp(j\omega t) + K\ \overline{q}\exp(j\omega t) = \overline{0} \tag{17}$$

$$\left\{-\omega^2 M + K\right\}\overline{q}\exp(j\omega t) = \overline{0} \tag{18}$$

$$\left\{-\omega^2 M + K\right\}\overline{q} = \overline{0} \tag{19}$$

$$\left\{K - \omega^2 M\right\}\overline{q} = \overline{0} \tag{20}$$

Equation (20) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{ K - \omega^2 M \right\} = 0 \tag{21}$$

The eigenvectors are found via the following equations.

$$\left\{ K - \omega_1{}^2\ M \right\} \bar{q}_1 = \bar{0} \tag{22}$$

$$\left\{ K - \omega_2{}^2\ M \right\} \bar{q}_2 = \bar{0} \tag{23}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{24}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{25}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\ \ \bar{q}_1 \mid \ \bar{q}_2\ \ ] \tag{26}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{27}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T\ M\ \hat{Q} = I \tag{28}$$

$$\hat{Q}^T\ K\ \hat{Q} = \Omega \tag{29}$$

where

superscript T represents transpose

I is the identity matrix

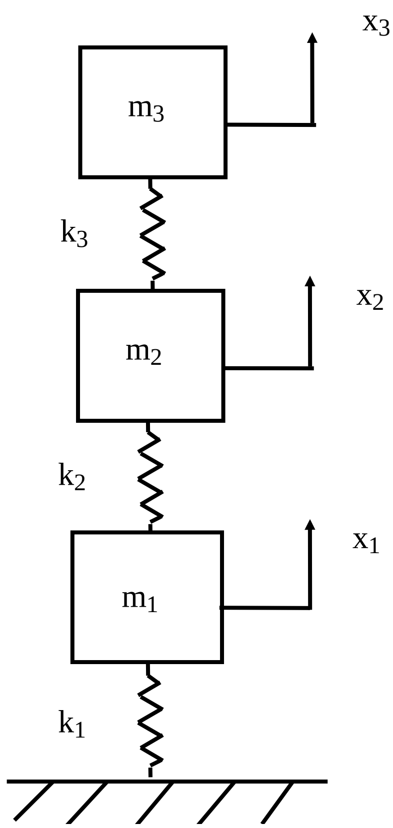
Ω is a diagonal matrix of eigenvalues

Note that

$$Q = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{30}$$

$$Q^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{31}$$

System with Three-degrees-of-freedom



The mass and stiffness matrices are given in equations (32) and (33), respectively. Both matrices are symmetric.

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ & m_1 & 0 \\ & & m_3 \end{bmatrix} \tag{32}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ & k_1 + k_2 & -k_3 \\ & & k_3 \end{bmatrix} \tag{33}$$

System with n degrees-of-freedom

Consider a system with $n \geq 5$.

The mass and stiffness matrices are given in equations (34) and (35), respectively. Both matrices are symmetric.

$$M = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & m_2 & 0 & \cdots & 0 & 0 & 0 \\ & & m_3 & \cdots & 0 & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & m_{n-2} & 0 & 0 \\ & & & & & m_{n-1} & 0 \\ & & & & & & m_n \end{bmatrix} \tag{34}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 & 0 & 0 \\ & k_2 + k_3 & -k_3 & \cdots & 0 & 0 & 0 \\ & & k_3 + k_4 & \cdots & 0 & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & k_{n-1} + k_{n-2} & -k_{n-1} & 0 \\ & & & & & k_{n-1} + k_n & -k_n \\ & & & & & & k_n \end{bmatrix} \tag{35}$$