THE STATE SPACE METHOD FOR SOLVING SHOCK AND VIBRATION PROBLEMS Revision A

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State Space Model

The state space method is used to transform a single second-order ordinary differential equation (ODE) into two first-order equations.

It is particularly useful for the numerical solution of vibration problems, including non-linear problems.

Free Vibration

The governing second-order ODE for a single-degree-of-freedom subjected to free vibration is

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = 0 \tag{A-1}$$

Equation (A-1) is taken from Reference 1.

Let

$$x_1 = x \tag{A-2}$$

$$x_2 = \dot{x}_1 \tag{A-3}$$

By substitution,

$$\dot{x}_2 + 2\xi\omega_n x_2 + \omega_n^2 x_1 = 0 \tag{A-4}$$

The resulting pairs are

$$\dot{x}_1 = x_2 \tag{A-5}$$

$$\dot{x}_2 = -2\xi\omega_n x_2 - \omega_n^2 x_1 \tag{A-6}$$

The pair of equations can be expressed in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{A-7}$$

Take the Laplace transform of each equation.

$$sX_1(s) - x_1(0) = X_2(s) \tag{A-8}$$

$$sX_2(s) - x_2(0) = -\omega_n^2 X_1(s) - 2\xi\omega_n X_2(s) \tag{A-9}$$

Rearrange,

$$sX_1(s) - X_2(s) = x_1(0) \tag{A-10}$$

$$\left[s + 2\xi\omega_n\right] X_2(s) + \omega_n^2 X_1(s) = x_2(0) \tag{A-11}$$

Reassemble in matrix form.

$$\begin{bmatrix} s & -1 \\ \omega_n^2 & s + 2\xi\omega_n \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{A-12}$$

The determinant of the coefficient matrix is

$$\det\begin{bmatrix} s & -1 \\ \omega_n^2 & s+2\xi\omega_n \end{bmatrix} = s^2 + 2\xi\omega_n s + \omega_n^2 \tag{A-13}$$

Solve for X_1 using Cramer's method.

$$X_1 = \left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} \det\begin{bmatrix} x_1(0) & -1 \\ x_2(0) & s+2\xi\omega_n \end{bmatrix} \tag{A-14}$$

$$X_1(s) = \frac{(s+2\xi\omega_n)x_1(0) + x_2(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{A-15}$$

Recall that

$$x_1 = x \tag{A-16}$$

Thus

$$X(s) = \left\{\frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{A-17}$$

The inverse Laplace transform is

$$x(t) = \exp(-\xi\omega_n t)\left\{[x(0)]\cos(\omega_d t) + \left[\frac{\dot{x}(0) + (\xi\omega_n)\,x(0)}{\omega_d}\right]\sin(\omega_d t)\right\}, \quad \xi < 1$$

(A-18)

where

$$\omega_d = \omega_n\sqrt{1-\xi^2} \tag{A-19}$$

<u>Base Excitation</u>

The relative displacement for a single-degree-of-freedom system subjected to base excitation is

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{B-1}$$

Equation (B-1) is taken from Reference 2.

Let the base excitation be a sinusoidal function.

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -A\sin(\omega t) \tag{B-2}$$

Let

$$z_1 = z \tag{B-3}$$

$$z_2 = \dot{z}_1 \tag{B-4}$$

By substitution,

$$\dot{z}_2 + 2\xi\omega_n z_2 + \omega_n^2 z_1 = -A\sin(\omega t) \tag{B-5}$$

The resulting pairs are

$$\dot{z}_1 = z_2 \tag{B-6}$$

$$\dot{z}_2 = -2\xi\omega_n z_2 - \omega_n^2 z_1 - A\sin(\omega t) \tag{B-7}$$

The pair of equations can be expressed in matrix form as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -A\sin(\omega t) \end{bmatrix} \tag{B-8}$$

Take the Laplace transform of each equation.

$$sZ_1(s) - z_1(0) = Z_2(s) \tag{B-9}$$

$$sZ_2(s) - z_2(0) = -\omega_n^2 Z_1(s) - 2\xi\omega_n Z_2(s) - \frac{A\omega}{s^2 + \omega^2} \tag{B-10}$$

Rearrange,

$$sZ_1(s) - Z_2(s) = z_1(0) \tag{B-11}$$

$$[s + 2\xi\omega_n] Z_2(s) + \omega_n^2 Z_1(s) = z_2(0) - \frac{A\omega}{s^2 + \omega^2} \tag{B-12}$$

Reassemble in matrix form.

$$\begin{bmatrix} s & -1 \\ \omega_n^2 & s + 2\xi\omega_n \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} z_1(0) \\ z_2(0) - \dfrac{A\omega}{s^2 + \omega^2} \end{bmatrix} \tag{B-13}$$

The determinant of the coefficient matrix is

$$\det\begin{bmatrix} s & -1 \\ \omega_n^2 & s+2\xi\omega_n \end{bmatrix} = s^2 + 2\xi\omega_n s + \omega_n^2 \tag{B-14}$$

Solve for Z_1 using Cramer's method.

$$Z_1 = \left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} \det\begin{bmatrix} z_1(0) & -1 \\ z_2(0) - \dfrac{A\omega}{s^2+\omega^2} & s+2\xi\omega_n \end{bmatrix} \tag{B-15}$$

$$Z_1(s) = \frac{(s+2\xi\omega_n)z_1(0) + z_2(0) - \dfrac{A\omega}{s^2+\omega^2}}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{B-16}$$

$$Z_1(s) = \frac{(s+2\xi\omega_n)z_1(0) + z_2(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} - \frac{A\omega}{\left[s^2+\omega^2\right]\left[s^2 + 2\xi\omega_n s + \omega_n^2\right]} \tag{B-17}$$

Recall that

$$z_1 = z \tag{B-18}$$

Thus

$$Z(s) = \frac{(s+2\xi\omega_n)z_1(0) + z_2(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} - \frac{A\omega}{\left[s^2+\omega^2\right]\left[s^2 + 2\xi\omega_n s + \omega_n^2\right]} \tag{B-19}$$

The inverse Laplace transform is

$$z(t) = \exp(-\xi\omega_n t)\left\{[z(0)]\cos(\omega_d t) + \left[\frac{\dot{z}(0) + (\xi\omega_n)\,z(0)}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+ \frac{A}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2\right)\sin(\omega t)\right]$$

$$- \frac{\frac{A\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\sin(\omega_d t)\right]$$

for $\xi < 1$

(B-20)

References

1. T. Irvine, The Free Vibration of a Single-degree-of-Freedom System, Rev A, Vibrationdata, 2000.

2. T. Irvine, Response of a Single-degree-of-freedom system Subjected to a Classical Pulse Base Excitation, Revision A, Vibrationdata, 1999.