KINETIC & STRAIN ENERGY FORMULAS
Revision C

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Uniform Bar under a Constant Axial Load

The total strain energy U for a bar in tension or compression is

$$U = \frac{1}{2}\frac{\sigma^2}{E} L A \tag{1.1}$$

where

| | |
|---|---|
| $\sigma$ | is axial stress |
| $E$ | is Young's modulus of elasticity |
| $L$ | is length |
| $A$ | is cross-sectional area |

Equation (1.1) is taken from Reference 1.

The axial stress $\sigma$ is related to the applied load $P$ by

$$\sigma = \frac{P}{A} \tag{1.2}$$

Thus,

$$U = \frac{1}{2}\frac{P^2 L}{E A} \tag{1.3}$$

Derive an alternate formula. The displacement e is

$$e = \frac{P\,L}{E\,A} \tag{1.4}$$

Thus,

$$U = \frac{1}{2}\,P\,e \tag{1.5}$$

Bar under a General Axial Load

$$U = \frac{1}{2}\int_0^L \frac{P^2}{E\,A}\,dx \tag{2.1}$$

Equation (2.1) is taken from Reference 1.

Another form is

$$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx \tag{2.2}$$

Note that lower case u is the axial displacement. Equation (2.2) is taken from Reference 3.

Beam in Bending

The strain energy for a beam in bending undergoing a displacement y is

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx \tag{2.3}$$

An alternate form is

$$U = \frac{1}{2}\int_0^L \frac{1}{EI}\left[\,M(x)\,\right]^2 dx \tag{3.2}$$

$$T = \frac{1}{2} \rho \Omega^2 \int_0^L [y]^2 dx \tag{3.3}$$

where

$\rho$ = mass per length

$\Omega$ = angular natural frequency

<u>Circular Rod in Torsion</u>

The strain energy for a circular rod in torsion is

$$U = \frac{1}{2} \int_0^L \frac{T^2}{GJ} dx \tag{4.1}$$

where

$T$ is the torque

$J$ is the polar moment of inertia

$G$ is the shear modulus of elasticity

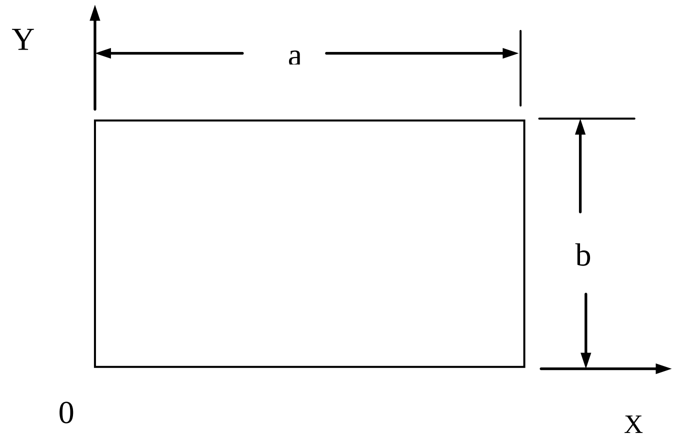
<u>Beam or Rod in Shear</u>

The shear strain energy is

$$U = \frac{1}{2} \int_0^L \frac{V^2}{GA} dx \tag{5.1}$$

where $V$ is the shear force.

Rectangular Plate in Bending




Let Z represent the out-of-plane displacement. The total strain energy of the plate is

$$U = \frac{D}{2}\int_0^b\int_0^a \left[ \left(\frac{\partial^2 Z}{\partial X^2}\right)^2 + \left(\frac{\partial^2 Z}{\partial Y^2}\right)^2 + 2\mu\left(\frac{\partial^2 Z}{\partial X^2}\right)\left(\frac{\partial^2 Z}{\partial Y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 Z}{\partial X\,\partial Y}\right)^2 \right] dXdY \tag{6.1}$$

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{6.2}$$

where

| | | |
|---|---|---|
| $E$ | = | elastic modulus |
| $h$ | = | plate thickness |
| $\mu$ | = | Poisson's ratio |

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^b \int_0^a Z^2 \, dX \, dY \tag{6.3}$$

where

| | | |
|---|---|---|
| h | = | thickness |
| $\rho$ | = | mass per volume |
| $\Omega$ | = | angular natural frequency |

An alternate form which accounts for rotary inertia is

$$T = \frac{\rho}{2} \int_0^b \int_0^a \left( \frac{h^3}{12} \dot{\alpha}^2 + \frac{h^3}{12} \dot{\beta}^2 + h \dot{z}^2 \right) dx \, dy \tag{6.4}$$

where

| | | |
|---|---|---|
| $\alpha$ | = | angular rotation about the x-axis |
| $\beta$ | = | angular rotation about the y-axis |

Equation (6.4) is taken from Reference 5.

Rectangular Plate in Bending & Shear, Stress-Strain Formulation

The following is taken from Reference 4.

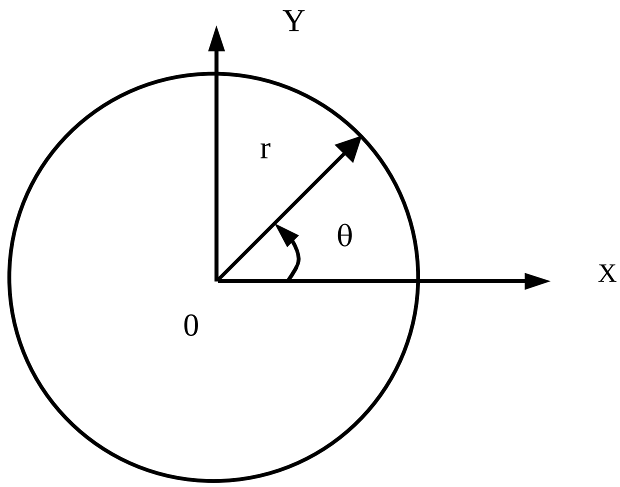
$$U = \frac{1}{2}\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix} \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} \right\} dz\, dA$$

$$+ \frac{k}{2}\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \gamma_{yz} & \gamma_{zx} \end{bmatrix} \begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} \right\} dz\, dA \qquad (7.1)$$

where

| | |
|---|---|
| $h$ | is the plate thickness |
| $A$ | is the surface area |
| $k$ | is the shear factor |
| $z$ | is the axis perpendicular to the plate |

Circular Plate in Bending




The strain energy for a circular plate in bending undergoing an out-of-plane displacement Z is

$$U = \frac{D}{2}\int_0^{2\pi}\int_0^R \left[ \left( \frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} \right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left( \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} \right) \right.$$

$$\left. + 2(1-\mu)\left\{ \frac{\partial}{\partial r}\left( \frac{1}{r}\frac{\partial Z}{\partial \theta} \right) \right\}^2 \right] r \; dr \, d\theta$$

(8.1)

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2}\int_0^{2\pi}\int_0^R Z^2 \; r \, dr \, d\theta \qquad (8.2)$$

where

h = thickness

ρ = mass per volume

Ω = angular natural frequency

Axial Spring

The strain energy is

$$U = \frac{1}{2} k x^2 \tag{9.1}$$

where $k$ is the spring stiffness and $x$ is the displacement.

Ring, In-Plane, Extensional Displacement

| | |
|---|---|
| u | is the radial displacement |
| E | is the elastic modulus |
| A | is the cross-sectional area |
| R | is the radius |

$$U = \frac{\pi A E}{r} \; u^2 \tag{10.1}$$

Equation (10.1) is taken from Reference 2.

The kinetic energy T is

$$T = \frac{\rho A}{2} \; \dot{u}^2 \; 2\pi r \tag{10.2}$$

References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5. J. Rao, Dynamics of Plates, Alpha Science International, 1999.